CHAPTER – 1
BINARY OPERATIONS

NOTES

Binary Operation

Let A be any non- empty set. Then, a function $*: A \times A \to A$ is called a binary operation or binary composition or an internal composition on the set A.

External Binary operation

Let A and S be any non- empty sets. Then, a function $*: A \times S \to S$ is called an external binary operation on S over A.

Algebraic Structure

A set equipped with one or more binary operations (internal or external) is known as an algebraic structure.

Note: Let $*$ be any operation on a non-empty set A, then $*$ is called binary operation on A, if $a*b \in A$, for every $a,b \in A$. It is called **closure law** and we say that ' A is closed with respect to* or " A satisfies closure law with respect to *'.

Associativity and Commutativity

Let $*$ be a binary operation on a non-empty set A, then.

(i) $*$ is called **commutative** , if $a*b = b*a$, for every $a,b \in A$.

(ii) $*$ is called **associative**, if $(a*b)*c = a*(b*c)$, for every $a,b,c \in A$.

Identity Element

For an algebraic structure (A, $*$) ,an element $e \in A$ (if it exists) is called an **identity element** , if $a*e = e*a = a$, for every $a \in A$.

Inverse of an Element

Let $*$ be a binary operation on a non-empty set A and let $e \in A$ be the identity element of $*$ on A. Then, an element $a \in A$ is called **invertible (or inversible)**, if there exists an element $b \in A$ such that $a*b = b*a = e$. The element b is called the inverse of a and it denoted as a^{-1}.

Distributivity

Let * and $\circ$ be two binary operations on a set A. Then , * is distributive over $\circ$, if $a*(b \circ c) = (a*b) \circ (a*c)$ (left distributive law) and $(b \circ c)*a = (b*a) \circ (c*a)$, (right distributive law) for every $a,b,c \in A$.

Theorem: The identity element for an algebraic structure, if it exists, is unique.

Theorem: If (A , *) is an algebraic structure with identity, in which the binary operation * is associative, then the inverse of an element of A , if it exists, is unique.

Theorem: Let (A , *) is an algebraic structure with identity, in which the binary operation * is associative. If a and b are two invertible elements of A, then $a*b$ is also invertible and $(a*b)^{-1}=b^{-1}*a^{-1}$.

Subsets closed under a binary operation

Definition: Let * be a binary operation on a set A and H be a subset of A. Then , H is said to be closed under *, if for every pair $(a,b)\in H\times H \Rightarrow a*b\in H$.

Composition Table (or Operation Table)

Let A be a non-empty set and let * be an operation on A. Then , we can completely describe the operation with the help of a table called***composition table*** (or ***operation table***). Let $A=\{a_1,a_2,...,a_n\}$ be a finite set and * be an operation on A .Then we can construct composition table as given below:

*	a_1	a_2	. . .	a_i	. . .	a_n
a_1	a_1*a_1	a_1*a_2	. . .	a_1*a_i	. . .	a_1*a_n
a_2	a_2*a_1	a_2*a_2	. . .	a_2*a_i	. . .	a_2*a_n
. . .	. . .	. . .	. . .	. . .	. . . .	. . .
a_i	a_i*a_1	a_i*a_2	. . .	a_i*a_i	. . .	a_i*a_n
. . .	. . .	. . .	. . .	. . .	. . .	. . .
a_n	a_n*a_1	a_n*a_2	. . .	a_n*a_i	. . .	a_n*a_n

- The elements of the left-most column are called heads of the corresponding rows.
- The elements of the top-most row are called heads of the corresponding columns.

  We can infer the following result from the composition table:

  (i) * is a binary operation if all the entries of the table belong to the set A .

  (ii) * is commutative if the composition table is symmetric about the diagonal joining the upper left corner and the lower right corner.

(iii) An element $e \in A$ is the identity element if the row headed by e coincides with the top-most row and the column headed by e coincides with the left-most column.

(iv) Let $e \in A$ be the identity element. If e appears in the entry of the table (not on the top-most row and the left-most column), then the heads of that particular row and column are **invertible** and are inverses of each other.

Some Special Operations

1. Addition modulo *n*

Let n be a positive integer, we define the operation 'addition modulo n' denoted by '$+_n$' as follows:

$a +_n b$ =the least non-negative remainder when $a + b$ is divided by n; $a, b \in Z$

For example,

$2 +_6 5 =$ the least non-negative remainder when $2 + 5$ $i.e. 7$ is divided by 6

$= 1$

2. Multiplication modulo *n*

Let n be a positive integer, we define the operation ' multiplication modulo n ' denoted by '$\times_n$' as follows:

$a \times_n b =$ the least non-negative remainder when $a \times b$ is divided by n; $a, b \in Z$.

For example,

$2 \times_6 5$ =the least non-negative remainder when 2×5 $i.e. 10$ is divided by

$= 4$

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