MATHEMATICS

for

CLASS IX

(ENGLISH MEDIUM)

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FOREWORD

The Board developed text-books under the National Curriculum Framework, 2005 to keep abreast with the national change for the schools of Manipur. The Board since its inception had been trying to promote education for betterment and quality.

The book has been developed in line with the NCF, 2005. Utmost care has been taken to make it suitable to the local needs and schools of Manipur. Every effort has been made to make the book worthwhile. In the course of preparation, a series of meetings was held with the authors, reviewers etc. to bring it to the present form.

I sincerely thank the authors, reviewers and all others who had helped to make the book presentable and suitable for use by the students.

The Board would welcome any suggestions for further improvement of the text-book.

Dr. Chithung Mary Thomas
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CHAPTER

1 NUMBER SYSTEM

1.1 Introduction

Starting from the natural numbers, the number system has been successively extended. From the system of natural numbers we pass onto that of the whole numbers by introducing the number zero. Again by using the idea of signed numbers, the negatives of the natural numbers are introduced and we pass onto the system of integers. Further with the introduction of positive and negative fractions, the number system is again extended to get the system of rational numbers. From the way of extension at each stage, it is clear that ***every natural number is a whole number, every whole number is an integer and every integer is a rational number***. In this chapter we aim at extending the system of rational numbers to form what is called the system of real numbers by introducing irrational numbers.

Let us first recall some fundamental properties of the rational numbers. You know that 0, 1, $\frac{2}{3}$, $\frac{-3}{2}$ etc. are all rational numbers. In fact, a number that can be put into the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$, is a rational number. The condition $q \neq 0$ indicates the exclusion of the case of division by zero. The result of performing any one of the four basic operations of arithmetic (i.e. addition, subtraction, multiplication and division) in respect of any two rational numbers (excluding, of course division by zero) is again a rational number. In class VII you have learnt that if $\frac{p}{q}$ and $\frac{r}{s}$ are any two rational numbers such that $\frac{p}{q} < \frac{r}{s}$ then $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$. Thus , between two unequal rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ there lies a rational number $\frac{p+r}{q+s}$.

Let us again consider two rational numbers, denoted by a and b such that $a < b$. Now

$$a < b \Rightarrow \quad a + b < b + b \quad \text{(adding } b \text{ to both sides)}$$

$$\Rightarrow \quad a + b < 2b$$

$$\Rightarrow \quad \frac{a+b}{2} < b \text{ (dividing both sides by 2) .... (1)}$$

$$\text{Again, } a < b \Rightarrow \quad a + a < a + b$$

$$\Rightarrow \quad 2a < a + b$$

$$\Rightarrow \quad a < \frac{a+b}{2} \quad \text{............. (2)}$$

Combining (1) and (2), we obtain

$$a < \frac{a+b}{2} < b$$

i.e. $\frac{a+b}{2}$ is a rational number lying between a and b.

We have thus shown that between two unequal rational numbers a and b there lies another rational number $\frac{a+b}{2}$. For instance between the rational numbers 1 and 2 we can insert a rational number $\frac{1+2}{2}$ i.e. $\frac{3}{2}$ so that

$$1 < \frac{3}{2} < 2$$

Again, we can insert $\frac{1}{2}\left(1+\frac{3}{2}\right)$ i.e. $\frac{5}{4}$ between 1 and $\frac{3}{2}$ and $\frac{1}{2}\left(\frac{3}{2}+2\right)$ i.e. $\frac{7}{4}$ between $\frac{3}{2}$ and 2 so that

$$1 < \frac{5}{4} < \frac{3}{2} < \frac{7}{4} < 2$$

Proceeding in the same way, in the next step we obtain

$$1 < \frac{9}{8} < \frac{5}{4} < \frac{11}{8} < \frac{3}{2} < \frac{13}{8} < \frac{7}{4} < \frac{15}{8} < 2$$

The process may go on indefinitely and infinite number of rational numbers may be found each greater than 1 but less than 2. Thus, there lie an infinite number of rational numbers between two unequal rational numbers 1 and 2. You may now conclude that, ***there are an infinite number of rational numbers between two unequal rational numbers.***

Remark : In the system of natural numbers or whole numbers or integers, given any number, say *a,* you can find the next greater number $a + 1$ so that no number in the system is left between a and $a + 1$. For instance between 3 and 4 (= 3 + 1) no integer has been left. However, in the system of rational numbers you cannot do so. Given a rational number say *a*, you cannot find the next greater rational number. Because if you choose a rational number say *b,* to be the next higher number you have left an infinite number of other rational numbers lying between *a* and *b*.

Example 1 : Find four rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$.

Solution 1 : We know that the rational number $\frac{p+r}{q+s}$ lies between $\frac{p}{q}$ and $\frac{r}{s}$. Hence we proceed as follows

$$\frac{1}{3} < \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{3} < \frac{1+1}{3+2} < \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{3} < \frac{2}{5} < \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{3} < \frac{3}{8} < \frac{2}{5} < \frac{3}{7} < \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{3} < \frac{4}{11} < \frac{3}{8} < \frac{2}{5} < \frac{3}{7} < \frac{1}{2}$$

So, four required rational numbers are $\frac{4}{11}, \frac{3}{8}, \frac{2}{5}$ and $\frac{3}{7}$.

Solution 2 : One rational number that lies between $\frac{1}{3}$ and $\frac{1}{2}$ is

$$\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right) \quad \text{i.e. } \frac{5}{12}. \text{ Thus } \frac{1}{3} < \frac{5}{12} < \frac{1}{2}.$$

Similarly, we can find rational numbers between $\frac{1}{3}$ and $\frac{5}{12}$, between $\frac{5}{12}$ and $\frac{1}{2}$ and write

$$\frac{1}{3} < \frac{1}{2}\left(\frac{1}{3}+\frac{5}{12}\right) < \frac{5}{12} < \frac{1}{2}\left(\frac{5}{12}+\frac{1}{2}\right) < \frac{1}{2}$$

i.e. $$\frac{1}{3} < \frac{3}{8} < \frac{5}{12} < \frac{11}{24} < \frac{1}{2}$$

Finally inserting one more rational number, say between $\frac{1}{3}$ and $\frac{3}{8}$ we write

$$\frac{1}{3} < \frac{17}{48} < \frac{3}{8} < \frac{5}{12} < \frac{11}{24} < \frac{1}{2}$$

so that four of the desired rational numbers are

$$\frac{17}{48}, \frac{3}{8}, \frac{5}{12} \text{ and } \frac{11}{24}.$$

Solution 3 : We first write the given fractions with common denominator as follows:

$$\frac{1}{3} = \frac{2}{6}$$
$$\frac{1}{2} = \frac{3}{6}$$

Then we multiply both numerator and denominator of each of the new fractions by a suitable number so that the difference between the resulting numerators is greater than or equal to the number of rational numbers required plus one. Here, the suitable number is 5 and we write

$$\frac{1}{3} = \frac{2}{6} = \frac{10}{30}$$
$$\frac{1}{2} = \frac{3}{6} = \frac{15}{30}$$

It is now readily seen that $\frac{11}{30}, \frac{12}{30}, \frac{13}{30}, \frac{14}{30}$ are rational numbers all lying between $\frac{1}{3}$ and $\frac{1}{2}$. Thus four required rational numbers are $\frac{11}{30}, \frac{2}{5}, \frac{13}{30}, \frac{7}{15}$.

EXERCISE 1.1

1. What is a rational number ? Is it true that every integer is a rational number?

2. If a and b are two unequal rational numbers, show that $\frac{a+b}{2}$ is a rational number lying between a and b.

3. Insert four rational numbers between 2 and 3.

4. Find five rational numbers between $\frac{1}{4}$ and $\frac{1}{3}$.

5. Find six rational numbers between $\frac{1}{3}$ and $\frac{2}{3}$.

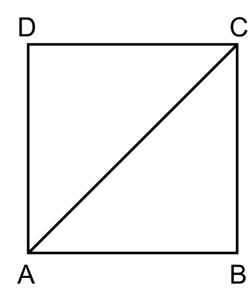
1.2 Irrational Numbers

You have seen that the collection of rational numbers is infinite i.e. there are infinitely many rational numbers. And given two unequal rational numbers, however small their difference may be, there are an infinite number of rational numbers between them. Thus, there are infinitely many rational numbers between 0 and 1, also between 0 and $\frac{1}{2}$ and between 0 and $\frac{1}{8}$ etc. Further, all the rational numbers may be represented by points on a number line. Think of a number line on which each of the rational numbers is represented by a point. Infinite number of points on the line should have been consumed in representing the rational numbers. A pertinent question that arises is "Whether all the points on the number line are exhausted or are there points still left on the line not representing any rational number ?" In short, "Are there points on the number line which do not represent any rational number ?"

The answer to this question is affirmative. There are points on the number line (in fact, infinitely many of them) which do not represent rational numbers. Around 400 BC, this fact was known to the Pythagoreans, the followers of the great Greek Mathematician and philosopher Pythagoras (569 BC – 479 BC). You may recall Pythagoras theorem which states that "The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides." This theorem enables us to determine the diagonals of a rectangle of given length and breadth and hence the diagonals of a square of given side.

Consider a square ABCD whose side is of one unit in length. The diagonal AC is of length given by

$$AC^2 = AB^2 + BC^2 \text{ (Pythagoras Theorem)}$$
$$= 1^2 + 1^2$$
$$= 2$$
$$\therefore \quad AC = \sqrt{2}$$



Let us now prove that *"There is no rational number whose square is 2"* or equivalently, "$\sqrt{2}$ *is not a rational number."*

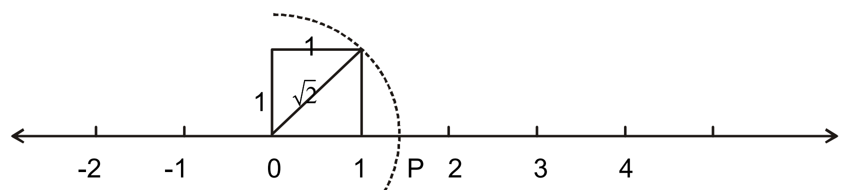
Suppose $\sqrt{2}$ is rational. Then there exist integers p and q such that $q \neq 0$, p, q are coprime and $\frac{p}{q} = \sqrt{2}$.

$$\therefore \quad \frac{p^2}{q^2} = 2 \text{ i.e. } \frac{p^2}{q} = 2q \quad \text{... (1)}$$

Since p and q have no common factor other than 1, therefore the left hand side of (1) is a fraction which cannot be reduced to simpler form whereas the right hand side is an integer. The equality holds only when $q = 1$ in which case the left hand side is an integer. But when $q = 1$ we get $p^2 = 2$. This is a contradiction since there is no integer whose square is 2. [Observe that $1^2 < 2 < 2^2$ and there is no other integer between 1 and 2].

This contradiction is due to our assumption that $\sqrt{2}$ is rational. Hence $\sqrt{2}$ is not rational.

It is now seen that the number indicating the length of a diagonal of a square whose side is one unit in length, is not a rational number. Accordingly the number represented on the number line by the point P where OP (the point O represents the number zero) is equal to the length of a diagonal of a unit square is not a rational number. This indicates that there are points on the number line not representing rational numbers. And the system of rational numbers is inadequate to cover all the points on the number line. As such the necessity of formulating a more comprehensive system of numbers



called real numbers is felt so that to each point on the number line there corresponds a number of the system and vice versa.

That, formulation of the system of real numbers may be done in various ways has been shown by Mathematicians like Dedekind, Cantor, Weierstrass etc. It is beyond the scope of this book to discuss systematically the definition of real numbers. However a round-about definition may be given on the basis of Dedekind-Cantor Axiom which states:

"To every real number there corresponds a unique point on the number line and to every point on the number line there corresponds a unique real number."

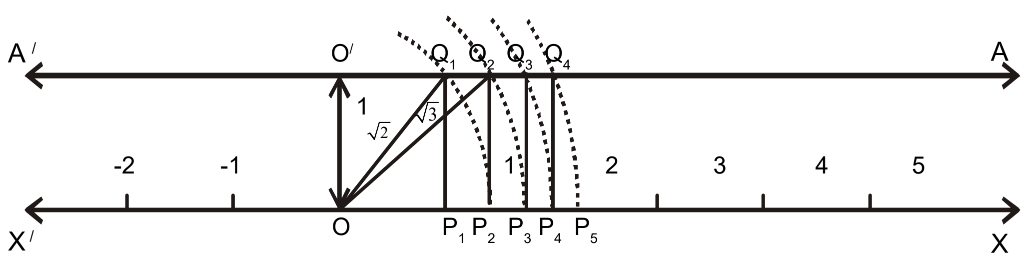
Definition : Irrational numbers are numbers represented on the number line by points other than those representing rational numbers.

Definition : Real numbers are numbers which are either rational or irrational.

Thus, irrational numbers are real numbers which are not expressible as a ratio of two integers. Examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. The number π, denoting the ratio of circumference of a circle to its diameter is also an irrational number. The square root of any positive integer other than the square numbers 1^2, 2^2, 3^2, 4^2, ... may be shown to be irrational (the proof being analogous to that of $\sqrt{2}$). Similarly the cube root of any positive integer different from the integers 1^3, 2^3, 3^3, 4^3, ... is an irrational number.

1.3 Representation of Square Roots of Positive Integers on Number Line :

Let XOX' be a number line, O being the point that represents the number zero. Taking a suitable scale, points are marked on $\overrightarrow{OX}$, unit distance apart, to represent the integers 1, 2, 3, 4 etc. A straight line $AO'A'$ is drawn parallel to XOX' at a unit distance from it. Through the point P_1 representing 1, we draw P_1Q_1 perpendicular to $\overrightarrow{OX}$ meeting $\overrightarrow{O'A}$ at Q_1. With centre O and radius OQ_1 an arc is drawn intersecting $\overrightarrow{OX}$ at P_2. Then P_2 represents $\sqrt{2}$. Through P_2, a line P_2Q_2 is drawn perpendicular to $\overrightarrow{OX}$ intersecting $\overrightarrow{O'A}$ at Q_2. With centre O and radius OQ_2 an arc is drawn intersecting $\overrightarrow{OX}$ at P_3. Then P_3 represents $\sqrt{3}$. Again a line perpendicular to $\overrightarrow{OX}$ is drawn through P_3



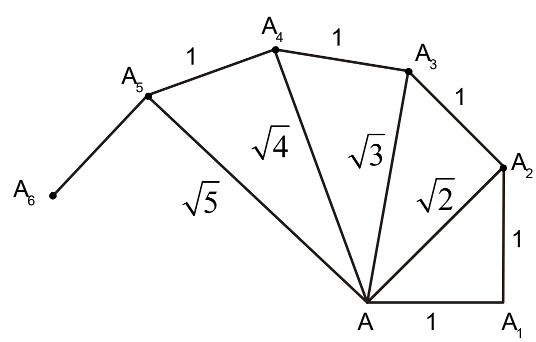
intersecting $\overrightarrow{O'A}$ at Q_3. With centre O and radius OQ_3 an arc is drawn intersecting $\overrightarrow{OX}$ at P_4. Clearly P_4 represents $\sqrt{4}$ i.e. P_4 coincides with the point representing 2.

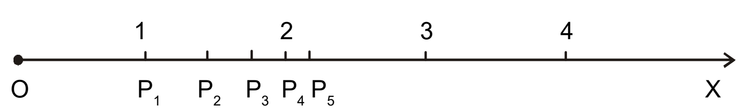
Proceeding in this way we can find successively the other points P_5, P_6, P_7 etc. representing $\sqrt{5}, \sqrt{6}, \sqrt{7}$ etc. Having found the point P_n representing $\sqrt{n}$, we can find the next point P_{n+1} by drawing a perpendicular to $\overrightarrow{OX}$ through P_n meeting $\overrightarrow{O'A}$ at the point Q_n and then drawing an arc with centre O and radius OQ_n to intersect $\overrightarrow{OX}$ at this point P_{n+1}.

The points P_1, P_2, P_3 etc. representing $\sqrt{1}, \sqrt{2}, \sqrt{3}$ etc. may also be found on the number line by constructing what is known as *square root spiral.*

Draw a line segment AA_1 of unit length. Also draw a line segment A_1A_2 of unit length, perpendicular to AA_1.

Join AA_2 and draw line segment A_2A_3 of unit length perpendicular to AA_2. Join AA_3 and draw line segment A_3A_4 of unit length perpendicular to AA_3. Proceeding in this way, we find successively the points A_1, A_2, A_3 etc. such that AA_1, AA_2, AA_3 etc. are of lengths $\sqrt{1}, \sqrt{2}, \sqrt{3}$ etc.

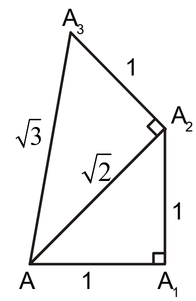




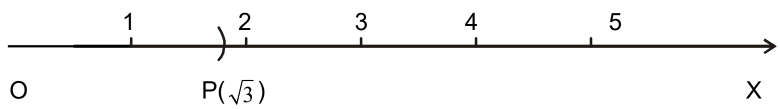
On the number ray $\overrightarrow{OX}$ we can mark points P_1, P_2, P_3 etc. in such a way that $OP_1 = AA_1$, $OP_2 = AA_2$, $OP_3 = AA_3$ etc. These points P_1, P_2, P_3 etc. represent the numbers $\sqrt{1}, \sqrt{2}, \sqrt{3}$ etc.

Example 2. Represent $\sqrt{3}$ on the number line.

Solution : Take a line segment AA_1 of unit length and draw a line segment A_1A_2 also of unit length, perpendicular to AA_1.



Join AA_2 and draw line segment A_2A_3 of unit length perpendicular to AA_2. Join AA_3. On the number ray $\overrightarrow{OX}$, mark a point P such that OP = AA_3. Then P is the point P($\sqrt{3}$) representing $\sqrt{3}$ on the number line.

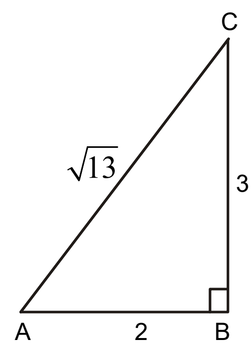


In case of any integer which can be expressed as sum of two square numbers, a short cut method can be used to locate its square root, as illustrated below :

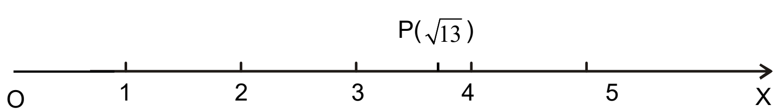
Example 3 : Represent $\sqrt{13}$ on the number line.

Solution : It is seen that $13 = 4 + 9 = 2^2 + 3^2$.

Draw a line segment AB of length 2 units. Draw a line segment BC of length 3 units, perpendicular to AB. Join AC. Now



$$AC^2 = AB^2 + BC^2$$
$$= 2^2 + 3^2$$
$$= 13$$
$$\therefore \quad AC = \sqrt{13}.$$



Mark the point P on the number ray $\overrightarrow{OX}$ such that OP = AC. Then P is the required point P($\sqrt{13}$) representing $\sqrt{13}$ on the number line.

1.4 Real Numbers and their Representation on the Number Line

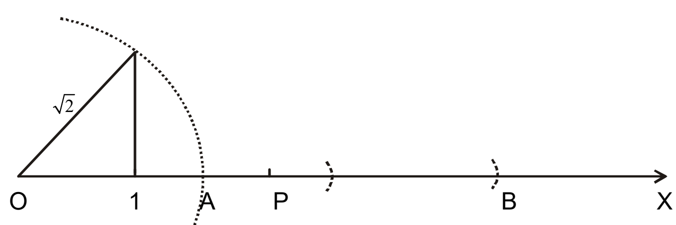
The rational numbers and irrational numbers taken together form the system of real numbers. In this system there is a number corresponding to each point on the number line. Also to each number of the system there corresponds a point on the number line. In other words, there is a correspondence between the system of real numbers and the totality of all points on the number line. The collection of all real numbers is denoted by **R**. In view of the correspondence, a number line provides a complete geometrical picture of **R**.

A point on the number line is termed as a rational point or an irrational point according as it represents a rational number or an irrational number. We see that infinite number of points lie between any two different points on the number line. This indicates that between two unequal real numbers there lie infinitely many real numbers. Of these real numbers lying between two given unequal real numbers, infinitely many are rational and infinitely many are irrational. You can now guess the juxtaposition of rational points and irrational points on the number line.

You have studied the representation of rational numbers on the number line in Class VII. In general, to represent a rational number $\frac{m}{n}$ (where m, n are integers and $n > 0$) we take the point M representing the integer m on the number line and then divide OM (O represents zero) into n equal parts. If P is the point of division nearest to O then it represents $\frac{m}{n}$. Thus, given a real number x, if it is rational we can locate on the number line, the point P(x) representing x, using the illustrated process. However if the given number is irrational there is no generalised process to find the exact position of the point representing the same, on the number line. Only in some exceptional cases when the given number can be put into the form $r\sqrt{m} + s\sqrt{n}$; r, s being rational and m, n being positive integers, the exact position of the point can be located. The representation in such a case is as illustrated in the following examples.

Example 4 : Represent $\frac{3\sqrt{2}}{2}$ on the number line.

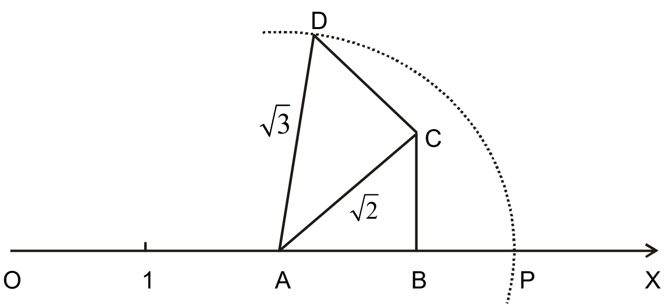
Solution : The point A representing $\sqrt{2}$ is located on the number line.



The segment OB = 3 × OA is measured to get the point B representing $3\sqrt{2}$. Then OB is bisected at P. The point P, so obtained, represents $\frac{3\sqrt{2}}{2}$.

Example 5 : Locate the point representing $2+\sqrt{3}$ on the number line.

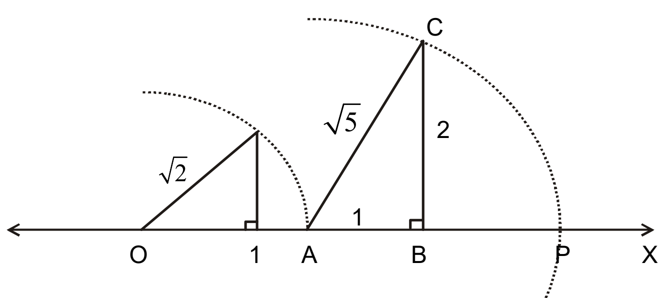
Solution : First, the points A, B representing 2, 3 respectively are located on the number line. Then we draw successively segments BC and CD of unit length, perpendicular to AB and AC respectively.



With A as centre and AD as radius an arc is drawn, intersecting $\overrightarrow{AX}$ at P. Clearly P represents $2+\sqrt{3}$ on the number line.

Example 6 : Represent $2+\sqrt{5}$ on the number line.

Solution : The point A representing $\sqrt{2}$ is located on the number line. A point B is marked on $\overrightarrow{AX}$ such that AB = 1 (unit). A line segment BC of length 2 (units) is drawn perpendicular to AB. With centre A and radius AC, an arc is drawn cutting $\overrightarrow{AX}$ at P. As OA = $\sqrt{2}$ and AP = $\sqrt{5}$, the point P therefore, represents $\sqrt{2}+\sqrt{5}$ on the number line.



There are infinitely many irrational numbers that cannot be expressed in the form $r\sqrt{m}+s\sqrt{n}$. In fact, infinitely many irrational numbers are there, which cannot be expressed by using known arithmetical symbols such as digits, radicals, powers etc. There is no well known process to locate the exact position of the point representing such an irrational number. However, given an irrational number, we may find an approximate position of the point representing the same on the number line, by expressing the number in decimal expansion.

We assume that every real number has a decimal representation. In previous classes, you have learnt that every rational number has either a terminating or a non-terminating recurring decimal representation and conversely every terminating or recurring decimal represents a rational number.

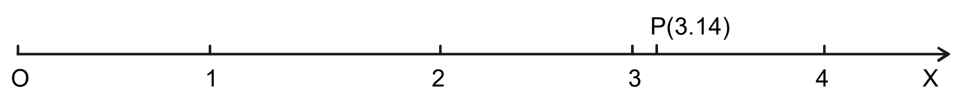
There are non-terminating and non-recurring decimals such as

$$0.202002000200002\ ...$$

(Here, in this decimal there are only two distinct digits namely 2 and 0, and 2's are separated by one zero in the beginning, next by two zeros, next by three zeros and so on).

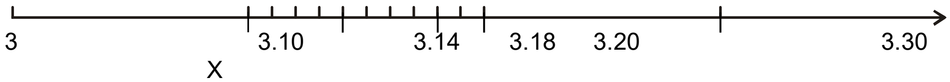
What numbers will such decimals represent ? Of course, irrational numbers, since they cannot represent rational numbers. Conversely each irrational number will have a non-terminating and non-recurring decimal representation, for if the decimal is terminating or recurring the corresponding number is rational. From this consideration, we can also define irrational numbers as non-terminating and non-recurring decimals.

Consider the irrational number π, denoting the ratio of the circumference of a circle to its diameter. Its decimal expansion (upto four places of decimal) is 3.1416. To represent the number π on the number line, we first approximate its value correct to a desired number of decimal places and use the approximate value to find the corresponding



point. For instance, the value of π correct to two decimal places is 3.14 and we locate the point P on the number line representing the rational number 3.14 and use this point P as the point P(π) representing the number π. We know that P is not the exact position of the point P(π) but very close to it.

To find the point P(3.14) we are to divide the unit length between 3 and 4 into 100 equal parts and take first 14 points. In fact, the point P(3.14) divides the segment joining 3.1 and 3.2 in the ratio 4:6 i.e. 2:3. With due magnification, the point representing 3.14 is shown below :



The knowledge of the form of decimal expansion of irrational numbers enables us to find irrational numbers lying between two given real numbers.

Example 7 : Write any three rational numbers and any three irrational numbers lying between $\frac{2}{5}$ and $\frac{3}{5}$.

Solution : We know that

$$\frac{2}{5} = 0.4 \quad \text{and} \quad \frac{3}{5} = 0.6$$

Three rational numbers between them are 0.45, 0.46, 0.5. And decimal expansion of three irrational numbers lying between them are as given below :

(i) 0.45045004500045 ...

(ii) 0.5050050005 ...

(iii) 0.5151151115 ...

(The above decimals are non-terminating and non-recurring, hence they are irrational numbers and all three of them lie between 0.4 and 0.6)

EXERCISE 1.2

1. Show that there exist points on the number line not representing rational numbers.

2. Prove that there is no rational number whose square is 3.

3. Prove that $\sqrt{5}$ is an irrational number.

4. Prove that $\sqrt{7}$ is not a rational number.

5. Represent the following number on the number line.

(i) $\sqrt{5}$ **(ii)** $\sqrt{7}$ **(iii)** $\sqrt{11}$ **(iv)** $\sqrt{20}$ **(v)** $\sqrt{29}$

6. Locate on the number line the point representing the following number :

(i) $2\sqrt{3}$ **(ii)** $3\sqrt{2}$ **(iii)** $\frac{\sqrt{5}}{2}$ **(iv)** $3 + \sqrt{2}$ **(v)** $2 + \sqrt{5}$

(vi) $4 - \sqrt{5}$ **(vii)** $\sqrt{2} + \sqrt{3}$ **(viii)** $\sqrt{5} - \sqrt{2}$

7. Write any three rational numbers and any three irrrational numbers lying between 2.1 and 2.2

8. Write any four irrational numbers lying between $\frac{1}{3}$ and $\frac{1}{2}$.

1.5 Existence of $\sqrt{x}$ for a given Positive Real Number *x*.

If n is a natural number, then $\sqrt{n}$ stands for the positive real number p such that p^2 = n. Thus, $\sqrt{n} = p$ means that $p^2 = n$ and $p > 0$. Similar definition may be given for $\sqrt{x}$ where x is a positive real number.

Let x be a positive real number. Then $\sqrt{x} = y$ means that $y^2 = x$ and $y > 0$.

It can be shown geometrically the existence of $\sqrt{x}$ for any real number $x > 0$ and the representation of the same on the number line. The idea involved in the process is based on the well known algebraic identity :

$$(a + b)^2 - (a - b)^2 = 4ab$$

Taking $a = x$ and $b = 1$, the identity becomes

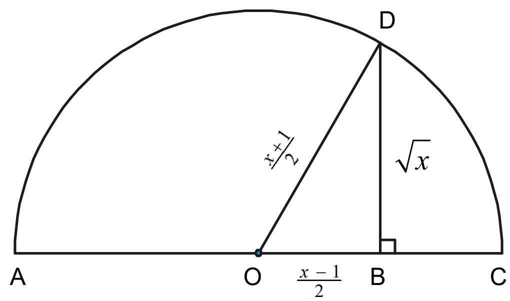
$$(x + 1)^2 - (x - 1)^2 = 4x$$

$$\Rightarrow \quad \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = x$$

$$\Rightarrow \quad \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2 = \left(\sqrt{x}\right)^2$$

Consider a right triangle whose hypotenuse and one side are of lengths $\frac{x+1}{2}$ and $\frac{x-1}{2}$ respectively. Then from the above identity it follows that the remaining side of the triangle is of length $\sqrt{x}$. To construct one such right triangle we proceed as under :

Points A, B, C are marked on a line such that AB = x units and BC = 1 unit. AC is bisected at O and a semi-circle is drawn with O as centre and AC as diameter. A line is drawn through B perpendicular to AC, intersecting the semi-circle at D. Then OBD is the desired right triangle in which the hypotenuse OD = $\frac{x+1}{2}$ and OB = OC − BC = $\frac{x+1}{2} - 1 = \frac{x-1}{2}$.



$$\text{Now,}\quad BD^2 = OD^2 - OB^2$$

$$= \left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2$$

$$= x$$

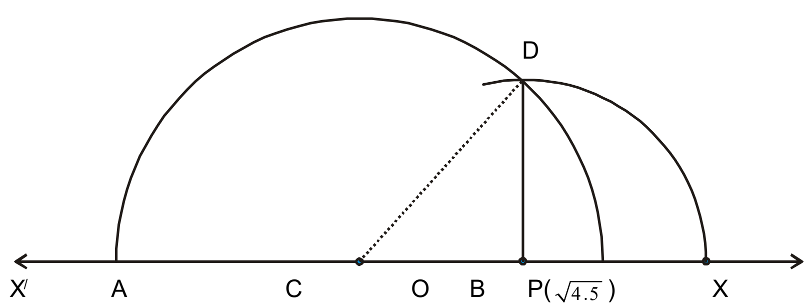
$$\therefore\quad BD = \sqrt{x}$$

This is the visual and geometric proof for the existence of $\sqrt{x}$ for any positive real number x.

Since we can construct a line segment of length $\sqrt{x}$ units for any given positive real number x, therefore we can represent $\sqrt{x}$ on the number line using the above illustrated process.

Example 8 : Represent $\sqrt{4.5}$ on the number line.

Solution : On the number line XOX′, points A and B are marked representing the numbers, – 4.5 and 1 respectively. Clearly A and B lie on opposite sides of O and the length AB is of 5.5 units. Bisect AB at C and draw the semi-circle having C as centre



and AB as the diameter. Draw OD perpendicular to $\overrightarrow{OX}$ meeting the semi-circle at D.

Again draw an arc, with centre O and radius OD to intersect $\overrightarrow{OX}$ at the point P. Then P represents $\sqrt{4.5}$ on the number line.

1.6 The *n*th Root of a Real Number

Let us now extend the idea of square roots to cube roots, fourth roots, fifth roots etc. Generalising the idea, the *n*th root of a given positive real number is defined when *n* is a positive integer.

Since $2^3 = 2 \times 2 \times 2 = 8$, therefore $\sqrt[3]{8} = 2$. Thus, cube root of 8 i.e. $\sqrt[3]{8}$ is the positive number 2 whose cube is 8. If x is a positive real number, then $\sqrt[3]{x} = y$ means $y^3 = x$ and $y > 0$.

Again $3^4 = 3 \times 3 \times 3 \times 3 = 81$ so that $\sqrt[4]{81} = 3$. Thus fourth root of 81 i.e. $\sqrt[4]{81}$ is the positive number 3 whose fourth power is 81.

If x is a positive real number, then $\sqrt[4]{x} = y$ means $y^4 = x$ and $y > 0$.

In general, if x is a positive real number and n is a positive integer then $\sqrt[n]{x} = y$ means $y^n = x$ and $y > 0$.

Here, $\sqrt[n]{x}$ is called the n^{th} root of the positive number x and it is a positive real number by definition. As in the case of rational numbers, the n^{th} root of a real number x is also expressed as $x^{\frac{1}{n}}$. Thus,

$$\sqrt[n]{x} = x^{\frac{1}{n}} = y \quad \text{provided} \quad y^n = x \quad \text{and} \quad y > 0.$$

1.7 Operations on Real Numbers and Laws of Exponents

In earlier classes you have studied that the system of rational numbers satisfy closure property with respect to addition and multiplication i.e. the sum and product of two rational numbers are rational numbers. Also rational numbers satisfy associative and commutative laws of addition and multiplication. Further, multiplication distributes over addition in this system. All these laws (closure, associativity, commutativity and distributivity) hold in the case of real numbers also. And the existence of the additive inverse or negative, $-x$ for any real number x and the multiplicative inverse or reciprocal, $\frac{1}{x}$ for any non-zero real number x enables us to perform the operations of subtraction and division (except by zero) in the system of real numbers.

Unlike the case of rational numbers, the sum or product of two irrational numbers need not be an irrational number. For example, $\sqrt{3}+\left(-\sqrt{3}\right)$ and $\sqrt{3}\times\sqrt{3}$ are not irrational. Also $\frac{\sqrt{3}}{\sqrt{3}}$ and $\sqrt{3}-\sqrt{3}$ are not irrational, whereas $\sqrt{3}+\sqrt{2}$, $\sqrt{3}-\sqrt{2}$, $\sqrt{3}\times\sqrt{2}$ and $\frac{\sqrt{3}}{\sqrt{2}}$ are all irrational.

The sum, difference, product and quotient of two non-zero real numbers, one rational and other irrational are all irrational. For example, the numbers $2+\sqrt{3}$, $2-\sqrt{3}$, $2\sqrt{3}$, $\frac{2}{\sqrt{3}}$ and $\frac{\sqrt{3}}{2}$ are all irrational.

Consider the case of $2+\sqrt{3}$. Let $2+\sqrt{3} = x$. Clearly x is a real number which is either rational or irrational. Suppose x is rational. Then $x - 2$, being difference of two rational numbers must be rational. But $x - 2 = \sqrt{3}$ so that a rational number $x - 2$ is found to be equal to an irrational number $\sqrt{3}$. This is a contradiction. Hence x i.e. $2+\sqrt{3}$ is not a rational number. So $2+\sqrt{3}$ is irrational.

Using similar reasoning, it may be shown that $2-\sqrt{3}$, $2\sqrt{3}$, $\frac{2}{\sqrt{3}}$ and $\frac{\sqrt{3}}{2}$ are all irrational numbers. In general if r is a non-zero rational number and s an irrational number, then $r + s$, $r - s$, rs, $\frac{s}{r}$ and $\frac{r}{s}$ are all irrational numbers.

Example 9 : Find the sum of $3+2\sqrt{2}$ and $5-3\sqrt{2}$

Solution :

$$3+2\sqrt{2} + 5-3\sqrt{2} = 3 + 5 + 2\sqrt{2} - 3\sqrt{2}$$
$$= 8 + (2 - 3)\sqrt{2}$$
$$= 8 + (-1)\sqrt{2}$$
$$= 8 - \sqrt{2}.$$

Example 10 : Find the product of $3\sqrt{2} - 2\sqrt{3}$ and $2\sqrt{2}+5\sqrt{3}$

Solution :

$$(3\sqrt{2} - 2\sqrt{3})(2\sqrt{2}+5\sqrt{3})$$
$$= 3\sqrt{2}\left(2\sqrt{2} + 5\sqrt{3}\right)-2\sqrt{3}\left(2\sqrt{2} + 5\sqrt{3}\right)$$
$$= 6\sqrt{4}+15\sqrt{6} - 4\sqrt{6} - 10\sqrt{9}$$
$$= 12 + \left(15 - 4\right)\sqrt{6} - 30$$
$$= -18 + 11\sqrt{6}$$

Let us now recall the terms base, exponent and power.

You know that $2^3 = 2 \times 2 \times 2 = 8$. In the notation 2^3, the number 2 is the base, 3 is the exponent or index and 2^3 is a power of 2. The laws of exponents for rational base and integral exponents may be written as follows :

If m, n are integers and x, y are non-zero rational numbers, then

(i) $x^m \times x^n = x^{m+n}$

(ii) $\frac{x^m}{x^n} = x^{m-n}$

(iii) $(xy)^m = x^m y^m$

(iv) $(x^m)^n = x^{mn}$

Taking $m = n$ in (ii) we get

$$\frac{x^n}{x^n} = x^{n-n} \quad \text{or } 1 = x^0$$

Thus, $\boxed{x^0 = 1}$ (for any non-zero number x)

Again, putting $m = 0$ in (ii) we get

$$\frac{x^0}{x^n} = x^{0-n} \quad \text{or} \quad \frac{1}{x^n} = x^{-n}$$

Thus, $\boxed{x^{-n} = \frac{1}{x^n}}$

and this relation gives the meaning of a power where the exponent is negative.

Let us now study the meaning of powers where the exponent is a rational number. Consider the power $4^{5/2}$ where the exponent $\frac{5}{2}$ is a rational number. We may simplify this power in two different ways as follows :

(i) $4^{5/2} = \left(4^{\frac{1}{2}}\right)^5$

$= \left(\sqrt{4}\right)^5$

$= 2^5 = 32$

(ii) $4^{5/2} = \left(4^5\right)^{1/2} = \sqrt{4^5}$

$= \sqrt{4 \times 4 \times 4 \times 4 \times 4}$

$= 4 \times 4\sqrt{4} = 32$

We may now introduce powers where the base is a positive real number and the exponent is a rational number.

If x is a positive real number and p, q are integers relatively prime to each other, then

$$x^{p/q} = \left(\sqrt[q]{x}\right)^p = \sqrt[q]{x^p}$$

Thus, $2^{4/3} = \sqrt[3]{2^4} = \sqrt[3]{2 \times 2 \times 2 \times 2} = 2\sqrt[3]{2}$

and $3^{-5/2} = \frac{1}{3^{5/2}} = \frac{1}{\sqrt{3^5}} = \frac{1}{\sqrt{3 \times 3 \times 3 \times 3 \times 3}} = \frac{1}{9\sqrt{3}}$

The laws of exponents as stated above, including the deductions hold good when bases are positive real numbers and exponents are rational numbers, positive or negative.

Example 11 : Simplify : $\sqrt{147} + \frac{\sqrt{3}}{3} - \frac{\sqrt{363}}{6}$

Solution : $\sqrt{147} + \frac{\sqrt{3}}{3} - \frac{\sqrt{363}}{6}$

$= \sqrt{3 \times 7 \times 7} + \frac{\sqrt{3}}{3} - \frac{1}{6}\sqrt{3 \times 11 \times 11}$

$= 7\sqrt{3} + \frac{\sqrt{3}}{3} - \frac{11}{6}\sqrt{3}$

$= \left(7 + \frac{1}{3} - \frac{11}{6}\right)\sqrt{3}$

$= \frac{42+2-11}{6}\sqrt{3}$

$= \frac{33\sqrt{3}}{6} = \frac{11\sqrt{3}}{2}$

Example 12 : Simplify : $3\sqrt[3]{2} + 7\sqrt[3]{16} - \sqrt[3]{250}$

Solution : We have, $\sqrt[3]{16} = \sqrt[3]{2 \times 2 \times 2 \times 2} = 2\sqrt[3]{2}$

and $\sqrt[3]{250} = \sqrt[3]{2 \times 5 \times 5 \times 5} = 5\sqrt[3]{2}$

$\therefore$ $3\sqrt[3]{2} + 7\sqrt[3]{16} - \sqrt[3]{250}$

$= 3\sqrt[3]{2} + 7 \times 2\sqrt[3]{2} - 5\sqrt[3]{2}$

$= (3 + 14 - 5)\sqrt[3]{2} = 12\sqrt[3]{2}$

Example 13: Find (i) $64^{1/3}$ (ii) $81^{1/4}$ (iii) $125^{2/3}$

Solution : **(i)** $64^{1/3} = \left(2^6\right)^{1/3} = 2^{6/3} = 2^2 = 4$

(ii) $81^{1/4} = \left(3^4\right)^{1/4} = 3$

(iii) $125^{2/3} = \left(5^3\right)^{2/3} = 5^{3\times 2/3} = 5^2 = 25$

Example 14: Simplify : (i) $4^{-\frac{3}{2}}$ (ii) $16^{\frac{5}{4}}-8^{\frac{2}{3}}$

(iii) $\left(\frac{1}{27}\right)^{-\frac{4}{3}}\times 81^{-\frac{3}{4}}$ (iv) $\frac{1}{6^{-2}} \div 216^{\frac{2}{3}}$

Solution: **(i)** $4^{-\frac{3}{2}} = \frac{1}{4^{3/2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{2^3} = \frac{1}{8}$

(ii) $16^{\frac{5}{4}} - 8^{\frac{2}{3}} = \left(2^4\right)^{\frac{5}{4}} - \left(2^3\right)^{\frac{2}{3}}$

$= 2^{4\times\frac{5}{4}} - 2^{3\times\frac{2}{3}}$

$= 2^5 - 2^2$

$= 32 - 4 = 28$

(iii) $\left(\frac{1}{27}\right)^{-\frac{4}{3}} \times 81^{-\frac{3}{4}} = 27^{\frac{4}{3}} \times \left(\frac{1}{81}\right)^{\frac{3}{4}} = \left(3^3\right)^{\frac{4}{3}} \times \frac{1}{\left(3^4\right)^{\frac{3}{4}}}$

$= 3^4 \times \frac{1}{3^3} = 3^{4-3} = 3.$

(iv) $\frac{1}{6^{-2}} \div 216^{\frac{2}{3}} = 6^2 \div 216^{\frac{2}{3}}$

$= 6^2 \div \left(6^3\right)^{\frac{2}{3}}$

$= 6^2 \div 6^2$

$= 6^{2-2}$

$= 6^0 = 1$

1.8 Rationalisation of Real Numbers

Consider the real number $\frac{1}{2-\sqrt{2}}$. Can you locate the point on the number line representing this number ? It is difficult to locate. However if we express the number with a rational denominator, then we may have an idea to locate the point. So, our aim is, first to express the number with a rational denominator. To do so we proceed as follows :

Multiplying both numerator and denominator by $2+\sqrt{2}$ (how would you guess this factor ?) we have

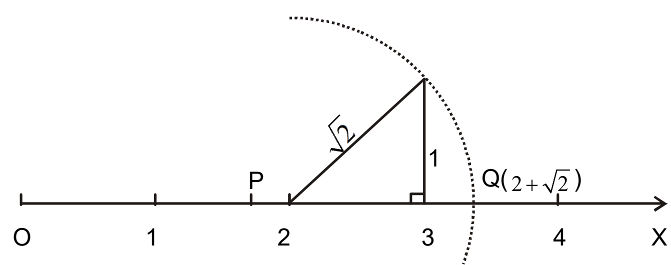
$$\frac{1}{2-\sqrt{2}} = \frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$= \frac{2+\sqrt{2}}{2^2-(\sqrt{2})^2}$$

$$= \frac{2+\sqrt{2}}{4-2}$$

$$= \frac{2+\sqrt{2}}{2}$$

As done earlier, the number may now, be represented on the number line. The representation is shown below :



The point P representing $\frac{2+\sqrt{2}}{2}$ lies between 1 and 2.

A point to be emphasized in the above process is the choice of the multiplying factor $2+\sqrt{2}$. Given the denominator $2-\sqrt{2}$ we choose this multiplying factor $2+\sqrt{2}$ by using the well known identity $(a+b)(a-b) = a^2 - b^2$ so that the product turns out to be a rational number.

If the product of two irrational numbers is a rational number then each is called a *rationalising factor* of the other. Thus $2+\sqrt{2}$ and $2-\sqrt{2}$ are rationalising factors of one another. Similarly $a+b\sqrt{x}$ and $a-b\sqrt{x}$ are rationalising factors of one another. Also $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ are rationalising factors of each other.

To rationalise the denominators of the numbers of the type $\frac{1}{a+b\sqrt{x}}$ and $\frac{1}{\sqrt{x}+\sqrt{y}}$ we choose $a-b\sqrt{x}$ and $\sqrt{x}-\sqrt{y}$ respectively as the rationalising factors and multiply both numerator and denominator of each fraction by the corresponding rationalising factor.

Example 15 : Express with rational denominator

(i) $\frac{3\sqrt{15}}{2\sqrt{5}}$ **(ii)** $\frac{1}{\sqrt{2}+\sqrt{3}}$ **(iii)** $\frac{1}{4-\sqrt{5}}$

(iv) $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$ **(v)** $\frac{5-\sqrt{7}}{3+2\sqrt{7}}$

Solution : **(i)** $\frac{3\sqrt{15}}{2\sqrt{5}} = \frac{3}{2}\sqrt{\frac{15}{5}} = \frac{3\sqrt{3}}{2}$

(ii) $\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{2}}$

$= \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$

$= \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2} = \sqrt{3}-\sqrt{2}$

(iii) $\frac{1}{4-\sqrt{5}} = \frac{4+\sqrt{5}}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{4+\sqrt{5}}{4^2-(\sqrt{5})^2}$

$= \frac{4+\sqrt{5}}{16-5} = \frac{4+\sqrt{5}}{11}$

(iv) $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} = \frac{(3\sqrt{2}-2\sqrt{3})^2}{(3\sqrt{2}+2\sqrt{3})(3\sqrt{2}-2\sqrt{3})}$

$= \frac{(3\sqrt{2})^2+(2\sqrt{3})^2-2\times 3\sqrt{2}\times 2\sqrt{3}}{(3\sqrt{2})^2-(2\sqrt{3})^2}$

$= \frac{18+12-12\sqrt{6}}{18-12}$

$= \frac{30-12\sqrt{6}}{6} = 5-2\sqrt{6}$

(v) $\dfrac{5-\sqrt{7}}{3+2\sqrt{7}} = \dfrac{(5-\sqrt{7})(3-2\sqrt{7})}{(3+2\sqrt{7})(3-2\sqrt{7})}$

$= \dfrac{5(3-2\sqrt{7})-\sqrt{7}(3-2\sqrt{7})}{3^2-(2\sqrt{7})^2}$

$= \dfrac{15-10\sqrt{7}-3\sqrt{7}+2\times 7}{9-28} = \dfrac{29-13\sqrt{7}}{-19}$

$= \dfrac{-29+13\sqrt{7}}{19}$

Example 16 : Simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

Solution : $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}+\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}} = \dfrac{\sqrt{2}+\sqrt{3}-\sqrt{5}+\sqrt{2}+\sqrt{3}+\sqrt{5}}{(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})}$

$= \dfrac{2\sqrt{2}+2\sqrt{3}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2}$

$= \dfrac{2(\sqrt{2}+\sqrt{3})}{2+3+2\sqrt{6}-5}$

$= \dfrac{2(\sqrt{2}+\sqrt{3})}{2\sqrt{6}}$

$= \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{6}}$

$= \dfrac{\sqrt{6}(\sqrt{2}+\sqrt{3})}{\sqrt{6}\times\sqrt{6}} = \dfrac{\sqrt{12}+\sqrt{18}}{6}$

$= \dfrac{2\sqrt{3}+3\sqrt{2}}{6}$

EXERCISE 1.3

1. Construct a line segment of length $\sqrt{5.6}$.

2. Represent $\sqrt{6.5}$ on the number line.

3. **Classify the following numbers as rational or irrational :**

(i) $3+\sqrt{2}$ **(ii)** $3-\sqrt{2}$

(iii) $2\sqrt{5}$ **(iv)** $\frac{1}{\sqrt{7}}$

(v) $\left(3+\sqrt{3}\right)-\sqrt{3}$ **(vi)** $\left(13+\sqrt{7}\right)+\left(4-\sqrt{7}\right)$ **(vii)** $\frac{4\sqrt{21}}{5\sqrt{21}}$

(viii) $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{3}-\sqrt{2}\right)$

4. **Simplify :**

(i) $\sqrt{18}+\sqrt{64}$ (ii) $\sqrt{43}+\sqrt{12}$ (iii) $\sqrt{5}+\sqrt{80}$

(iv) $\sqrt{50}-\sqrt{32}$ (v) $\sqrt{363}-\sqrt{147}$ (vi) $2\sqrt{45}-5\sqrt{20}+\sqrt{80}$

(vii) $2\sqrt{12}+\sqrt{75}-7\sqrt{3}$ (viii) $\sqrt{7}\times\sqrt{14}$ (ix) $2\sqrt{3}\times 3\sqrt{2}$

(x) $8\sqrt{12}\times 3\sqrt{24}$ (xi) $3\sqrt{10}\div 4\sqrt{15}$ (xii) $6\sqrt{12}\div 3\sqrt{27}$

5. **Multiply :**

(i) $3\sqrt{5}+2\sqrt{3}$ by $4\sqrt{5}-5\sqrt{3}$

(ii) $\sqrt{2}+\sqrt{5}+\sqrt{7}$ by $\sqrt{2}+\sqrt{5}-\sqrt{7}$

(iii) $\sqrt{3}-\sqrt{7}+2\sqrt{5}$ by $\sqrt{3}+\sqrt{7}-2\sqrt{5}$

6. **Express the following avoiding fractional or negative exponents :**

(i) $3^{\frac{2}{5}}$ (ii) $5^{-\frac{3}{4}}$ (iii) $\frac{5}{2^{-\frac{4}{3}}}$

(iv) $3^{-\frac{2}{3}}\times 3^{-\frac{1}{2}}$ (v) $5^{-\frac{2}{3}}\div 2\sqrt{5^{-3}}$

7. **Express the following avoiding radical signs and negative exponents :**

(i) $\left(\sqrt{3}\right)^{5}$ (ii) $\left(\sqrt[3]{5}\right)^{-4}$

(iii) $\frac{1}{\sqrt[5]{7^{-3}}}$ (iv) $\sqrt[3]{2^{4}}\times\left(\sqrt[6]{2}\right)^{-2}$

(v) $\sqrt[4]{3^{-5}}\div\left(\sqrt[6]{3}\right)^{-9}$

8. **Find the value of**

(i) $16^{-\frac{3}{4}}$ (ii) $\sqrt[5]{32^{3}}$ (iii) $\sqrt{9^{5}}$

(iv) $125^{-\frac{2}{3}}$ (v) $\left(\frac{1}{343}\right)^{-\frac{2}{3}}$

9. Express with rational denominator :

(i) $\frac{\sqrt{3}+1}{4+3\sqrt{3}}$ **(ii)** $\frac{3+4\sqrt{2}}{5-3\sqrt{2}}$ **(iii)** $\frac{\sqrt{5}+2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ **(iv)** $\frac{\sqrt{5}+\sqrt{3}}{4-\sqrt{15}}$

(v) $\frac{1}{1+\sqrt{6}-\sqrt{7}}$ **(vi)** $\frac{4}{1+\sqrt{2}+\sqrt{3}}$ **(vii)** $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

10. Simplify :

(i) $\frac{\sqrt{18}+\sqrt{27}}{\sqrt{75}-\sqrt{48}-\sqrt{32}+\sqrt{50}}$ **(ii)** $\frac{\sqrt{18}}{\sqrt{3}+\sqrt{6}}-\frac{\sqrt{48}}{\sqrt{2}+\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

(iii) $\frac{1}{\sqrt{5}-\sqrt{3}}-\frac{1}{\sqrt{5}+\sqrt{3}}$ **(iv)** $\frac{2}{4+3\sqrt{2}}+\frac{7}{3-\sqrt{2}}-\frac{31}{1+4\sqrt{2}}$

(v) $\frac{1}{1+\sqrt{2}-\sqrt{3}}-\frac{1}{1+\sqrt{2}+\sqrt{3}}$ **(vi)** $\frac{11}{1-\sqrt{3}+\sqrt{5}}-\frac{11}{1+\sqrt{3}+\sqrt{5}}$

(vii) $3\sqrt{35}+\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}-\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

11. Show that the following are rational

(i) $\frac{\left(\sqrt{3}-\sqrt{2}\right)\left(5+2\sqrt{6}\right)}{\sqrt{3}+\sqrt{2}}$ **(ii)** $\frac{5-\sqrt{5}}{3+\sqrt{5}}+2\sqrt{5}$

(iii) $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}$

(iv) $x^2 - xy + y^2$ where $x = \frac{\sqrt{2}-1}{\sqrt{2}+1}$ and $y = \frac{\sqrt{2}+1}{\sqrt{2}-1}$

ANSWER

Exercise 1.1

1. A number of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$. Yes, it is true.

3. Four rational numbers between 2 and 3 may be taken in many ways. One way of to write $2 = \frac{10}{4+1} = \frac{10}{5}$; $3 = \frac{15}{4+1} = \frac{15}{5}$ and take $\frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}$.

4. Taking $\frac{1}{4} = \frac{3}{12} = \frac{18}{72}$, $\frac{1}{3} = \frac{4}{12} = \frac{24}{72}$
we get the five rationals $\frac{19}{72}, \frac{20}{72}, \frac{21}{72}, \frac{22}{72}, \frac{23}{72}$.

5. Use $\frac{p}{q} < \frac{r}{s} \Rightarrow \frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$ and get the rationals

$\frac{2}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}$

Exercise 1.2

1. Take the point P on the number line such that OP is equal to the length of a diagonal of a unit square.

2. Proceed as in case of $\sqrt{2}$.

5. Proceed as in Example 2 and 3.

6. Proceed as in Example 4 and 5.

7. Rationals : 2.11 , 2.12 , 2.13

Irrationals : 2.11010010001 ...; 2.12020020002 ... ; 2.13030030003....

8. 0.34040040004 ... ; 0.353353335 ... ; 0.4040040004 ... ;

0.4141141114 ... .

Exercise 1.3

1. Refer Example 8.

2. Proceed as in Example 8.

3. **(i)** Irrational **(ii)** Irrational **(iii)** Irrational **(iv)** Irrational

(v) Rational (vi) Rational (vii) Rational (viii) Rational

4. (i) $8+3\sqrt{2}$ (ii) $6\sqrt{3}$ (iii) $5\sqrt{5}$ (iv) $\sqrt{2}$

(v) $4\sqrt{3}$ (vi) 0 (vii) $2\sqrt{3}$ (viii) $7\sqrt{2}$

(ix) $6\sqrt{6}$ (x) $288\sqrt{2}$ (xi) $\frac{\sqrt{6}}{4}$ (xii) $\frac{4}{3}$

5. (i) $30-7\sqrt{15}$ (ii) $2\sqrt{10}$ (iii) $-24+4\sqrt{35}$

6. (i) $\sqrt[5]{3^2}$ (ii) $\frac{1}{\sqrt[4]{5^3}}$ (iii) $5\sqrt[3]{2^4}$

(iv) $\frac{1}{\sqrt[6]{2^7}}$ (v) $\frac{1}{2}\sqrt[6]{5^5}$

7. (i) $3^{\frac{5}{2}}$ (ii) $\frac{1}{5^{\frac{4}{3}}}$ (iii) $7^{\frac{3}{5}}$

(iv) 2 (v) $3^{\frac{1}{4}}$

8. (i) $\frac{1}{8}$ (ii) 8 (iii) 243

(iv) $\frac{1}{25}$ (v) 49

9. (i) $\frac{5-\sqrt{3}}{11}$ (ii) $\frac{29+29\sqrt{2}}{7}$ (iii) $\frac{-1+\sqrt{15}}{2}$ (iv) $7\sqrt{5}+9\sqrt{3}$

(v) $\frac{6+\sqrt{6}+\sqrt{42}}{12}$ (vi) $2+\sqrt{2}-\sqrt{6}$ (vii) $\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

10. (i) 3 (ii) 0 (iii) $\sqrt{3}$ (iv) 0

(v) $\frac{\sqrt{6}}{2}$ (vi) $4\sqrt{15}-6\sqrt{3}$ (vii) $\sqrt{35}$

SUMMARY

Main points studied in this chapter may be summarised as follows :

1. A number which can be put in the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$, is called a rational number.

2. The number indicating the length of a diagonal of a unit square is not rational.

3. There are points on the number line which do not represent rational numbers and such points represent irrational numbers.

4. A real number which cannot be expressed as a ratio of two integers, is an irrational number.

5. The decimal expansion of an irrational number is non-terminating and non-recurring. Conversely, a non-terminating and non-recurring decimal represents an irrational number.

6. The rational numbers and irrational numbers taken together, form the system of real numbers.

7. To each point on the number line there corresponds a unique real number and to each real number there corresponds a unique point on the number line.

8. The sum, difference, product and quotient of two irrational numbers need not be irrational.

9. If m, n are integers, $n > 0$ and x is a positive real number,

then $x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$.

10. If x, y are positive real numbers and m, n are rational numbers, then

(i) $x^m.x^n = x^{m+n}$ **(ii)** $\frac{x^m}{x^n} = x^{m-n}$

(iii) $(xy)^m = x^m.y^m$ **(iv)** $(x^m)^n = x^{mn}$

11. To rationalise the denominator of $\frac{1}{a + b\sqrt{x}}$ we multiply both numerator and denominator by $a - b\sqrt{x}$.

12. To rationalise the denominator of $\frac{1}{\sqrt{x} + \sqrt{y}}$ we multiply both numerator and denominator by $\sqrt{x} - \sqrt{y}$.

****

It may also be noted that when $n = 0$ and $a_0 \neq 0$, the polynomial becomes a_0 (a non-zero constant) and its degree is 0. In other words, a non-zero constant is a polynomial of degree zero. Whereas when $n = 0$ and $a_0 = 0$ we get a zero polynomial of which the degree is not defined as the condition $a_n \neq 0$ is violated.

A polynomial is said to be in the **standard form** when its terms are arranged in ascending or descending powers of the variable.

Definition : A polynomial in which the coefficient of the highest degree term is 1 is called a **Monic Polynomial**.

For example, $x^3 + x^2 + 1$, $x^4 - 3x^3 + x - 2$ are monic polynomials. Monic polynomial of degree zero is 1.

2.4 Some Special Names of Polynomials

(i) ***Linear Polynomial :*** A polynomial of degree one is called a linear polynomial. For example, $4x$, $x + 3$, $\sqrt{5}x - 2$, $4 - 3x$ etc. are linear polynomials.

The general form of a linear polynomial is $ax + b$, where a and b are constants and $a \neq 0$.

(ii) ***Quadratic Polynomial :*** A polynomial of degree two is called a quadratic polynomial.
For example, $2x^2 + 5$, $x^2 + \frac{2}{5}x$, $6 - y - y^2$ etc. are quadratic polynomials.

In general, a quadratic polynomial in x is of the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$.

(iii) ***Cubic Polynomial :*** A polynomial of degree three is called a cubic polynomial. For example, $5x^3 - 1$, $y^3 - 5y + 2$ etc. are cubic polynomials.

The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$.

(iv) ***Biquadratic Polynomial :*** A polynomial of degree four is called a biquadratic (or quartic) polynomial and its general form is $ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, e are constants and $a \neq 0$.

For example, $5x^4 + 4x^3 - 2x^2 + x - 1$ is a biquadratic polynomial.

So far we have dealt with polynomials in one varaible only. We can also have polynomials in more than one variable. For example, $x^3 + y^3 + z^3 - 3xyz$ is a polynomial in three variables x, y and z. Similarly, $s^2 + st + t^2$ and $p^2 + q^2 + pqr$ are polynomials in two and three variables respectively. We shall study such polynomials in detail later.

EXERCISES 3.2

1. Find the distance of the following points from the origin.

(a) (2, 5) **(b)** (6, –10) **(c)** (–6, 12)

2. Find the distance between the pair of points

(a) (1, 3), (7, 2) **(b)** (7, –2), (3, –1)

(c) (10, 4), (–1, –2) **(d)** (–1, 3), (4, –2)

3. The coordinates of A are (–4, 8) and those of B are (x, 3). Find x if AB = 13.

4. Show that the points A(2, 7), B(3, 0), C(–4, –1) are vertices of an isosceles triangle and find the length of the base.

5. Show that the points A(4, 3), B(1, 2), C(1, 0), D(4, 1) are vertices of a parallelogram and find the lengths of its diagonals.

6. Show that (2, 1) is the centre of the circumcircle of the triangle whose vertices are (–3, –9), (13, –1) and (–9, 3).

7. Show that the points (4, 4), (5, -1), (-6, 2) are the vertices of a right angled triangle.

8. Show that the points (p, p), $(-p, -p)$, $(p\sqrt{3}, -p\sqrt{3})$ are the vertices of an equilateral triangle.

ANSWER

1. (a) $\sqrt{29}$ (b) $2\sqrt{34}$

   (c) $3\sqrt{20}$

2. (a) $\sqrt{37}$ (b) $\sqrt{17}$ (c) $\sqrt{157}$ (d) $5\sqrt{2}$

3. –8, –16 4. 10 5. $\sqrt{10},\ 3\sqrt{2}$

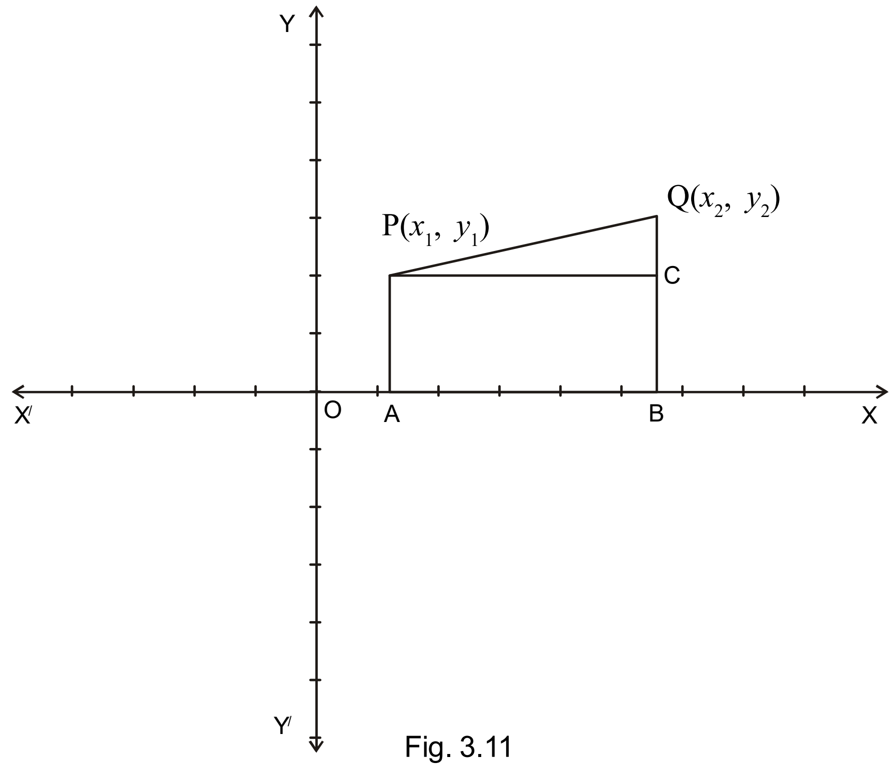


Fig. 3.11

Draw $\overline{PA}$ and $\overline{QB}$ perpendicular to the X-axis. Then OA = x_1, AP = y_1, OB = x_2, BQ = y_2. Draw $\overline{PC}$ perpendicular to $\overline{QB}$ meeting $\overline{QB}$ at C. Then ABCP is a rectangle and so, BC = AP = y_1.

$\therefore$ QC = BQ − BC

= $y_2 - y_1$

Again, PC = AB = OB − OA = $x_2 - x_1$.

In ΔPCQ, $\angle PCQ = 90^0$.

$\therefore$ by Pythagoras theorem

$$PQ^2 = PC^2 + QC^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Step I : Take the co-ordinate axes on the plane so that the origin is at a suitable position, preferably at the middle of the plane.

Step II : Choose the scale on the axes so that the point corresponding to the given co-ordinates may be shown in the plane.

Step III : Check the sign of the abscissa. If it is positive, take the required units starting from O along the positive direction of the X-axis. If it is negative take the required units starting from O along the negative direction of X-axis. If it is zero, it remains at O.

Step IV : Name the point obtained in step III, A, say.

Step V : Check the sign of the ordinate. If it is positive, take the required units starting from A along the positive direction of the Y-axis. If it is negative, take the required units starting from A along the negative direction of the Y-axis. If it is zero, it remain at A.

Step VI : Name the point obtained in step V, P, say.

Then, P is the required point on the plane with the given co-ordinates. Such a process of locating a point with given co-ordinates is called **plotting a point**.

EXERCISE 3.1

1. In which quadrants, do the following points lie :

(a) (–2, 7) **(b)** (–2, –3) **(c)** (1, 6) **(d)** (5, –3)

2. Locate the points given below on the Cartesian plane and also state the quadrants in which they lie :

(a) (2, 5) **(b)** (–4, –1) **(c)** (–1, 5) **(d)** (2, –6)

3. On which axis do the following points lie :

(a) (1, 0) **(b)** (0, 5) **(c)** (–3, 0)

(d) (0, –2) **(e)** (0, 0)

The above observations can also be put in a table as below :

QUADRANT / COORDINATES	1st	2nd	3rd	4rd
Abscissa	+	–	–	+
Ordinate	+	+	–	–

Let us again find the coordinates of the points A, B, C, D marked on the axes as in Fig. 3.8.

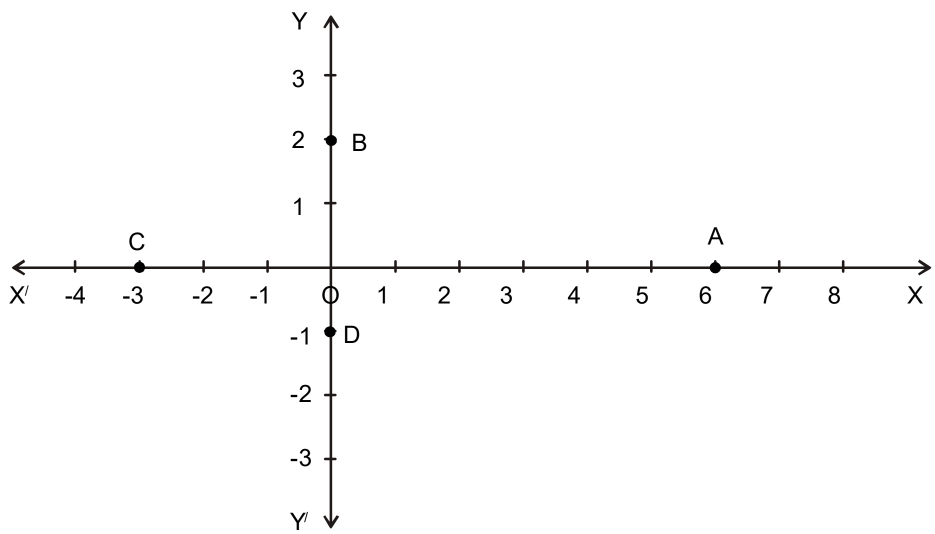


Fig. 3.8

The distance of the point A from the Y-axis is 6-units and the distance from the X-axis is 0 unit. So, the co-ordinates of the point A are (6, 0). Similarly, the co-ordinates of B, C, D are (0, 2), (–3, 0), (0, –1) respectively. Observe that A and C lie on the X-axis and their ordinates are zeroes. Again, B and D lie on the Y-axis and their abscissas are zeroes. Thus,

if a point lies on the X-axis, its ordinate is zero and if a point lies on the Y-axis, its abscissa is zero.

Note that the origin lies on both the axes. So, both its ordinate and abscissa are zeros. Hence, the co-ordinates of the origin are (0, 0).

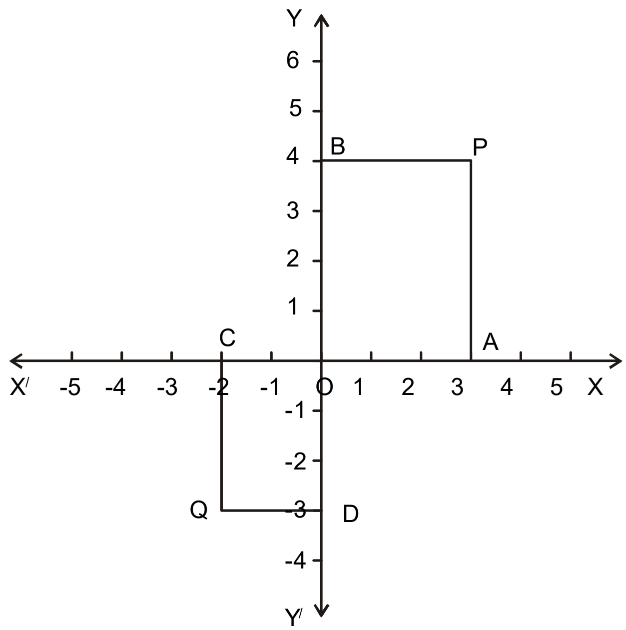


Fig. 3.5

Draw perpendiculars PA, QC on the X-axis and PB, QD on the Y-axis. Measure PA, PB, QC, QD. Observe that PB lies along the positive direction of the X-axis and PB = 3 units. So, the abscissa of P is +3 or simply 3. Whereas, QD lies along the negative direction of the X-axis and QD = 2 units. So, the abscissa of Q is –2. Similarly, PA lies along the positive direction of Y-axis with PA = 4 units whereas QC lies along the negative direction of Y-axis with QC = 3 units. So, the ordinate of P is 4 whereas that of Q is – 3. Hence, the co-ordinates of P are (3, 4) and the co-ordinates of Q are (–2, –3).

3.5 Quadrants

The co-ordinate axes $\overleftrightarrow{X'OX}$ and $\overleftrightarrow{Y'OY}$ divide the plane into four regions. Each region is called a quadrant. Thus, the co-ordinate axes divide the plane into four quadrants.

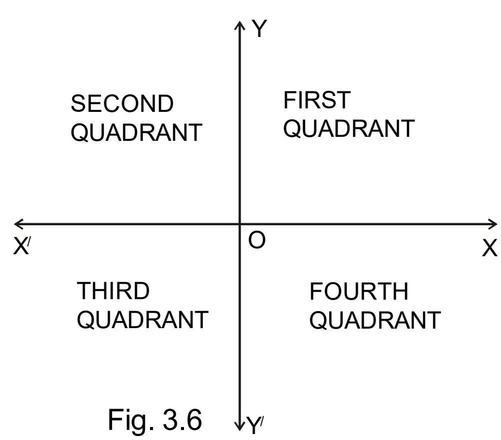


Fig. 3.6

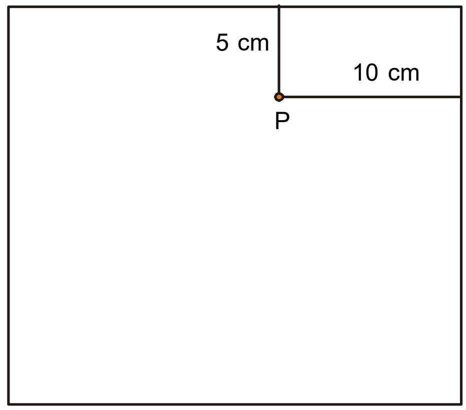


Fig. 3.3

You may say that it is near the top edge of the paper or you may further say that it is near both the top edge and right edge of the paper. This also does not specify the exact location of the dot.

3.3 Cartesian Co-ordinates

*Rene Descartes, the great French mathematician and philosopher propounded a system of describing the position of a point in a plane. In honour of Descartes, this system used for describing the position of a point in a plane is known as the Cartesian System of Co-ordinates.

3.4 Rectangular Cartesian Coordinate System

To fix the position of a point P in a plane, take two fixed perpendicular lines, conventionally one horizontal and other vertical, on the plane intersecting at a point. Let $\overleftrightarrow{X'X}$ be the horizontal line, $\overleftrightarrow{Y'Y}$ be the vertical line and O be the point of their intersection. Take the line $\overleftrightarrow{X'OX}$ as a number line in which O is the origin, positive numbers are represented along $\overrightarrow{OX}$ and negative numbers along $\overrightarrow{OX'}$ in a certain scale. Similarly, take the line $\overleftrightarrow{Y'OY}$ as a number line in which O is the origin, positive numbers are represented along $\overrightarrow{OY}$ and negative numbers along $\overrightarrow{OY'}$ in the same scale. The horizontal line $\overleftrightarrow{X'OX}$ and vertical line $\overleftrightarrow{Y'OY}$ are together called the co-ordinate axes,

* Pronounce as Rene′ \ dā′-kärt

ANSWERS

3. **(i)** (5, 0) and (0, 3) **(ii)** 6 **4.** (3, 4)

5. 5 **6.** $6x - y + 4 = 0$ **7.** $y = kx$

8. $x + y = 125$

9. (i) 104^0F **(ii)** 45^0C **(iii)** 32^0F and -17.8^0C (approx.) **(iv)** -40^0

10. (i) 40^g **(ii)** 63^0

SUMMARY

In this chapter, you have studied the following points :

1. An equation of the form $ax + by + c = 0$, where a, b, c are constants (real numbers) such that both a and b are not zeroes, is called a linear equation in two variables.

2. A linear equation in two variables has infinitely many solutions.

3. The graph of every linear equation in two variables is a straight line.

4. Every point on the graph of a linear equation in two variables is a solution of the linear equation, and every solution of a linear equation is a point on the graph of the linear equation.

5. The graph of $x = c$ is a straight line parallel to the Y-axis and passing through $(c, 0)$.

6. The graph of $y = c$ is a straight line parallel to the X-axis and passing through $(0, c)$.

7. The graph of $x = 0$ is the Y-axis.

8. The graph of $y = 0$ is the X-axis.

9. The graph of an equation of the type $y = kx$, where k is a constant, always passes through the origin.

******

(0, 0), (2, 4), (4, 8), (–1, –2)

Plotting the points represented by the above ordered pairs and joining these points we get a line $\overleftrightarrow{AB}$ as shown in Fig. 4.5. This straight line is the graph of the above equation.

We see that the point (5, 10) lies on the line $\overleftrightarrow{AB}$.

Hence, when the distance travelled by the body is 5 units, the work done is 10 units.

Remarks : The graph of the equation of the form $y = kx$ is a line which always passes through the origin.

EXERCISE 4.2

1. Draw the graph of the following equation :

(i) $x = 4$ **(ii)** $x = -5$ **(iii)** $y = 3$ **(iv)** $y = -4$

(v) $x + y = 0$ **(vi)** $x - y = 0$ **(vii)** $x + 3y = 0$

2. Draw the graph of the following equation :

(i) $3x + 4y = 12$ **(ii)** $x + 2y = 6$ **(iii)** $2x - 5y = 10$

(iv) $3x = 4y - 12$ **(v)** $\frac{x}{5} + \frac{y}{6} = 1$ **(vi)** $\frac{x}{5} + \frac{y}{6} = 1$

(vii) $4x + 3y = 13$ **(viii)** $2x + 3y = 11$

(ix) $x - 3y = 8$ **(x)** $3x - y = 8$

3. Draw the graph of $\frac{x}{5} + \frac{y}{3} = 1$.

(i) Find the co-ordinates of the points where the graph intersects the two co-ordinate axes.

(ii) From the graph find the value of y when $x = -5$.

4. Using the same unit and the same axes, draw the graphs of the following equations:

$2x - y = 2$ and $3x + 2y = 17$

Find the co-ordinates of the point where the graphs intersect.

Remarks : **(i)** The graph of the equation $y = c$, where c is a constant, is a straight line parallel to the X-axis. It cuts the Y-axis at the point $(0,c)$. The ordinate of any point on this line is c.

(ii) The graph of $y = 0$ is the X-axis.

Example 3 : Draw the graph of $3x + 5y = 15$.

Solution : $3x + 5y = 15$ is a linear equation in two variables x and y. Some of the ordered pairs satisfying this equation are (0,3), (5,0), (10, –3), (–5, 6) etc. Plotting the points represented by these ordered pairs and joining them we get a straight line $\overleftrightarrow{AB}$ as shown in Fig. 4.4. The straight line $\overleftrightarrow{AB}$ is the graph of the equation $3x + 5y = 15$.

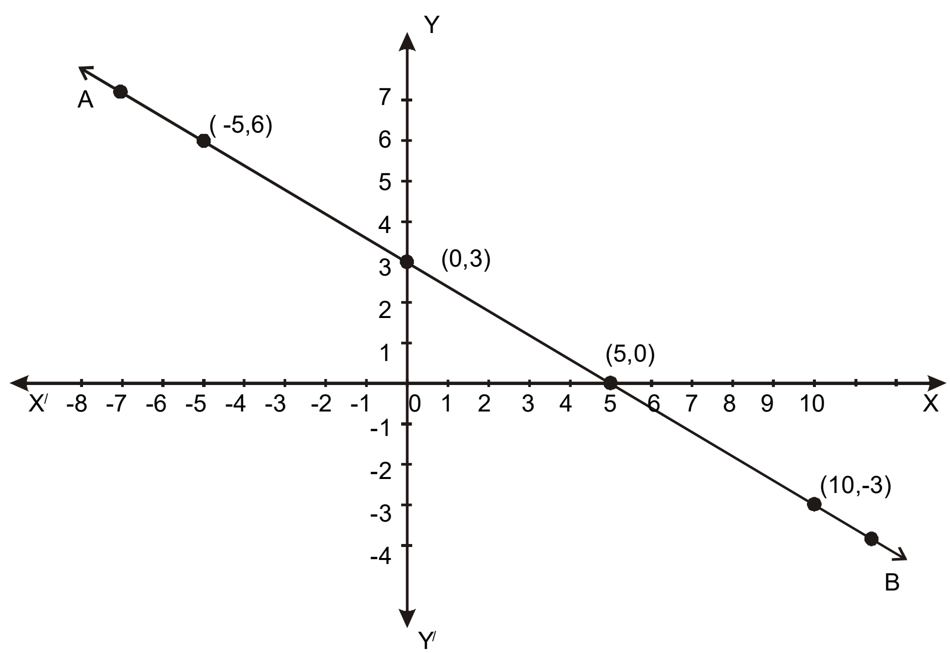


Fig. 4.4

Example 4 : If the work done by a constant force acting on a body is directly proportional to the distance travelled by the body, the constant of proportionality being the constant force, express this in the form of an equation in two variables and draw the graph of the same by taking the constant force as 2 units. Also read from the graph the work done when the distance travelled by the body is 5 units.

So, the linear equation (i) is geometrically represented by the line $\overleftrightarrow{AB}$. This line is called the graph of the linear equation. Thus, we conclude that the graph of a linear equation is a straight line whose points make up the aggregate of solutions, of the equation. To obtain the graph of a linear equation in two variables, it is enough to plot two points corresponding to two solutions of the equation and draw a line through them. However, it is better to plot at least three such points so that you can immediately check the correctness of the graph.

Example 1 : Draw the graph of $x = 5$.

Solution : $x = 5$ is a linear equation in one variable. However, it can be written as

$x + 0.y = 5$ ... (i)

which is a linear equation in two variables. Some of the ordered pairs which satisfy equation (i) are given below :

(5, 0), (5, 2), (5, 5), (5, –3) etc.

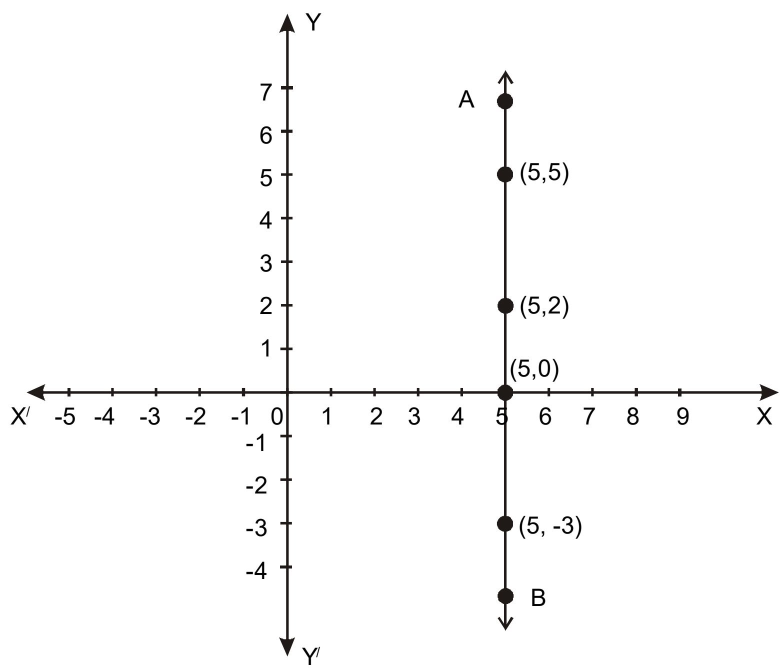


Fig. 4.2

The points represented by the above ordered pairs are plotted in the cartesian plane. Joining these points we get a straight line $\overleftrightarrow{AB}$ (Fig. 4.2). The line $\overleftrightarrow{AB}$ thus obtained is the graph of the equation $x = 5$.

(iii) Two tables and three chairs cost ₹ 1200.

(iv) Two numbers are in the ratio 2 : 3.

3. **Find four different solutions for each of the following equations in two variables:**

(i) $x = 3y$ **(ii)** $4x + 3y = 12$ **(iii)** $x + 2y = 6$

(iv) $5x + 2y = 0$ **(v)** $3x + 4 = 0$ **(vi)** $2y - 5 = 0$

4. **Check which of the following are solutions of the equation $3x - 2y = 6$ and which are not :**

(i) (2, 0) **(ii)** (0, 3) **(iii)** (3, 0)

(iv) (0, –3) **(v)** (–2, –6) **(vi)** (4, 3)

5. **Find the value of *k* if**

(i) (1, 2) is a solution of $3x + 2y = k$

(ii) (2, –3) is a solution of $kx - 3y + 5 = 0$.

ANSWERS

1. **(i)** –6 **(ii)** 3 **(iii)** 1 **(iv)** 4 **(v)** 13

(vi) 4 **(vii)** $\frac{3}{2}$ **(viii)** 12 **(ix)** 1 **(x)** 7

2. **(i)** $x - 8y = 0$ **(ii)** $x - y = 5$

(iii) $2x + 3y = 12000$ **(iv)** $3x - 2y = 0$

3. **(i)** (0, 0), (3, 1), (–6, –2), $(1, \frac{1}{3})$ **(ii)** (0, 4), (3, 0), (–3, 8), $(1, \frac{8}{3})$

(iii) (0, 3), (6, 0), (2, 2), (–2, 4) **(iv)** (0, 0), (2, –5), (–2, 5), $(1, -\frac{5}{2})$

(v) $(-\frac{4}{3}, 0), (-\frac{4}{3}, 1), (-\frac{4}{3}, 2), (-\frac{4}{3}, 3)$

(vi) $(0, \frac{5}{2}), (1, \frac{5}{2}), (2, \frac{5}{2}), (3, \frac{5}{2})$

4. **(i)** Yes **(ii)** No **(iii)** No **(iv)** Yes **(v)** Yes **(vi)** Yes

5. **(i)** 7 **(ii)** –7

in two variables. In other words, a linear equation in two variables has infinitely many solutions. However, an arbitrary ordered pair need not be a solution of the equation.

Example 1 : The cost of a notebook is twice that of a pen. Write a linear equation to represent this statement.

Solution : Let the cost of a notebook be ₹ x and that of a pen be ₹ y.

Then we have,

$$x = 2y$$

$$\therefore \quad x - 2y = 0.$$

This is the linear equation which represents the given statement.

Example 2 : Write each of the following equations in the form $ax + by + c = 0$ ($a > 0$) and indicate the values of a, b and c in each case : (i) $3x - 4y = 5$, (ii) $x + 5 = 2y$ and (iii) $3x = y$.

Solution : **(i)** The equation $3x - 4y = 5$ can be written as $3x - 4y - 5 = 0$, which is of the form $ax + by + c = 0$. Here $a = 3$, $b = -4$ and $c = -5$.

(ii) The equation $x + 5 = 2y$ can be written as $x - 2y + 5 = 0$, which is of the form $ax + by + c = 0$. Here $a = 1$, $b = -2$ and $c = 5$.

(iii) The equation $3x = y$ can be written as $3x - y + 0 = 0$, which is of the form $ax + by + c = 0$, where $a = 3$, $b = -1$ and $c = 0$.

Example 3 : Write each of the following as an equation in two variables :

(i) $4x = 3$ (ii) $x = -2$ (iii) $2y = 5$

Solution : **(i)** $4x = 3$ can be written as $4x + 0.y = 3$ or $4x + 0.y - 3 = 0$.

(ii) $x = 2$ can be written as $x + 0.y - 2 = 0$.

(iii) $2y = 5$ can be written as $0.x + 2y - 5 = 0$.

Example 4 : Find any three different solutions for each of the following equations :

(i) $2x + 3y = 6$ (ii) $3x + 4y = 0$ (iii) $2x + 5 = 0$.

Solution : **(i)** Taking $x = 0$, the equation $2x + 3y = 6$ reduces to $3y = 6$, which gives $y = 2$. Therefore, (0, 2) is a solution of the equation $2x + 3y = 6$.

CHAPTER

4

LINEAR EQUATIONS IN TWO VARIABLES

4.1 Introduction

You have studied linear equations in one variable in the previous classes. Recall that a linear equation in one variable is an equation involving only linear polynomials in one variable. Thus, equations of the form $ax = b$, $ax + b = c$, $ax + b = cx + d$ (where $a \neq 0$, b, c, d are constants) are linear equations in one variable. For example, $x + 2 = 0$, $2x - \sqrt{3} = 0$, $5x + 3 = 3x - 5$, $\frac{y}{2} - 3 = \frac{y}{3} + 3$ etc. are linear equations in one variable. You also know that such equations have a unique (i.e. one and only one) solution. Now, let us see some examples of solving linear equations in one variable (i.e. finding the value of the variable satisfying the equation).

Example 1 : Solve $4x + 3 = 15$

Solution :

$4x + 3 = 15$

or $4x = 15 - 3$ (by transposition)

or $4x = 12$

$\therefore$ $x = \frac{12}{4} = 3.$

Example 2 : Solve $4x + 3 = 2x + 5$

Solution : $4x + 3 = 2x + 5$

or $4x - 2x = 5 - 3$ (by transposition)

or $2x = 2$

$\therefore$ $x = 1.$

Example 3 : Solve $\frac{y}{3} + 1 = \frac{y}{4} - 2$

Solution : $\frac{y}{3} + 1 = \frac{y}{4} - 2$

or $\frac{y}{3} - \frac{y}{4} = -2 - 1$ (by transposition)

or $\frac{y}{12} = -3$

$\therefore$ $y = -3 \times 12 = -36.$

CHAPTER

5 INTRODUCTION TO EUCLID'S GEOMETRY

5.1 Introduction

Geometry is one of the three major components of Mathematics, the other two being Algebra and Analysis. Geometry deals with space and related concepts. Euclidean Geometry is one of the oldest treatises that dominated this branch of Mathematics for a long period of more than 2000 years. But by the end of the 19^{th} century mathematicians found some flaws in the form of logical inconsistencies among some of Euclid's axioms and postulates. All these led to the development of other forms of Geometry called non-Euclidean Geometries which do not depend on axioms and postulates. Hyperbolic Geometry and Elliptic Geometry are two examples.

However, the importance of Euclidean Geometry is still undisputed and it forms the basis of study of this important branch of Mathematics.

5.2 Origin of Geometry

It is commonly believed that the beginnings of Geometry are the figures made on sand by the farmers of the by-gone eras who cultivated the Nile Valley. The figures were made to preserve the records of the demarcations of the boundaries of the fields of the farmers which were obliterated by the frequent floods caused by the water of the mighty Nile. This perhaps justifies the etymological meaning of the word Geometry which is the combination of two Greek words, Geo – meaning the Earth and metrien – meaning to measure.

Records of the knowledge of Geometry by many ancient civilizations are now available. Ancient civilizations of the Babyloneans, Egyptians, Greeks, Chinese etc. show ample evidence that the peoples of those times knew application of Geometry and its principles.

In the Indian subcontinent also excavations in Harappa and Mohenjo-Daro indicate that the Indus Valley civilization about (3000 B.C.) knew the applications of Geometry. Drawings and constructions for the performance of Vedic rites like construction of altars and fireplaces in specific patterns show that people of that time knew some properties of figures like triangles, circles, trapeziums etc.

But all these records do not show any evidence of a systematic study of the subject.

It was only the Greek Mathematicians Thales (640 B.C. – 540 B.C.) and Pythagoras (about 572 B.C.) who gave some ideas of deductive reasoning behind the construction works of important monuments of their time.

A systematic study based on a logically consistent foundation was possible only after Euclid (325 B.C. – 265 B.C.) developed his treatise on the subject called Elements in thirteen chapters each forming a book.

As stated earlier these books form the basis of the study of the subject Geometry.

5.3 Concept of Dimension

It is a matter of common knowledge that right from the early days till now natural objects and phenomena give inspirations to man particularly to scientists and philosophers. From these inspirations they could formulate and establish various principles and laws of nature.

From the objects available around them the ancient thinkers of Greece who were contemporaries of Euclid, developed the idea of solid objects and their abstract geometrical models. They also conceived Geometry as the abstract model of the things around them.

A solid object has shape, size and it occupies a definite space. It has boundaries called surfaces and surfaces are bounded by lines which may be straight or curved and the lines end in points. One particular attribute of a solid body is that it can be moved from place to place without any deformation of its shape and size.

In order to give generalization of these, the concept of dimension was brought in.

A solid has three dimensions viz. length, breadth and thickness; a surface has two viz. length and breadth and a line has only one viz. length. A point has no dimension.

Though Euclid in his elements did not mention it categorically, the notion of dimensions is implicit in his definitions of points, lines, surfaces and solids.

It was only in 1912 that the French mathematician Jules Henry Poincare (1854-1912) made a definition of the concept of dimensionality.

Poincare observed that a line is one (1) dimensional because a pair of points on it can be separated by a point which is of (1–1) i.e. 0 dimension.

A plane is two (2) dimensional because a pair of points in it can be separated by a line which is of (2–1) i.e. 1 dimension and not by a point which is of lesser dimension.

Similarly, a solid is three (3) dimensional because only a plane which is of (3–1) i.e. 2 dimensions can separate two points in it and not by a line or a point which are of lesser dimensions.

5.4 Study of Geometry

To study Geometry in a systematic way Euclid made a scheme. This consists of

(i) giving some entities in the form of definitions

(ii) enunciating nine general axioms

(iii) stating five postulates which are axioms related to Geometry.

On the backdrop of one or more of the above three parts of the scheme he proved a number of propositions some of which are so important that they are taken as standard theorems of Geometry.

However, mathematicians of later age found some inconsistencies in the proof of some of his theorems even though better proofs were not available.

Thus, forgetting the proof part, the truth of these propositions are retained as axioms needing no proof.

In the following sections we shall discuss these various aspects and also the inherent criticisms and their presently accepted forms.

5.5 Definition of Some Entities

A Point

A point in a plane is that whose position is known but having no length, no breadth and no thickness. In other words a point has no dimension. However its position is unique. A point is said to have no part.

A Line

A line straight or curved is an entity having length but no breadth and no thickness. A line is said to have one dimension.

The ends of a line are points.

A Surface

A surface is that entity having both length and breadth but no thickness. A surface has two dimensions. The edges of surfaces are lines.

A plane is a flat surface. Any straight line on a plane has all its points on it.

A Solid

A solid is a material body having length, breadth and thickness. A solid has three dimensions. A solid is bounded by surfaces.

These are only a few of the entities that Euclid introduced in his definition of terms.

5.6 Nine General Axioms

Axioms are self evident statements which need no proofs.

Euclid's nine general axioms are stated below :

Axiom 1 :

Things which are equal to the same thing are equal to one another.

The following are axioms related to the four basic operations of Arithmetic.

Axiom 2 : Addition

If equals are added to equals, the sum are equal.

Axiom 3 : Subtraction

If equals are taken away from equals, the remainders are equal.

Axiom 4 : Multiplication

Things which are multiples of equals are equal to one another.

Axiom 5 : Division

Things which are equal parts of the same are equal to one another.

Axioms related to lines, points and angles.

Axiom 6 :

If O is a point in a straight line AB, then a line OC, which turns about O from the position OA to the position OB must pass through one and only one position in which it is perpendicular to AB.

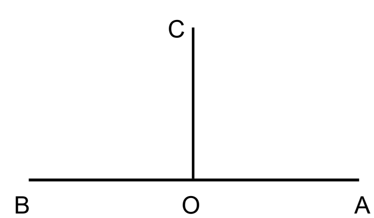


Fig. 5.1

Axiom 7 :

Every straight line of finite length i.e. a line segment has one and only one point of bisection.

Axiom 8 :

Every angle has one and only one internal bisector.

Axioms of Superposition

Axiom 9 :

Magnitudes (entities) which can be made to coincide with one another one equal.

5.7 Criticism on the Definitions

Though Euclid's definitions of certain terms like point, line and plane can give the distinguishing characters of the entities, they involve unexplained terms like part, length, breadth etc. To define them more undefined terms are to be brought in. In this way an endless chain of definitions arises.

To avoid all these, mathematicians now take a point, a line and a plane as undefined terms.

Further, the axiom of superposition is generally avoided because it incurs the movement of plane figures like a triangle to superpose on another triangle which is not possible in view of the definition of a solid and also a triangle being not a solid. However, equality by virtue of coincidence is meaningful as we shall find in later sections.

Thus a very important theorem of Euclid namely the SAS congruence theorem, the proof of which involves movement of a triangle is now taken as an axiom to avoid the inconsistency in its proof.

There are other cases like this that you will study in due course.

5.8 Euclid's Five Postulates

Though postulates and axioms are now taken as synonymous, Euclid called axioms related to geometry as postulates.

Postulate 1 :

(i) A straight line may be drawn from one point to any other point.

(ii) Given two distinct points there is one and only one line through them i.e. a unique line passes through two distinct points.

Postulate 2 :

A terminated line can be produced indefinitely.

A terminated line is now called a line segment.

Postulate 3 :

A circle can be drawn with any centre and any radius.

Postulate 4 :

All right angles are equal to one another.

Postulate 5 :

If a straight line falling on two straight lines makes the interior angles on the same side taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on the same side on which the sum of the angles is less than two right angles.

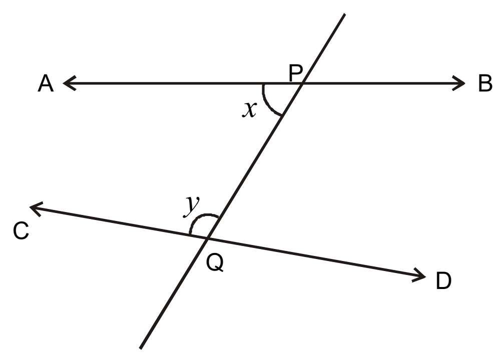


Fig. 5.2

This postulate is not as obvious as others. It requires demonstration.

In the figure, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines and $\overleftrightarrow{PQ}$ falls on both. If the sum of the two interior angles x and y on the left side of $\overleftrightarrow{PQ}$ is less than two right angles i.e. if $x + y < 2$ rt. angles then, the two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ when produced indefinitely will meet on the left side of $\overleftrightarrow{PQ}$.

There is another form of this postulate.

It states as follows :

Given a straight line and fixed point not on the line, there is only one line passing through the point and parallel to the given line.

This statement apparently seems to be different from postulate 5. But they are equivalent.

In the figure, $\overleftrightarrow{AB}$ is the given line and C is the given point. Let $\overleftrightarrow{PQ}$ be the straight line passing through C and parallel to $\overleftrightarrow{AB}$. A line through C meets $\overleftrightarrow{AB}$ at D. Now $\overleftrightarrow{CD}$ falls on them and the sum of the interior angles x and y is 2 right angles. If the sum is to be less than 2 right angles then, at least one of the angles x or y (or both) is to be reduced by turning the line (lines) about the point (points) C or D as the case may be. Surely the two lines will meet on the side of the line CD wherein the sum of the two interior angles is less than two right angles.

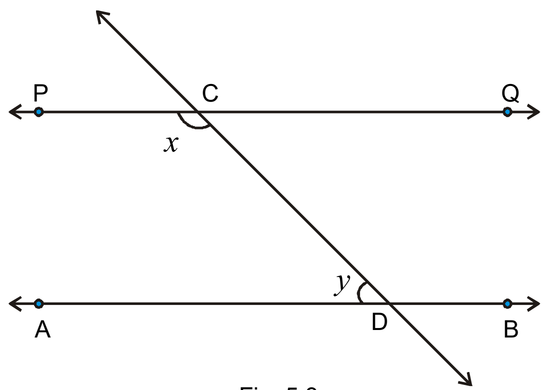


Fig. 5.3

Another form of the postulate is as follows :

Two distinct intersecting lines cannot be parallel to the same line.

Example 1 : C is a point on the line segment AB such that AC = CB, show that AC = $\frac{1}{2}$AB.

Solution : In the adjoining figure, from the given condition AC = CB.

Adding AC to the equals we get

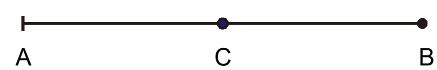


Fig. 5.4

AC + AC = AC + CB ; by Axiom 1 (axiom of addition of equals)

$\Rightarrow$ 2AC = AB, $\because$ (AC + CB) coincides with AB and therefore by axiom 9, they are equal.

Dividing by 2, $\frac{1}{2}$ 2AC = $\frac{1}{2}$ AB ; Axiom 5 (axiom of division by equals)

$\Rightarrow$ AC = $\frac{1}{2}$ AB.

Example 2 : AB is a line segment, C and D are two points on AB such that AC = BD, using appropriate axioms show that AD = CB.

Solution : As C and D are two points on $\overleftrightarrow{AB}$, one point say, D can be to the left or to the right of C.

Case I. When D is to the left of C.

As shown in the figure we see that

Fig. 5.5

AC coincides with (AD + DC). Thus by axiom 9 they are equal. Similarly, BD and (DC + CB) are equal.

Thus, AC = BD

$\Rightarrow$ AD + DC = DC + CB

$\Rightarrow$ AD = CB, (Axiom 3, subtraction of equals)

Case II. When D is to the right of C.

Fig. 5.6

Now, AC = BD

$\Rightarrow$ AC + CD = BD + DC , (Axiom 2, addition of equals)

$\Rightarrow$ AD = CB, (Axiom 9, Axiom of coincidence)

We shall now establish a very basic theorem of Geometry.

Theorem : **Two distinct straight lines cannot have more than one point in common.**

Proof : Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two distinct straight lines.

Two possibilities arise

(i) When the two lines are parallel.

In this case the two lines do not meet each other. In other words there is no point common to both the straight lines.

(ii) When the two lines are not parallel they will meet at one point say, at the point P. Now P is a point common to both the lines.

We shall show that P is the only point common to both the lines.

If possible let Q be another point common to the lines. In other words both the points P and Q are on the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Then $\overleftrightarrow{AB}$ passes through the points P and Q. So also $\overleftrightarrow{CD}$ passes through P and Q.

Thus $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are distinct lines passing through two different points.

This violates postulate 1 (ii) which states that there is one and only one line passing through two distinct points.

Hence, it is not possible to have another point common to both of them.

Thus two distinct lines cannot have more than one point in common.

EXERCISE 5.1

1. Using the postulate of drawing a circle, show that an equilateral triangle can be drawn with a given line segment as a side.

2. Given a line segment AB and a point C not on it, using appropriate axiom show that there is only one perpendicular from C on AB.

3. Using Euclid's postulate 5 show that there exists a line parallel to a given line.

******

SUMMARY

In this chapter the following points have been studied.

1. If a ray stands on a line, then the sum of the two adjacent angles so formed is 180^0 and its converse.

2. If two lines intersect, the vertically opposite angles are equal.

3. If a transversal intersects two parallel lines

(a) each pair of corresponding angles are equal.

(b) each pair of alternate angles are equal.

(c) each pair of interior angles on the same side of the transversal are supplementary.

4. Lines which are parallel to a given line are parallel.

5. The sum of the angles of a triangle is 180^0.

6. If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

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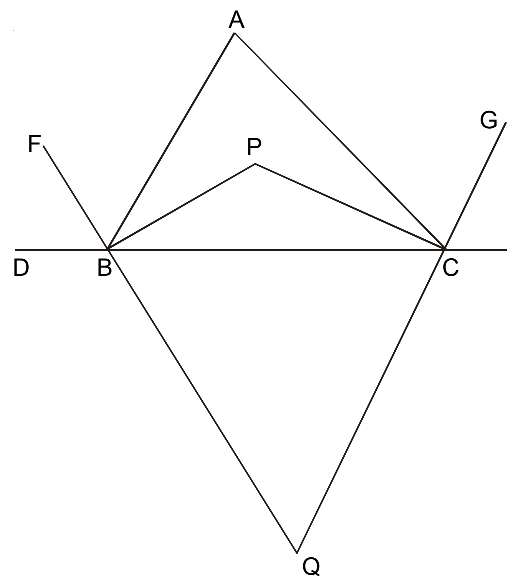


Fig. 6.42

Proof : It is given that

$\angle PBC = \frac{1}{2}\angle B$ and $\angle PCB = \frac{1}{2}\angle C$

$\therefore \quad \angle PBC + \angle PCB = \frac{1}{2}(\angle B + \angle C)$ ——— (i)

Now, $\angle CBQ = \angle DBF$

(vertically opposite angles)

$= \frac{1}{2}\angle ABD$

$= \frac{1}{2}(\angle A + \angle C)$

[$\because$ $\angle ABD$ is external angle and $\angle A$, $\angle C$ are corresponding interior opposite angles]

Similarly, $\angle BCQ = \frac{1}{2}(\angle A + \angle B)$

$\therefore \quad \angle CBQ + \angle BCQ = \angle A + \frac{1}{2}(\angle B + \angle C)$

$= \angle A + \angle B + \angle C - \frac{1}{2}(\angle B + \angle C)$

$= 180^0 - \frac{1}{2}(\angle B + \angle C)$ (using angle sum property)

——— (ii)

Using (i) and (ii),

$\angle PBC + \angle CBQ = (\angle PCB + \angle BCQ) = 180^0$

$\Rightarrow \quad \angle PBQ + \angle PCQ = 180^0$

But $\angle BPC + \angle BQC + \angle PBQ + \angle PCQ = 360^0$

(sum of the angles of a quadrilateral)

$\Rightarrow \quad \angle BPC + \angle BQC + 180^0 = 360^0$

Hence, $\angle BPC + \angle BQC = 180^0$.

Exterior Angles of a Triangle :

The side BC of the ΔABC is produced to D (Fig. 3.39). Then ∠ACD is called an exterior angle of ΔABC. ∠BAC and ∠ABC are called interior opposite angles with respect to the exterior angle ∠ACD. Similarly, if AC is produced to E, ∠BCE is an exterior angle and ∠BAC, ∠ABC are the corresponding interior opposite angles (Fig. 6.39). Thus, at each vertex of a triangle, there are two exterior angles of equal measures, being vertically opposite angles. Thus the total number of exterior angles of a triangle is six.

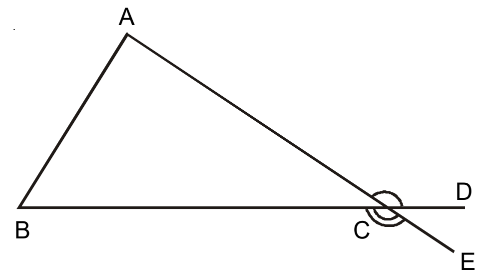


Fig. 6.39

Theorem 6.12 : ***If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.***

Given : ∠ACD is exterior angle obtained by producing BC to D and ∠ABC, ∠BAC are the corresponding two interior opposite angles (Fig. 6.40).

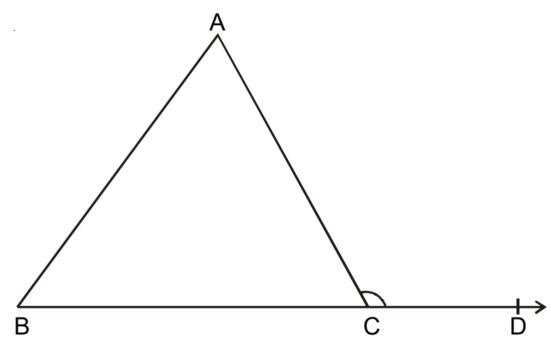


Fig. 6.40

To prove : ∠ACD = ∠ABC + ∠BAC

Proof : By the angle sum property of a triangle,

$\angle ABC + \angle BAC + \angle BCA = 180^0$ ... (i)

Again, ∠BCA and ∠ACD are a pair of linear angles.

$\therefore \angle BCA + \angle ACD = 180^0$ ... (ii)

From (i) and (ii)

$\angle ABC + \angle BAC + \angle BCA = \angle BCA + \angle ACD$

$\Rightarrow \angle ABC + \angle BAC = \angle ACD$

Hence, ∠ACD = ∠ABC + ∠BAC.

2. In Fig. 6.36, $l \parallel m$ and transversal n intersects l and m at A and B respectively. If $\angle 1 : \angle 2 = 3 : 2$, determine all the eight angles.

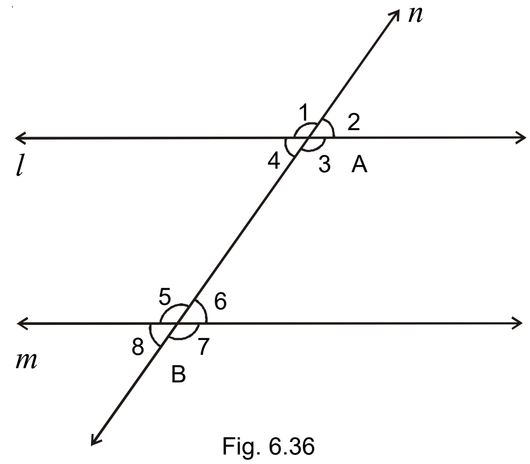


Fig. 6.36

3. In Fig. 6.37, AB || DE. Find x.

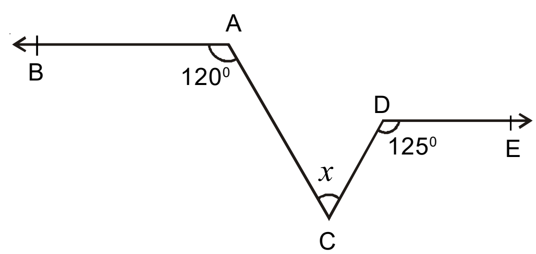


Fig. 6.37

4. If a transversal intersects two parallel lines, show that the bisectors of any pair of alternate angles are parallel.

5. Prove that lines which are perpendicular to the same line are parallel to one another.

6. Two unequal angles of a parallelogram are in the ratio 4 : 5. Find all the angles of the parallelogram in degrees.

7. Prove that if the two arms of an angle are perpendicular to the two arms of another angle, then the angles are either equal or supplementary.

8. If one angle of a parallelogram is 60^0, find the other angles.

$\Rightarrow \quad \angle APQ + y = 125^0$

$\Rightarrow \quad 68^0 + y = 125^0$

$\Rightarrow \quad y = 125^0 - 68^0 = 57^0$

$\therefore \quad x = 68^0, \; y = 57^0.$

Example 2 : Two plane mirrors *l* and *m* are placed parallel to each other as shown in Fig. 6.33. An incident ray AB to the first mirror is reflected twice in the direction CD. Prove that AB || CD.

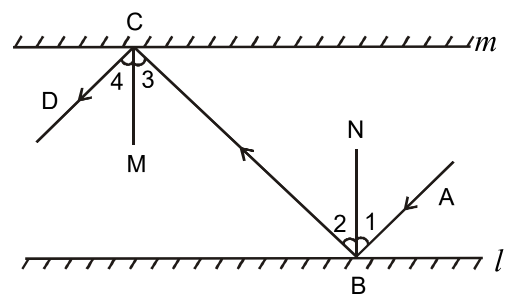


Fig. 6.33

Given : Two plane mirrors *l* and *m* such that *l* || *m*. An incident ray AB after reflections takes the path BC and CD; BN and CM are the normals to the plane mirrors *l* and *m* respectively (Fig. 6.33).

To prove : AB || CD

Proof : Since BN $\perp l$, CM $\perp m$ and $m \parallel n$,

$\therefore \quad CM \perp l.$

$\Rightarrow \quad BN \parallel CM.$

Thus, BN and CM are two parallel lines and a transversal BC intersects them at B and C respectively.

$\therefore \quad \angle 2 = \angle 3$ (alternate angles)

But $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (by laws of reflection)

Hence, $\angle 1 = \angle 2 = \angle 3 = \angle 4.$

$\angle ABC = \angle 1 + \angle 2 = \angle 3 + \angle 4 = \angle BCD.$

Now, lines AB and CD are intersected by transversal BC such that a pair of alternate angles ($\angle ABC$, $\angle BCD$) are equal. Therefore, AB || CD.

Theorem 6.8 : ***If a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are supplementary.***

Given : Two parallel lines AB and CD, and a transversal *l* intersecting them at E, F respectively (Fig. 6.29).

To prove : (i) $\angle 3 + \angle 6 = 180^0$

and (ii) $\angle 4 + \angle 5 = 180^0$.

Proof : It is given that

AB ∥ CD

$\therefore$ $\angle 2 = \angle 6$ (corresponding angles)

But $\angle 3 + \angle 2 = 180^0$ (linear pair)

$\therefore$ $\angle 3 + \angle 6 = 180^0$.

Similarly, $\angle 4 + \angle 5 = 180^0$.

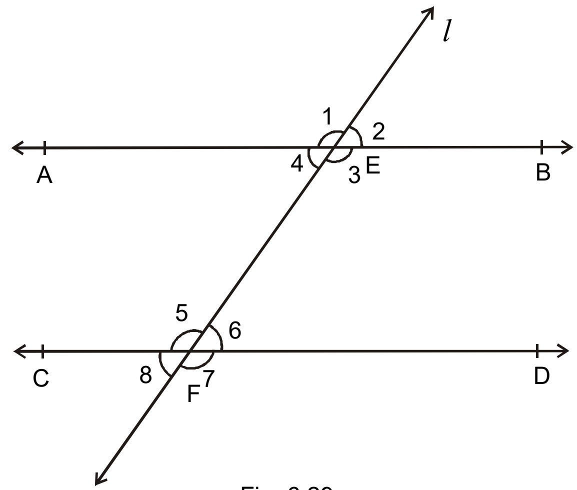


Fig. 6.29

Theorem 6.9 (*Converse of Theorem 6.8*) : ***If a transversal intersects two lines in such a way that a pair of interior angles on the same side of the transversal are supplementary, then the two lines are parallel.***

Given : A transversal *l* intersecting two lines AB, CD at E, F respectively such that the pair of interior angles $\angle 3$, $\angle 6$ on the same side of *l* are supplementary (Fig. 6.30).

To prove : AB ∥ CD.

Proof : It is given that

$\angle 3 + \angle 6 = 180^0$.

But $\angle 2 + \angle 3 = 180^0$ (linear pair)

$\therefore$ $\angle 3 + \angle 6 = \angle 2 + \angle 3$

$\Rightarrow$ $\angle 6 = \angle 2$.

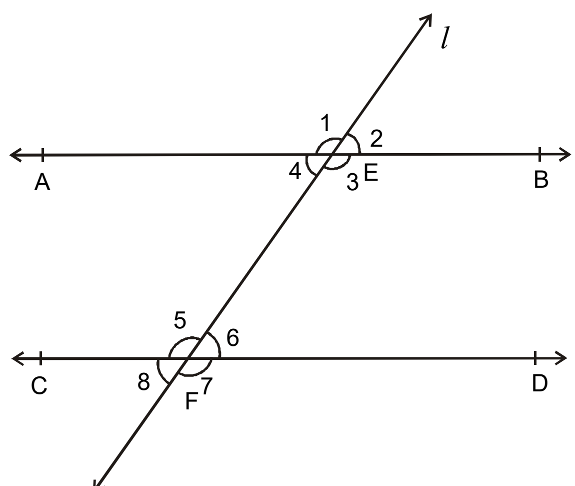


Fig. 6.30

i.e. a pair of corresponding angles are equal.

$\therefore$ AB ∥ CD. (corresponding angles axiom)

Repeat this activity with different pairs of parallel lines and transversal. But you will get the same fact. Thus, we have the following axiom.

Axiom 6.4. If a transversal intersects two parallel lines, then the angles in each pair of corresponding angles are equal.

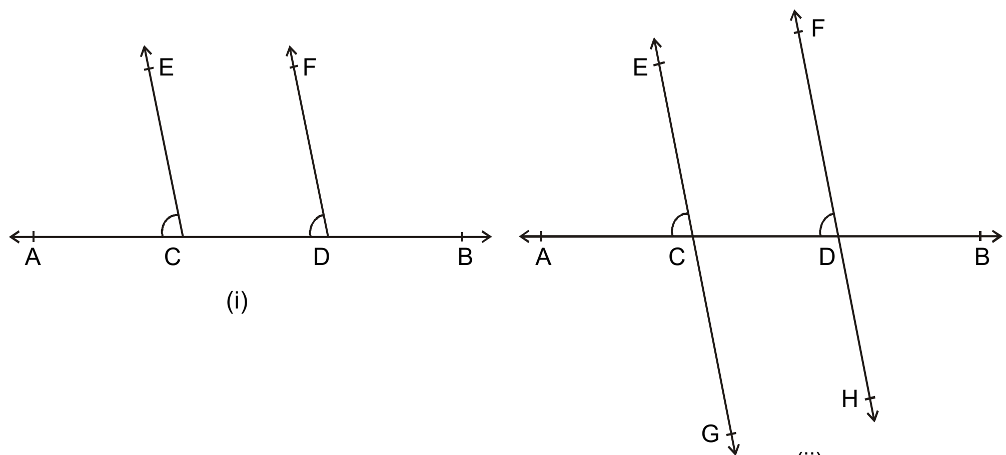


Fig. 6.26

What about the converse of this axiom ? If a transversal intersects two lines, such that the angles in a pair of corresponding angles are equal, then are the two lines parallel? To investigate this, draw line AB and take two points C and D on AB. Construct equal angles $\angle ACE$ and $\angle ADF$ [Fig. 6.26 (i)]. Produce EC and FD to get the lines EG and FH [Fig. 6.26 (ii)]. Observe that these lines do not intersect, however far, they are produced. You may also observe that these lines make equal intercepts on lines perpendicular to them. It follows that EG and FH are parallel i.e., the converse to the above axiom is also true. Thus,

Axiom 6.5 : If a transversal intersects two lines making a pair of corresponding angles equal, then the lines are parallel.

Axiom 6.4 and Axiom 6.5 are together known as **corresponding angles axiom.**

Using these two axioms, we can have the following theorems related to parallel lines and their transversal.

2. In the adjoining figure, $\angle AOC = (2y - 13)^0$ and $\angle BOC = (3y - 12)^0$, find all the four angles.

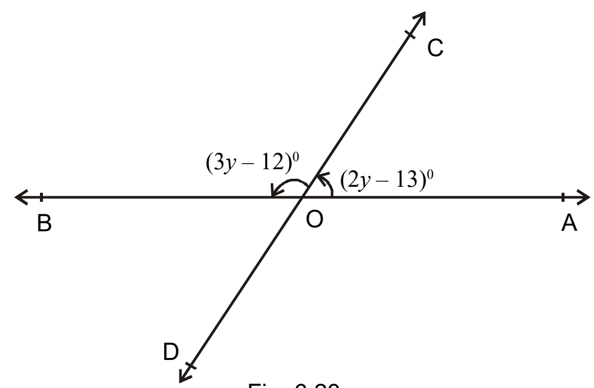


Fig. 6.23

3. Prove that the bisectors of a pair of vertically opposite angles are in the same straight line.

4. If two straight lines intersect each other, prove that the ray opposite to the bisector of one of the angles thus formed bisects the vertically opposite angle.

5. From a point O in a line AB, rays OC and OD are drawn on opposite sides of AB such that $\angle BOC = \angle AOD$. Prove that OC and OD are opposite rays.

ANSWER

1. 10 **2.** $\angle AOC = 90^0$, $\angle BOC = 111^0$, $\angle BOD = 69^0$, $\angle AOD = 111^0$.

6.4 Angles made by a Transversal with Two Lines

Recall that, a line which intersects two or more given lines at distinct points is called a **transversal** of the given lines.

When two straight lines are intersected by a transversal, eight angles are formed. For the sake of distinction, particular names are given to these eight angles.

Example 1 : In Fig. 6.20, $\angle ACU = \angle ABT$, $\angle BAC = 66^0$. Find $\angle ABC$, $\angle CBQ$, $\angle QBT$ and $\angle ABT$.

Solution : Here $\angle ACU = \angle ABT$

$\Rightarrow 180^0 - \angle ACU = 180^0 - \angle ABT$

$\Rightarrow (\angle ACU + \angle ACB) - \angle ACU$

$= (\angle ABT + \angle ABC) - \angle ABT$

$\Rightarrow \angle ACB = \angle ABC$ ... (i)

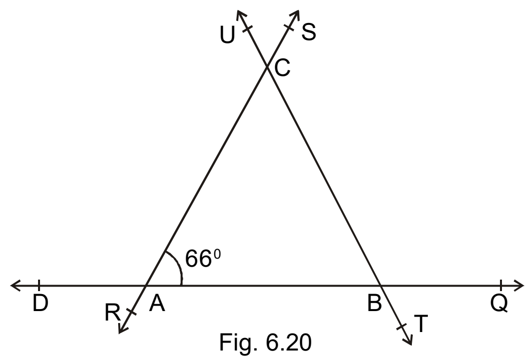


Fig. 6.20

By the angle sum property of a triangle,

$\angle CAB + \angle ABC + \angle ACB = 180^0$

$\Rightarrow \quad 66^0 + 2\angle ABC = 180^0$ [using (i)]

$\Rightarrow \quad 2\angle ABC = 180^0 - 66^0 = 114^0$

$\Rightarrow \quad \angle ABC = \frac{114^0}{2} = 57^0$

$\Rightarrow \quad \angle ABC = \angle ACB = 57^0$.

Now, $\angle ABC$ and $\angle CBQ$ form a linear pair.

$\therefore \quad \angle ABC + \angle CBQ = 180^0$

$\Rightarrow \quad 57^0 + \angle CBQ = 180^0$

$\Rightarrow \quad \angle CBQ = 180^0 - 57^0 = 123^0$

$\angle ABC$ and $\angle QBT$ are vertically opposite angles.

$\therefore \quad \angle QBT = \angle ABC$

$\Rightarrow \quad \angle QBT = 57^0$.

Again, $\angle ABT$ and $\angle CBQ$ are vertically opposite angles.

$\therefore \quad \angle ABT = \angle CBQ = 123^0$.

Hence, $\angle ABC = 57^0$, $\angle CBQ = 123^0$, $\angle QBT = 57^0$, $\angle ABT = 123^0$.

Adding (i) and (ii),

$\angle AOB + \angle BOD + \angle AOC + \angle COD = 360^0$

$\Rightarrow \quad \angle AOB + (\angle BOD + \angle COD) + \angle AOC = 360^0$

$\Rightarrow \quad \angle AOB + \angle BOC + \angle COA = 360^0$.

EXERCISE 6.1

1. In the adjoining figure, AB and AC are opposite rays. If $a - 3b = 20^0$, find a and b.

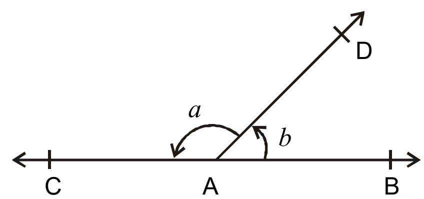


Fig. 6.16

2. In the adjoining figure, BAC is a line and $x : y : z = 5 : 6 : 7$. Find x, y and z.

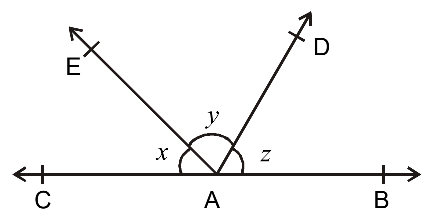


Fig. 6.17

3. Two angles of a linear pair are in the ratio 6 : 3. Find the measure of each of these angles.

4. Prove that the bisectors of two adjacent supplementary angles include a right angle.

5. In the ΔABC, $\angle ABC = \angle ACB$. If the side BC is produced both ways, prove that the exterior angles are equal.

6. The $\angle AOB$ is bisected by ray OC and ray OD is opposite to ray OC. Prove that $\angle AOD$ and $\angle BOD$ are equal.

ANSWER

1. 140^0, 40^0 **2.** 50^0, 60^0, 70^0 **3.** 120^0 , 60^0.

Adding together, $\angle ACD + \angle BCD = \angle ACE + \angle BCE$

$= 90^0 + 90^0$ [$\because$ CE $\perp$ AB,

$= 180^0$ $\angle ACE = 90^0 = \angle BCE$]

$\therefore$ $\angle ACD + \angle BCD = 180^0$.

Theorem 6.2 ***(Converse of Theorem 6.1) : If the sum of two adjacent angles is 180^0, then their non-common arms are two opposite rays.***

Given : Two adjacent angles $\angle BAD$ and $\angle CAD$, such that $\angle BAD + \angle CAD = 180^0$.

To Prove : AB and AC are two opposite rays.

Construction : Suppose AB and AC are not opposite rays then draw ray AE opposite to ray AB, so that BAE is a straight line (Fig. 6.12).

Proof : Since ray AD stands on the line BAE,

$\angle BAD + \angle DAE = 180^0$ (by Theorem 6.1)

But BAD + CAD = 180^0 (given)

$\therefore$ $\angle BAD + \angle DAE = \angle BAD + \angle CAD$

$\Rightarrow$ $\angle DAE = \angle CAD$

$\Rightarrow$ $\angle DAE = \angle DAC$

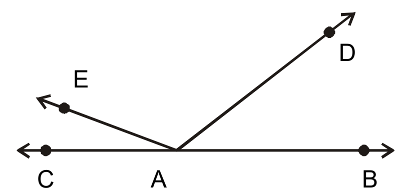


Fig. 6.12

It is possible only when ray AE coincides with ray AC.

Hence, ray AB and ray AC are two opposite rays.

Example 1 : OA and OB are two opposite rays. Ray OC stands on AB and $\angle AOC = 2x$ and $\angle BOC = 7x$. Find x.

Solution : Since OC stands on the line AOB,

$\angle AOC + \angle BOC = 180^0$ (linear pair)

$\Rightarrow$ $2x + 7x = 180^0$

$\Rightarrow$ $9x = 180^0$

$\therefore$ $x = 20^0$.

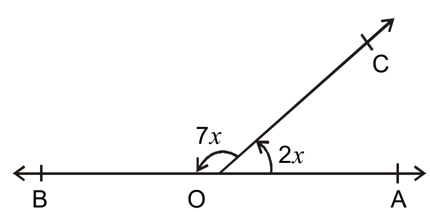


Fig. 6.13

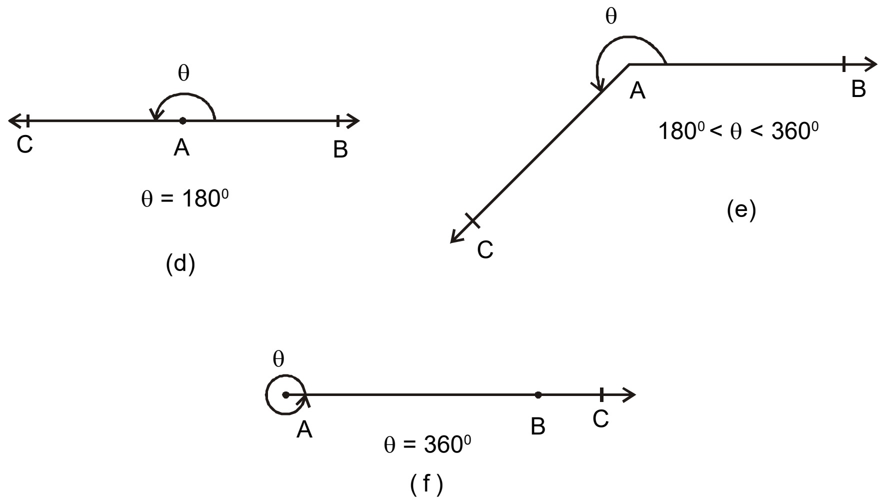


Fig. 6.7

Complementary Angles :

Two angles, the sum of whose measures is 90^0, are called complementary angles. Each of the two complementary angles is called the complement of the other.

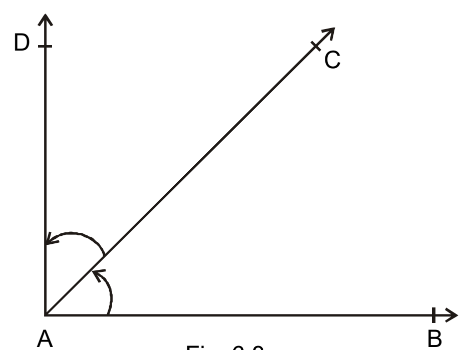


Fig. 6.8

In Fig. 6.8,

$\angle BAC + \angle CAD = 90^0$

So, $\angle BAC$, $\angle CAD$ are complementary angles.

Supplementary Angles :

Two angles, the sum of whose measures is 180^0, are called supplementary angles. Each of the two supplementary angles is called the supplement of the other.

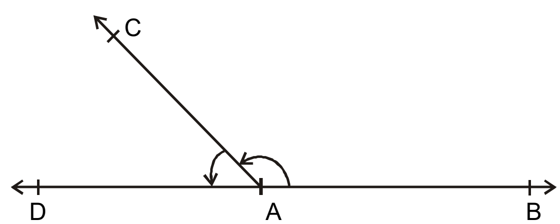


Fig. 6.9

In Fig. 6.9,

$\angle BAC + \angle CAD = 180^0$.

So, $\angle BAC$, $\angle CAD$ are supplementary angles.

Note :

(i) A line segment is a part of a line with two end points.

(ii) A ray is a part of a line with one end point.

(iii) A line is denoted as $\overleftrightarrow{AB}$ where A, B are any two distinct points on the line, a line segment with end points A, B is denoted as $\overline{AB}$ and its length by AB. A ray with initial point A is denoted as $\overrightarrow{AB}$. However, when the context is clear, we use the common symbol AB to denote these four entities.

Collinear Points : Three or more points are said to be collinear if there is a line which contains all of them.

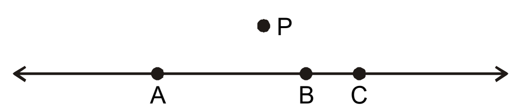


Fig. 6.4

In Fig. 6.4, the points A, B, C are collinear whereas the points A, P, B, C are not collinear i.e. they are non-collinear.

Concurrent Lines : Three or more lines are said to be concurrent if there is a point which lies on all of them.

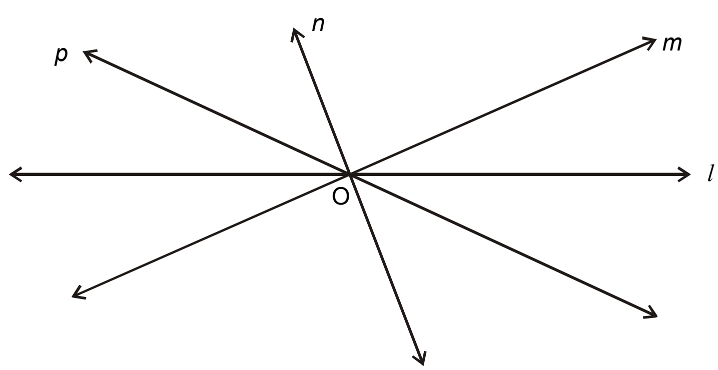


Fig. 6.5

In Fig. 6.5, the lines ***l, m, n, p*** are concurrent as the point O lies on all of them.

CHAPTER

6

LINES AND ANGLES

6.1 Introduction

In earlier classes, you have learnt about the concepts of points, lines and angles. You have also learnt the properties of the angles formed when two lines intersect each other, the properties of the angles formed when a line intersects two or more parallel lines at distinct points, the property of the sum of the three angles of a triangle and the property of an exterior angle of a triangle. All these properties were verified through activities. In this chapter, we will prove them with the help of deductive reasoning. Before this, let us recall the terms and definitions related to lines and angles learnt in the earlier classes.

6.2 Some Definitions

There are three basic concepts in geometry, namely point, line and plane. It is not possible to define these three concepts precisely. However, they can be visualised by certain practical situations.

Point :

It has neither length nor breadth, nor thickness, however, it has a unique position. It is denoted by a fine dot on a paper. A point is usually denoted by a capital letter such as A, B, P, Q, R, etc.

Thus, in Fig. 6.1 the dot A is the visualisation of a point.

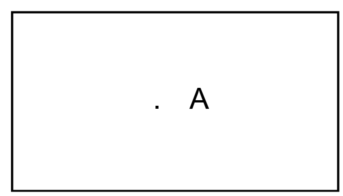


Fig. 6.1

Line :

A line has neither breadth nor thickness, however it has a sense of length. If we fold a leaf of a book lengthwise, the crease in the page gives an impression of a line.

SUMMARY

In this chapter, the following points have been studied :

1. Two triangles are congruent if any two sides and the included angle of one is equal to any two sides and the included angle of the other (SAS congruence).

2. Two triangles are congruent if any two angles and the included side of one is equal to any two angles and the included side of the other (ASA congruence).

3. If any two angles and a non-included side of one triangle are equal to corresponding angles and side of another triangle, the two triangles are congruent (AAS congruence).

4. If three sides of a triangle are respectively equal to the corresponding three sides of another triangle, the two triangles are congruent (SSS congruence).

5. In an isosceles triangle, the angles opposite to the equal sides are equal.

6. The sides opposite to the equal angles of a triangle are equal.

7. If in two right triangles, hypotenuse and one side of a triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent (RHS congruence).

8. In a triangle, angle opposite to the longer side is larger.

9. In a triangle, side opposite to the larger angle is longer.

10. Sum of any two sides of a triangle is greater than the third side.

****

Theorem 7.10 : ***The sum of any two sides of a triangle is greater than the third side.***

Example 10 : Prove that in a right triangle, the hypotenuse is the longest side.

Given : Right ΔABC in which $B = 90^0$. (Fig. 7.21)

To prove : AC is the longest side.

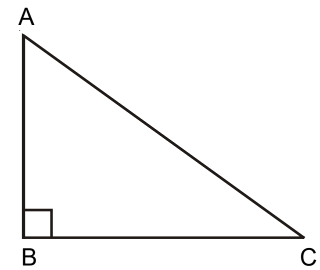


Fig. 7.21

Proof : By the angle sum property,

$\angle A + \angle B + \angle C = 180^0$

$\Rightarrow$ $\angle A + 90^0 + \angle C = 180^0$

$\Rightarrow$ $\angle A + \angle C = 90^0$

$\Rightarrow$ $\angle A < 90^0$ and $\angle C < 90^0$

$\Rightarrow$ $BC < AC$ and $AB < AC$ [$\because$ side opposite to the larger angle is longer]

$\therefore$ AC is the longest side.

Example 11: In Fig. 7.22, PQRS is a quadrilateral in which PQ is the longest side and RS is the shortest side. Prove that $\angle R > \angle P$.

Given : In the quadrilateral PQRS, PQ is the longest side and R is the shortest side.

To Prove : $\angle R > \angle P$.

Construction : Join PR.

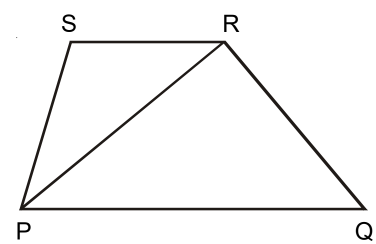


Fig. 7.22

Proof : Since PQ is the longest side,

$PQ > QR$.

$\therefore$ In ΔPQR, $\angle PRQ > \angle QPR$ ... (1)

[$\because$ angle opposite to the longer side is larger]

Again, SR is the shortest side.

$\therefore$ $SR < PS$ i.e., $PS > SR$

$\therefore$ In ΔPRS, $\angle PRS > \angle SPR$ ... (2)

[$\because$ angle opposite to the longer side is larger]

Take the points C_1, C_2, C_3 on $\overrightarrow{BX}$ as in Fig. 7.16. Observe that $BC < BC_1 < BC_2 < BC_3$. Measure the angles $\angle BAC$, $\angle BAC_1$, $\angle BAC_2$, $\angle BAC_3$ and compare. You will observe that $\angle BAC < \angle BAC_1 < \angle BAC_2 < \angle BAC_3$. Thus, you have observed that as the length of the side BC is increased, the measure of the angle $\angle A$ opposite to BC is also increased.

Take a scalene triangle say, ΔABC (Fig. 7.17).

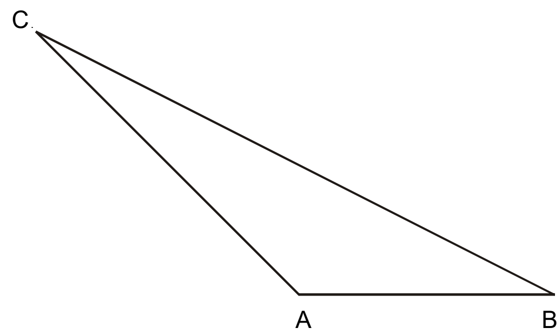


Fig. 7.17

Measure the sides AB, BC, CA. Observe that BC is the longest side and AB is the shortest side. Now, measure the angles of the ΔABC. You will observe that $\angle A$ is the largest and $\angle C$ is the smallest. Repeating this activity with different scalene triangles, we have the following theorem.

Theorem 7.8 : ***If two sides of a triangle are unequal, the angle opposite to the longer side is larger.***

Let us perform the following activity :

Draw a triangle say, ΔABC (Fig. 7.18). With A as centre and AC as radius, draw an arc of the circle as in Fig. 7.18.

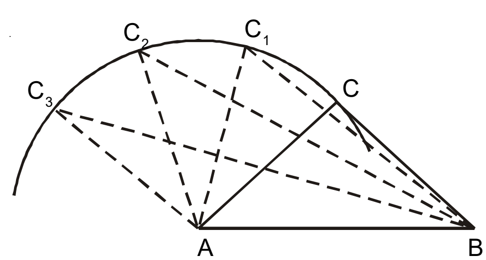


Fig. 7.18

In the rt. Δ^sABD and ACD

AB = AC (given)

$\angle$ADB = $\angle$ADC = 90^0 (given)

AD is common to both.

$\therefore$ by RHS congruence,

ΔABD $\cong$ ΔACD.

Example 9: **In Fig 7.15, LM = MN, QM = MR, ML $\perp$ PQ and MN $\perp$ PR. Prove that PQ = PR.**

Given : LM = MN, QM = MR,

ML $\perp$ PQ, MN $\perp$ PR.

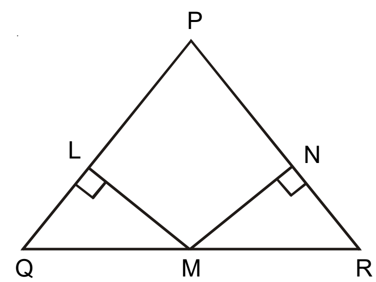


Fig. 7.15

To prove : PQ = PR

Proof : In rt. Δ^s LQM and NRM

LM = MN (given)

$\angle$QLM = $\angle$RNM = 90^0 (given)

QM = MR (given)

Here QM is the hypotenuse of ΔLQM and MR is the hypotenuse of ΔNRM.

$\therefore$ By RHS congruence,

ΔLQM $\cong$ ΔNRM

$\therefore$ $\angle$LQM = $\angle$NRM

$\Rightarrow$ $\angle$PQM = $\angle$PRM

$\Rightarrow$ PR = PQ [$\because$ sides opposite equal angles are equal]

6. Prove that medians of an equilateral triangle are equal.

7. Two lines AB and CD intersect at O such that BC is equal and parallel to AD. Prove that the lines AB and CD bisect each other at O.

8. Line segment AB is parallel to another equal line segment CD. O is the mid point of AD. Prove that **(i)** $\Delta AOB \cong \Delta DOC$ **(ii)** O is also the mid point of BC.

9. In ΔABC, the bisector AD of $\angle A$ is perpendicular to side BC. Prove that ΔABC is isosceles.

10. P is a point equidistant from two lines l and m intersecting at a point A. Prove that the line AP bisects the angle between the lines.

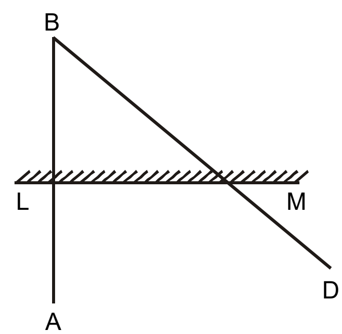
11. ABCD is a rectangle. P, Q, R and S are the mid points of AB, BC, CD and DA respectively. Prove that PQRS is a rhombus.

12. Prove that the diagonals of a rhombus bisect each other at right angles.

13. ABCD is a parallalogram and AC is one of its diagonals. Prove that $\Delta ABC \cong \Delta ACD$.

14. AB and AC are equal sides of an isosceles ΔABC. If the bisectors of $\angle ABC$ and $\angle ACB$ intersect each other at O, prove $\Delta AOB \cong \Delta AOC$.

15. The image of an object placed at a point A before a plane mirror LM is seen at the point B by an observer at D as in the adjoining figure. Prove that the image is as far behind the mirror as the object is in front of the mirror.



ANSWER

4. 40^0, 40^0 **5.** 40^0

Theorem 7.6 (SSS congruence)

If three sides of a triangle are equal respectively to the corresponding three sides of another triangle, then the two triangles are congruent.

Note **:** Even though, the direct proof of the above theorem is not given here, it can be proved using previous results.

Example 6: **In fig 7.12, AB = CD and AD = BC**

Prove that ΔADC ≅ ΔCBA

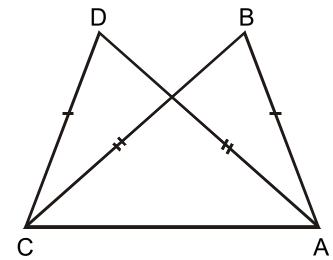


Fig 7.12

Given **:** AB = CD, AD = BC

To prove **:** ΔADC ≅ ΔCBA

Proof **:** In Δ^sADC and CBA

AB = CD (given)

BC = AD (given)

AC is common to both.

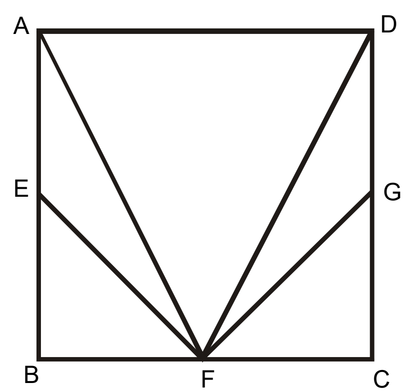
∴ by SSS congruence.

ΔADC ≅ ΔCBA.

Example 7: **ABCD is a square. E, F and G are mid points of side AB, BC, CD respectively. Prove that the triangles AEF and DGF are congruent.**

Given **:** In the square ABCD, E, F, G are mid points of AB, BC, CD respectively

To prove **:** ΔAEF ≅ ΔDGF



Proof **:** In Δ^sEBF and GCF.

BE = CG (= $\frac{1}{2}$ AB = $\frac{1}{2}$ CD)

BF = FC (= $\frac{1}{2}$ BC)

∠EBF = ∠GCF = 90^0

∴ by SAS congruence,

Theorem 7.5. ***The sides opposite to equal angles of a triangle are equal.***

Given : In ΔABC (Fig 7.8),

$\angle ABC = \angle ACB$.

To prove : $AC = AB$

Construction : Draw $AD \perp BC$.

Proof : In $\Delta^s ABD$ and ACD.

$\angle ABD = \angle ACD$ (given)

$\angle ADB = \angle ADC = 90^0$ (by construction)

AD is common to both

$\therefore$ by AAS congruence

$\Delta ABD \cong \Delta ACD$

$\therefore$ $AB = AC$.

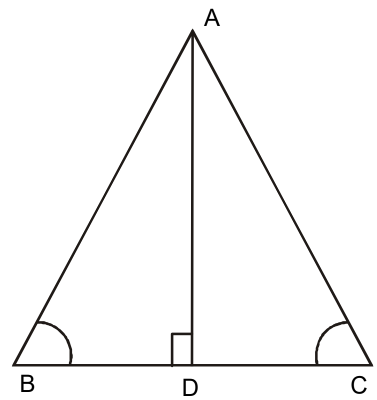


Fig 7.8

Example 4. **In Fig 7.9, AD is the bisector of $\angle A$ such that BD = DC. Prove that ΔABC is isosceles.**

Given : In ΔABC, $\angle BAD = \angle DAC$, $BD = DC$

To prove : ΔABC is isosceles

Construction : Produce AD to E such that AD = DE. Join CE.

Proof : In $\Delta^s ABD$ and EDC.

$BD = DC$ (given)

$\angle ADB = \angle EDC$ (vertically opposite angles)

$AD = DE$ (by construction)

$\therefore$ by SAS congruence, $\Delta ABD \cong \Delta ECD$ ...(1)

$\therefore$ $\angle BAD = \angle CED$.

$\Rightarrow$ $\angle DAC = \angle CED$
[It is given that $\angle DAC = \angle BAD$]

$\Rightarrow$ $EC = AC$ [sides opposite to equal angles]

$\Rightarrow$ $AB = AC$ (by (1))

$\therefore$ ΔABC is isosceles.

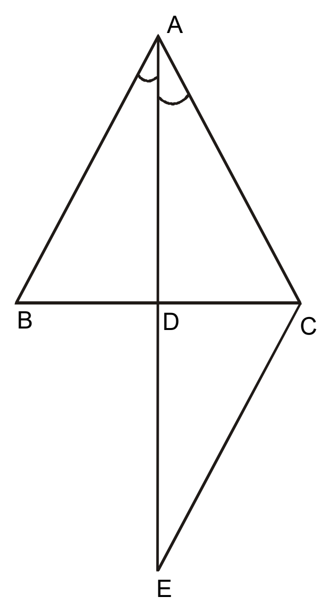


Fig 7.9

Case II. **AB < DE**

Take a point G on $\overline{ED}$ such that EG = BA (Fig 7.6)

Then in ΔABC and ΔGEF,

AB = GE

∠ABC = ∠GEF (given)

BC = EF (given)

∴ by SAS axiom,

ΔABC ≅ ΔGEF

∴ ∠BCA = ∠EFG

But ∠BCA = ∠EFD (given)

∴ ∠EFD = ∠EFG

Which is possible only when G coincides with D.

∴ AB = ED and so by case I, ΔABC ≅ ΔDEF.

Case III : **AB > DE**

Take a point H on $\overline{ED}$ produced (Fig 7.6) so that BA = EH. Then in Δ^sABC and HEF,

AB = EH

∠ABC = ∠HEF (given)

BC = EF (given)

∴ by SAS axiom, ΔABC ≅ ΔHEF

∴ ∠BCA = ∠EFH.

But ∠BCA = ∠EFD (given)

∴ ∠EFD = ∠EFH

Which is possible only when H coincides with D.

∴ AB = DE and so by case I, ΔABC ≅ ΔDEF.

Thus ΔABC ≅ ΔDEF.

AB is common to both.

∴ by SAS axiom.

$\Delta ABD \cong \Delta BAC$

∴ all the corresponding six parts of the Δ^sABD and BAC are equal.

So, BD = AC and $\angle ABD = \angle BAC$.

Using the SAS axiom, we can prove the following property of an isosceles triangle.

Theorem 7.2 : ***In an isosceles triangle, the angles opposite to the equal sides are equal.***

***Given* :** ΔABC in which AB = AC

***To prove* :** $\angle C = \angle B$

***Construction* :** Draw the bisector of $\angle A$ to meet BC at D (Fig 7.4)

***Proof* :** In Δ^sABD and ACD,

AB = AC (given)

$\angle BAD = \angle CAD$ (by construction)

AD is common to both.

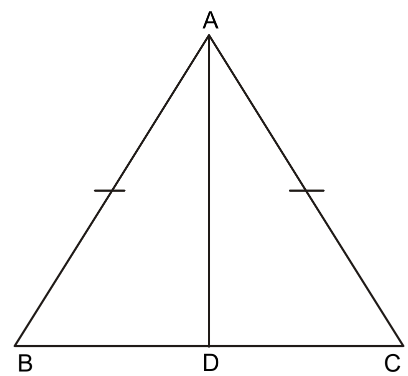


Fig 7.4

So, by SAS axiom, $\Delta ABD \cong \Delta ACD$.

∴ $\angle B = \angle C$.

Example 3 : In ΔABC, AB = AC, $\angle A = 80^0$. Find $\angle B$ and $\angle C$.

***Solution* :** It is given that

in ΔABC, AB = AC

∴ $\angle C = \angle B = x$ (say)

By angle sum property,

$\angle A + \angle B + \angle C = 180^0$

⇒ $80^0 + x + x = 180^0$

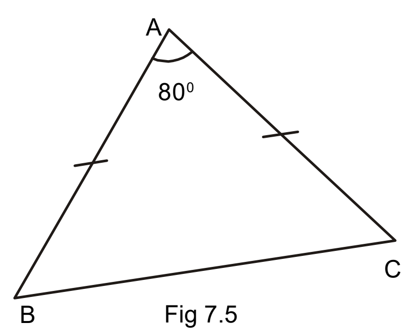


Fig 7.5

Draw a triangle ABC with AB = 6 cm, BC = 4 cm and the included angle $\angle B = 30^0$.

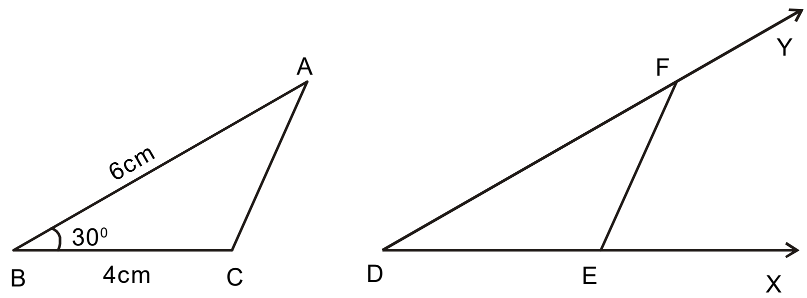


Fig 7.2

Take two lines $\overleftrightarrow{DX}$ and $\overleftrightarrow{DY}$ intersecting at D such that the included angle $\angle D = 30^0$ (Fig 7.2). Take a point E on $\overleftrightarrow{DX}$ such that DE = 4 cm and another point F on $\overleftrightarrow{DY}$ such that DF = 6 cm. Join EF to get ΔFDE. Now, in Δ^sABC and FDE,

AB = FD = 6 cm

BC = DE = 4 cm

$\angle ABC = \angle FDE = 30^0$ (by construction.)

So, there is SAS congruence criteria between ΔABC and ΔFDE. Let us check whether these two triangles are congruent or not. Make a trace copy of ΔABC and place it on ΔFDE such that vertex B falls on D, equal sides $\overline{BC}$ and $\overline{DE}$ concide and A falls on the same side of $\overline{DE}$ as F. You will observe that A falls exactly on F. Thus ΔABC cover exactly ΔFDE and so $\Delta ABC \cong \Delta FDE$. Thus, we have the first criterion for congruence of triangles as an axiom (appendix 1).

Axiom 7.1 (SAS congruence) :

Two triangles are congruent if any two sides and the included angle of one is equal to any two sides and the included angle of the other.

Note : There is no SSA congruence of triangles.

Footnote : Euclid proved the result in Axiom 7.1 and the result was regarded as a theorem instead of axiom. But there was a flaw in the proof. Mathematicians found that the result cannot be proved by using the previous results. So, it has been regarded as axiom.

SUMMARY

In this chapter, you have studied the following points :

1. The area enclosed by a closed curve is the amount of the surface enclosed by the curve.

2. The area of a rectangle a cm by b cm is ab cm^2.

3. A diagonal of a parallelogram divides it into two triangles of equal area.

4. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

5. The area of a parallelogram is the product of any of its sides and the corresponding altitude.

6. Triangles on the same base (or equal bases) and between the same parallels are equal in area.

7. Area of a triangle is half the product of any of its sides and the corresponding altitude.

8. Two triangles having equal areas and standing on the same base and on the same side of it lie between the same parallels.

9. If a parallelogram and a triangle are on the same base and between the same parallels, then the area of the triangle is equal to one-half of the area of the parallelogram.

10. A median of a triangle divides it into two triangles of equal area.

******

$\therefore$ area of ΔAEC = area of ΔABC

Adding area of ΔACD to both sides, we get

area of ΔAEC + area of ΔACD

= area of ΔABC + area of ΔACD

$\Rightarrow$ area of ΔAED = area of quad. ABCD.

EXERCISE 9.2

1. AD is a median of a triangle ABC and P is any point on AD. Show that area of ΔABP = area of ΔACP.

2. ABC is a triangle and DE is drawn parallel to BC, cutting the other sides at D and E. Join BE and CD. Prove that

(i) area of ΔDBC = area of ΔEBC.

(ii) area of ΔBDE = area of ΔCDE.

3. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

4. Show that the area of a rhombus is half the product of the lengths of its diagonals.

5. Prove that the area of a trapezium is half the product of the sum of the lengths of the parallel sides and distance between them.

6. The diagonals AC and BD of a quadrilateral ABCD intersect at O. If BO = OD, prove that area of ΔABC = area of ΔADC.

7. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. Prove that BDEF is a parallelogram whose area is half that of ΔABC and area of ΔDEF = $\frac{1}{4}$ (area of ΔABC).

8. Prove that the straight line joining the mid-points of two sides of a triangle is parallel to the third side.

9. Prove that the straight line joining the mid-points of the oblique sides of a trapezium is parallel to each of the parallel sides.

Also, area of ΔADC $= \frac{1}{2} \times DC \times AN$

$= \frac{1}{2} \times BD \times AN$

$[\because\ BD = DC]$ ... (ii)

From (i) and (ii), we have

area of ΔABD = area of ΔADC.

Example 2 : ABCD is a trapezium with AB || DC and diagonals AC and BD meet at O (Fig. 9.13). Prove that area of ΔAOD = area of ΔBOC.

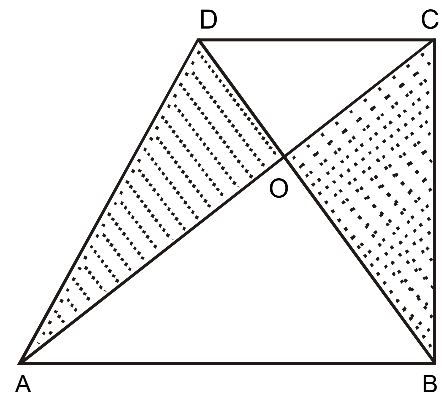


Fig. 9.13

Solution : Now, the Δ^s ABD and ABC are on the same base AB and between the same parallels AB and DC.

$\therefore$ area of ΔABD = area of ΔABC

$\Rightarrow$ area of ΔAOD + area of ΔABO

= area of ΔABO + area of ΔBOC

$\Rightarrow$ area of ΔAOD = area of ΔBOC.

Example 3 : ABCD is a parallelogram and P is any point on BC (produced) and AP meets CD in Q (Fig. 9.14). Prove that

(i) area of ΔACP = area of ΔABP – area of ΔAPD

(ii) area of ΔABP = area of quadrilateral ACPD

(iii) area of ΔQPD = area of ΔBCQ.

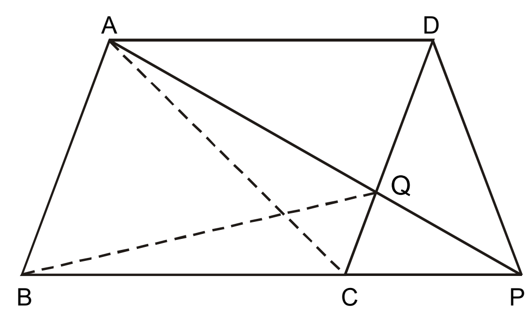


Fig. 9.14

Solution : **(i)** Area of ΔACD

= area of ΔAPD ...(i)

[Triangles on the same base and between the same parallels]

Also, area of ΔABC

= area of ΔACD [Diagonal divide a parl^m into equal areas]

$\therefore$ area of parallelogram ABCQ = area of parallelogram PBCD ... (i)

Now, AC is a diagonal of the parallelogram ABCD.

$\therefore$ area of ΔABC = area of ΔACQ

= $\frac{1}{2}$(area of parallelogram ABCQ) ... (ii)

Similarly, as BD is a diagonal of the parallelogram PBCD,

area of ΔDBC = $\frac{1}{2}$(area of parallelogram PBCD)

= $\frac{1}{2}$(area of parallelogram ABCQ) [From (i)] ... (iii)

From (ii) and (iii), we have

area of ΔABC = area of ΔDBC.

Note : You can also verify the above theorem by drawing several pairs of triangles on the same base and between the same parallels on the graph sheet. If you measure their areas by the method of counting the squares, each time you will find that the areas of the two triangles are (approximately) equal.

Corollary 1 : The area of a triangle is half the product of any of its sides and the corresponding altitude.

Given : ABC is a triangle, AN is the altitude corresponding to the side BC (Fig. 9.10).

To prove : Area of ΔABC = $\frac{1}{2}\times$ BC $\times$ AN

Construction : Complete the parallelogram ABCD.

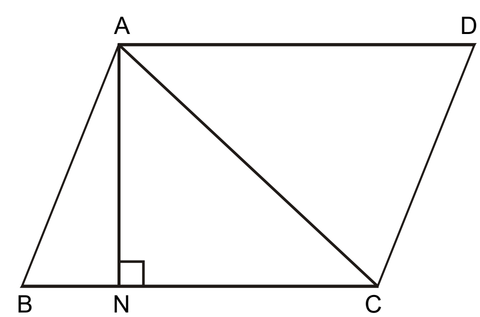


Fig. 9.10

Proof : Since AC is a diagonal of the parallelogram ABCD,

$\therefore$ area of ΔABC = $\frac{1}{2}$(area of parallelogram ABCD) ... (i)

Now, BC is a side of the parallelogram ABCD and AN is the corresponding altitude.

$\therefore$ area of parallelogram ABCD = BC $\times$ AN ... (ii)

From (i) and (ii), we have

area of ΔABC = $\frac{1}{2}\times$ BC $\times$ AN

4. Prove that the line segment joining the mid-points of a pair of opposite sides of a parallelogram divides it into two parallelograms of equal area.

5. Prove that three parallelograms formed by joining the mid-points of the three sides of a triangle are equal in area.

6. Prove that, of all the parallelograms of given sides, parallelogram which is a rectangle has the greatest area.

7. If O is an interior point of a parallelogram ABCD, prove that

(i) area of ΔOAB + area of ΔOCD

$= \frac{1}{2}$(area of parallelogram ABCD)

(ii) area of ΔOBC + area of ΔOAD

= area of ΔOAB + area of ΔOCD.

(***Hint :*** Through O, draw a line parallel to AB.)

8. ABCD is a parallelogram and P is any point on the side CD. Prove that area of ΔAPD + area of ΔBCP = area of ΔABP.

9. ABCD and ABPQ are parallelograms such that the points C, D, P, Q are collinear and R is any point on the side BP. Show that

(i) area of parallelogram ABCD = area of parallelogram ABPQ

(ii) area of ΔARQ = $\frac{1}{2}$(area of parallelogram ABCD).

ANSWER

1. 8.75 cm

Example 1 : If a parallelogram and a triangle are on the same base and between the same parallels, prove that the area of the triangle is equal to half the area of the parallelogram.

Solution : Let parallelogram ABCD and ΔABP be on the same base AB and between the same parallels AB and PC (Fig. 9.5).

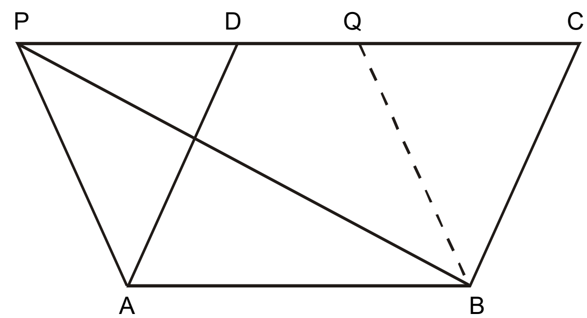


Fig. 9.5

To prove : Area of ΔABP

$= \frac{1}{2}$ (area of parl^m ABCD)

Construction : Complete the parallelogram ABQP.

Proof : Parallelograms ABQP and ABCD are on the same base AB and between the same parallels AB and PC.

$\therefore$ area of parl^m ABQP

= area of parl^m ABCD .... (i)

Since the diagonal BP divides the parallelogram ABQP into two triangles of equal area, therefore we have

area of ΔABP = $\frac{1}{2}$(area of parallelogram ABQP)

= $\frac{1}{2}$(area of parallelogram ABCD) [From (i)]

Example 2 : In a parallelogram ABCD, AB = 12 cm. The altitudes corresponding to the sides AB and AD are respectively 8 cm and 10 cm (Fig. 9.6). Find AD.

Solution : Area of the parallelogram ABCD

= AB × DM = AD × BN

i.e. 12 × 8 = AD × 10

$\therefore$ AD = $\frac{12 \times 8}{10}$ = 9.6 cm.

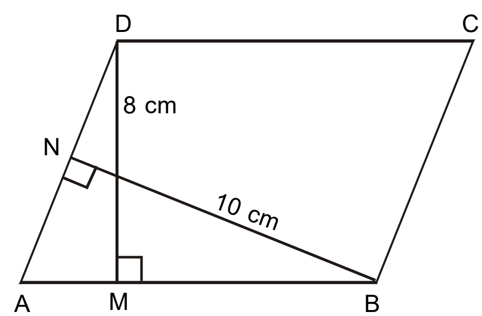


Fig. 9.6

Theorem 9.1 : ***A diagonal of a parallelogram divides it into two triangles of equal area.***

The proof is obvious and left as an exercise.

9.5 Base and Altitude of a Parallelogram

A base of a parallelogram is any of its sides and the length of the perpendicular drawn from any point on the base to the line containing the opposite side is called the corresponding altitude (or height).

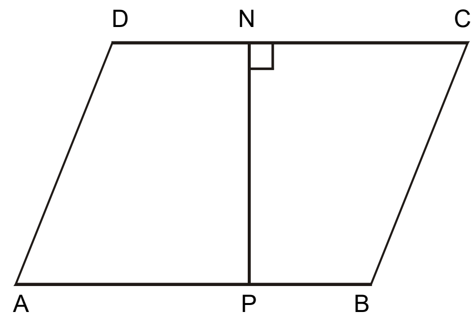


Fig. 9.2

For example, in the parallelogram ABCD (Fig. 9.2) the side AB may be taken as the base. Let P be any point on AB. Draw PN perpendicular to CD. Then the length of the line segment PN is the altitude of the parallelogram ABCD corresponding to the base AB.

9.6 Area of a Parallelogram

Two main theorems concerning area of a parallelogram are given below :

Theorem 9.2 : ***Parallelograms on the same base and between the same parallels are equal in area.***

Given : ABCD and ABEF are two parallelograms, having the same base AB and between the same parallels AB and FC.

To prove : Area of parallelogram ABCD
= area of parallelogram ABEF

Proof : In Δ^sBCE and ADF,

$\angle BCE = \angle ADF$

(corresponding angles from BC || AD and transversal FC)

$\angle BEC = \angle AFD$

(corresponding angles from BE || AF and transversal FC)

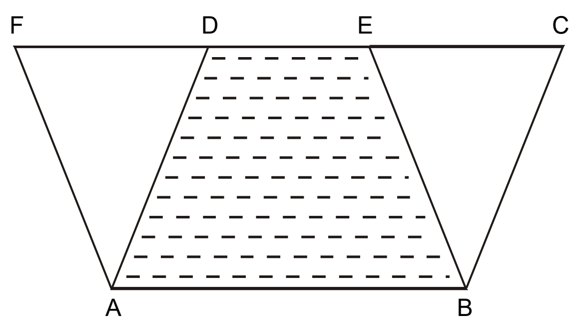


Fig. 9.3

CHAPTER

9 AREA

9.1 Introduction

Mathematics deals with two aspects of the topics within its limit of study. They are the qualitative and the quantitative aspects.

Whenever we discuss any subject matter, first the qualitative aspect including the defining properties is discussed. Then, to discuss the quantitative aspect the question of measurement, the standard units and other related problems are taken up gradually.

9.2 Idea of Area

We know that the length of a rod, the length and breadth of a rectangular field, the perimeter of a square field etc. can be measured in metres or centimetres as we like. Now imagine a room. You can measure its length and breadth. You can also find its height. All these can be done by using a measuring tape. But if you want to know the quantity of a carpet that will be necessary to cover the floor of the room, then you cannot measure the necessary amount by a measuring tape. Here comes the idea of area.

Any closed curve encloses an amount of surface. That amount of the surface is called the area enclosed by the closed curve. To make things easier we consider areas enclosed by rectangular and square figures in the beginning.

The amount of area enclosed by a square whose one side is 1 cm in length is taken as a unit of area and is denoted by 1 sq. cm or 1 cm^2. Similarly, the amount of area enclosed by a square whose one side is 1 metre long is also one unit of area called 1 sq. m or 1 m^2.

The amount of area enclosed by a rectangle 2 cm by 1 cm is 2 cm^2 because there are two small squares, one side of each being of length 1 cm. In short, in a rectangle 2 cm by 1 cm there are two units of area, each being 1 cm^2. Similarly, in a rectangle 4cm by 3 cm there are 4×3 i.e. 12 such units of area and we say that the area of the rectangle is 12 cm^2.

In this way, the area of a rectangle a cm by b cm is ab cm^2.

SUMMARY

In this chapter, you have studied the following points :

1. The area enclosed by a closed curve is the amount of the surface enclosed by the curve.

2. The area of a rectangle a cm by b cm is ab cm^2.

3. A diagonal of a parallelogram divides it into two triangles of equal area.

4. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

5. The area of a parallelogram is the product of any of its sides and the corresponding altitude.

6. Triangles on the same base (or equal bases) and between the same parallels are equal in area.

7. Area of a triangle is half the product of any of its sides and the corresponding altitude.

8. Two triangles having equal areas and standing on the same base and on the same side of it lie between the same parallels.

9. If a parallelogram and a triangle are on the same base and between the same parallels, then the area of the triangle is equal to one-half of the area of the parallelogram.

10. A median of a triangle divides it into two triangles of equal area.

******

$\therefore$ area of ΔAEC = area of ΔABC

Adding area of ΔACD to both sides, we get

area of ΔAEC + area of ΔACD

= area of ΔABC + area of ΔACD

$\Rightarrow$ area of ΔAED = area of quad. ABCD.

EXERCISE 9.2

1. AD is a median of a triangle ABC and P is any point on AD. Show that area of ΔABP = area of ΔACP.

2. ABC is a triangle and DE is drawn parallel to BC, cutting the other sides at D and E. Join BE and CD. Prove that

(i) area of ΔDBC = area of ΔEBC.

(ii) area of ΔBDE = area of ΔCDE.

3. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

4. Show that the area of a rhombus is half the product of the lengths of its diagonals.

5. Prove that the area of a trapezium is half the product of the sum of the lengths of the parallel sides and distance between them.

6. The diagonals AC and BD of a quadrilateral ABCD intersect at O. If BO = OD, prove that area of ΔABC = area of ΔADC.

7. D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. Prove that BDEF is a parallelogram whose area is half that of ΔABC and area of ΔDEF = $\frac{1}{4}$ (area of ΔABC).

8. Prove that the straight line joining the mid-points of two sides of a triangle is parallel to the third side.

9. Prove that the straight line joining the mid-points of the oblique sides of a trapezium is parallel to each of the parallel sides.

Also, area of ΔADC $= \frac{1}{2} \times$ DC × AN

$= \frac{1}{2} \times$ BD × AN

[∵ BD = DC] ... (ii)

From (i) and (ii), we have

area of ΔABD = area of ΔADC.

Example 2 : ABCD is a trapezium with AB || DC and diagonals AC and BD meet at O (Fig. 9.13). Prove that area of ΔAOD = area of ΔBOC.

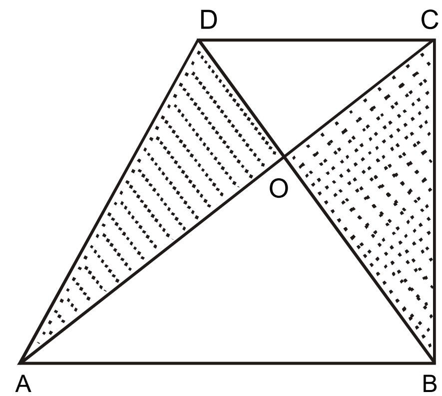


Fig. 9.13

Solution : Now, the Δ^s ABD and ABC are on the same base AB and between the same parallels AB and DC.

∴ area of ΔABD = area of ΔABC

⇒ area of ΔAOD + area of ΔABO

= area of ΔABO + area of ΔBOC

⇒ area of ΔAOD = area of ΔBOC.

Example 3 : ABCD is a parallelogram and P is any point on BC (produced) and AP meets CD in Q (Fig. 9.14). Prove that

(i) area of ΔACP = area of ΔABP – area of ΔAPD

(ii) area of ΔABP = area of quadrilateral ACPD

(iii) area of ΔQPD = area of ΔBCQ.

Solution : **(i)** Area of ΔACD

= area of ΔAPD ...(i)

[Triangles on the same base and between the same parallels]

Also, area of ΔABC

= area of ΔACD [Diagonal divide a parl^m into equal areas]

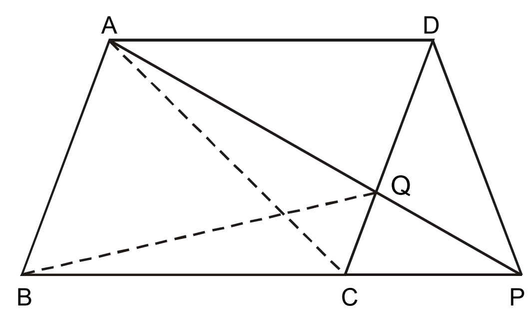


Fig. 9.14

$\therefore$ area of parallelogram ABCQ = area of parallelogram PBCD ... (i)

Now, AC is a diagonal of the parallelogram ABCD.

$\therefore$ area of ΔABC = area of ΔACQ

$= \frac{1}{2}$(area of parallelogram ABCQ) ... (ii)

Similarly, as BD is a diagonal of the parallelogram PBCD,

area of ΔDBC = $\frac{1}{2}$(area of parallelogram PBCD)

$= \frac{1}{2}$(area of parallelogram ABCQ) [From (i)] ... (iii)

From (ii) and (iii), we have

area of ΔABC = area of ΔDBC.

Note : You can also verify the above theorem by drawing several pairs of triangles on the same base and between the same parallels on the graph sheet. If you measure their areas by the method of counting the squares, each time you will find that the areas of the two triangles are (approximately) equal.

Corollary 1 : The area of a triangle is half the product of any of its sides and the corresponding altitude.

Given : ABC is a triangle, AN is the altitude corresponding to the side BC (Fig. 9.10).

To prove : Area of ΔABC = $\frac{1}{2} \times$ BC $\times$ AN

Construction : Complete the parallelogram ABCD.

Proof : Since AC is a diagonal of the parallelogram ABCD,

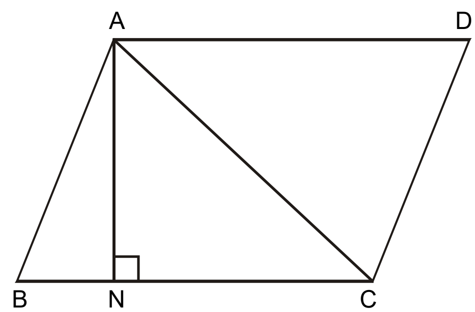


Fig. 9.10

$\therefore$ area of ΔABC = $\frac{1}{2}$(area of parallelogram ABCD) ... (i)

Now, BC is a side of the parallelogram ABCD and AN is the corresponding altitude.

$\therefore$ area of parallelogram ABCD = BC $\times$ AN ... (ii)

From (i) and (ii), we have

area of ΔABC = $\frac{1}{2} \times$ BC $\times$ AN

4. Prove that the line segment joining the mid-points of a pair of opposite sides of a parallelogram divides it into two parallelograms of equal area.

5. Prove that three parallelograms formed by joining the mid-points of the three sides of a triangle are equal in area.

6. Prove that, of all the parallelograms of given sides, parallelogram which is a rectangle has the greatest area.

7. If O is an interior point of a parallelogram ABCD, prove that

(i) area of ΔOAB + area of ΔOCD

$= \frac{1}{2}$(area of parallelogram ABCD)

(ii) area of ΔOBC + area of ΔOAD

= area of ΔOAB + area of ΔOCD.

(***Hint :*** Through O, draw a line parallel to AB.)

8. ABCD is a parallelogram and P is any point on the side CD. Prove that area of ΔAPD + area of ΔBCP = area of ΔABP.

9. ABCD and ABPQ are parallelograms such that the points C, D, P, Q are collinear and R is any point on the side BP. Show that

(i) area of parallelogram ABCD = area of parallelogram ABPQ

(ii) area of ΔARQ = $\frac{1}{2}$(area of parallelogram ABCD).

ANSWER

1. 8.75 cm

Example 1 : If a parallelogram and a triangle are on the same base and between the same parallels, prove that the area of the triangle is equal to half the area of the parallelogram.

Solution : Let parallelogram ABCD and ΔABP be on the same base AB and between the same parallels AB and PC (Fig. 9.5).

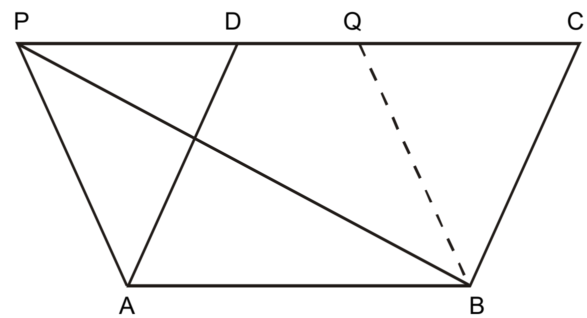


Fig. 9.5

To prove : Area of ΔABP

$= \frac{1}{2}$ (area of parl^{m} ABCD)

Construction : Complete the parallelogram ABQP.

Proof : Parallelograms ABQP and ABCD are on the same base AB and between the same parallels AB and PC.

∴ area of parl^{m} ABQP

= area of parl^{m} ABCD .... (i)

Since the diagonal BP divides the parallelogram ABQP into two triangles of equal area, therefore we have

area of $\Delta ABP = \frac{1}{2}$(area of parallelogram ABQP)

$= \frac{1}{2}$(area of parallelogram ABCD) [From (i)]

Example 2 : In a parallelogram ABCD, AB = 12 cm. The altitudes corresponding to the sides AB and AD are respectively 8 cm and 10 cm (Fig. 9.6). Find AD.

Solution : Area of the parallelogram ABCD

$= AB \times DM = AD \times BN$

i.e. $12 \times 8 = AD \times 10$

$\therefore\ AD = \dfrac{12 \times 8}{10} = 9.6$ cm.

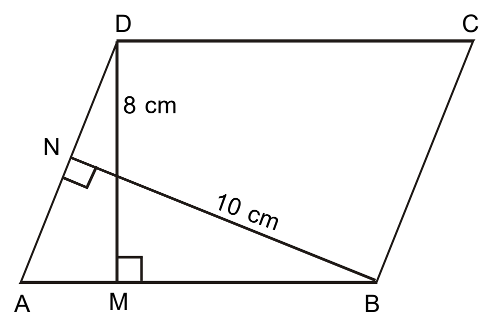


Fig. 9.6

Theorem 9.1 : ***A diagonal of a parallelogram divides it into two triangles of equal area.***

The proof is obvious and left as an exercise.

9.5 Base and Altitude of a Parallelogram

A base of a parallelogram is any of its sides and the length of the perpendicular drawn from any point on the base to the line containing the opposite side is called the corresponding altitude (or height).

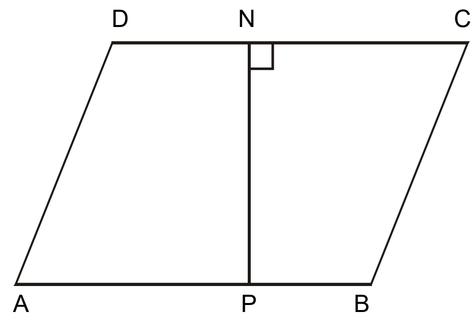


Fig. 9.2

For example, in the parallelogram ABCD (Fig. 9.2) the side AB may be taken as the base. Let P be any point on AB. Draw PN perpendicular to CD. Then the length of the line segment PN is the altitude of the parallelogram ABCD corresponding to the base AB.

9.6 Area of a Parallelogram

Two main theorems concerning area of a parallelogram are given below :

Theorem 9.2 : ***Parallelograms on the same base and between the same parallels are equal in area.***

Given : ABCD and ABEF are two parallelograms, having the same base AB and between the same parallels AB and FC.

To prove : Area of parallelogram ABCD
= area of parallelogram ABEF

Proof : In Δ^sBCE and ADF,

$\angle BCE = \angle ADF$

(corresponding angles from BC ∥ AD and transversal FC)

$\angle BEC = \angle AFD$

(corresponding angles from BE ∥ AF and transversal FC)

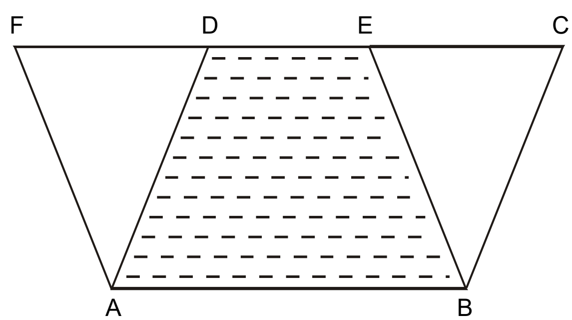


Fig. 9.3

CHAPTER

9 AREA

9.1 Introduction

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Whenever we discuss any subject matter, first the qualitative aspect including the defining properties is discussed. Then, to discuss the quantitative aspect the question of measurement, the standard units and other related problems are taken up gradually.

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The amount of area enclosed by a rectangle 2 cm by 1 cm is 2 cm^2 because there are two small squares, one side of each being of length 1 cm. In short, in a rectangle 2 cm by 1 cm there are two units of area, each being 1 cm^2. Similarly, in a rectangle 4cm by 3 cm there are 4×3 i.e. 12 such units of area and we say that the area of the rectangle is 12 cm^2.

In this way, the area of a rectangle a cm by b cm is ab cm^2.

Case I. The circumcircle meets AD at E.

Then, $\angle ACB$ and $\angle AEB$ are two angles in the same segment of a circle.

$\therefore \quad \angle ACB = \angle AEB$ (by theorem 10.9)

$\Rightarrow \quad \angle ADB = \angle AEB$ [It is given that $\angle ACB = \angle ADB$]

Which is impossible. (Why ?)

Case II. The circumcircle meets AD produced at F.

Then, $\angle ACB$ and $\angle AFB$ are two angles in the same segment of a circle.

$\therefore \quad \angle ACB = \angle AFB$ (by theorem 10.9)

$\Rightarrow \quad \angle ADB = \angle AFB$

Which is impossible. (Why ?)

So, D lies on the circumcircle.

Hence, A, B, C, D are concyclic.

> ***Note :*** The above theorem (Theorem 10.10) can be considered as the converse of the theorem 10.9.

10.7 Cyclic Quadrilaterals

In the previous class, you have studied the concepts of a cyclic quadrilateral. Recall that, a quadrilateral is cyclic if all its vertices lie on a circle. With the help of activities, you also have deduced that the opposite angles of a cyclic quadrilateral are supplementary. Here, a logical proof of the result is given.

Theorem 10.11 ***Opposite angles of a cyclic quadrilateral are supplementary.***

Given : Cyclic quadrilateral ABCD (Fig. 10.18)

To prove : $\angle A + \angle C = 180^0$ and

$\angle B + \angle D = 180^0$

Construction : Join AC and BD.

Proof : $\angle ADB$ and $\angle ACB$ are two angles in the same segment of a circle.

$\therefore \quad \angle ADB = \angle ACB = \angle 1$ (say)

Proof : Perpendiculars IE and IF are drawn from I on the sides AC and AB respectively.

Since I is a point on the bisector of $\angle B$, ID = IF.

Again, I is on the bisector of $\angle C$

$\therefore$ ID = IE

Thus ID = IE = IF

Hence, the circle touches the sides BC , CA, AB at D, E, F respectively.

Remark :

1. The distance of a point from a line is the length of the perpendicular segment drawn from the point upto the line.

2. The idea of a circle touching a line is not introduced hitherto. It may however be noted that a circle touches a line if the perpendicular distance of its centre from the line is equal to its radius.

Example 7 : The sides BC, CA, AB of a triangle ABC are 9 cm, 6.5 cm, 8.5 cm respectively. Construct (i) the circumcircle (ii) the incircle of the ΔABC.

Solution :

(i) ***Given*** **:** For a ΔABC, BC = 9 cm, CA = 6.5 cm and AB = 8.5 cm.

Required : To construct the circumcircle of the ΔABC.

Steps of construction :

1. The ΔABC is constructed such that BC = 9 cm, CA = 6.5 cm and AB = 8.5 cm.

2. Perpendicular bisectors of the sides AB and BC are drawn. They intersect at O.

3. With O as centre and OA as radius, a circle is drawn. This is the required circle.

Proof : By construction, O is equidistant from A, B and C. So, the circle drawn with O as centre and OA as radius will pass through all the three vertices A, B, C of the ΔABC.

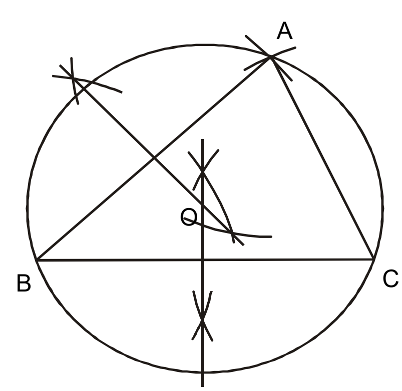


Fig. 11.13

ANSWERS

1. **(i)** 36000 cm^3, 6680 cm^2
(ii) 672 m^3, 472 m^2
(iii) 4800 m^3, 1720 m^2
(iv) 3672 cm^3, 1452 cm^2

2. **(i)** 864 cm^2, 1728 cm^3
(ii) 2166 m^2, 6859 m^3
(iii) 150 m^2, 125 m^3
(iv) 1014 m^2, 2197 m^3

3. 42 cm, 10584 cm^2 **4.** 8 cm, 512 cm^3 **5.** 51 m^2

6. **(i)** 660 cm^2 , 968 cm^2 , 2310 cm^3
(ii) 275 cm^2 , 352 cm^2 , 481.25 cm^3
(iii) 3080 cm^2 , 4312 cm^2 , 21560 cm^3
(iv) 1100 cm^2 , 1408 cm^2 , 3850 cm^3

7. 297 m^2 **8.** 660 m^2 **9.** 336336 cm^3 **10.** 528 cm^2

11. 50 m **12.** 2 m

13. **(i)** 154 m^3, 137.5 m^2, 176 m^2
(ii) 1232 dm^3, 550 dm^2, 704 dm^2
(iii) 12936 cm^3, 2310 cm^2, 3696 cm^2
(iv) 1232 cm^3, 550 cm^2, 704 cm^2
(v) 2816 cm^3, 2200 cm^2, 2816 cm^2

14. 9856 cm^3 **15.** 25 cm, 704 cm^2 **16.** 1232 m^3, ₹ 82,500

17. 5720 m^2 , 54208 m^3 **18.** 23408 cm^3 ; 2750 cm^2

19. 3.8 cm **20.** 4.2 cm 22. $1334\frac{2}{3}$ cm^3, 858 cm^2

23. 42 **24.** 565.2 cm^3 , 414.48 cm^2 **25.** 7 cm

6. **Find the curved surface area, the total surface area and the volume of a right circular cylinder whose radius r and height h are given by**

(i) $r = 7$ cm, $h = 15$ cm

(ii) $r = 3.5$ cm, $h = 12.5$ cm

(iii) $r = 14$ cm, $h = 35$ cm

(iv) $r = 7$ cm, $h = 25$ cm

7. The radius of a roller 1.4 m long, is 45 cm. Find the area it sweeps in 75 revolutions.

8. A garden roller of diameter 1 m is 2.1 m long. Find the area it covers in 100 revolutions.

9. A cylindrical metal pipe of thickness 1.4 cm and external diameter 56 cm is 14 m long. Find the volume of metal used in the construction of the pipe.

10. The volume of a right circular cylinder of height 24 cm is 924 cm^3. Find the area of the curved surface of the cylinder.

11. Find the depth of a well of radius 3.5 m if its capacity is equal to that of a rectangular tank of dimensions 25 m × 11 m × 7 m.

12. A well of diameter 3.5 m is dug 16 m deep. The earth taken out is spread evenly to form a rectangular platform of base 11 m × 7 m. Find the height of the platform.

13. **Find the volume, the curved surface area and total surface area of a cone, given that**

(i) radius of base = 3.5 m and height = 12 m

(ii) radius of base = 0.7 m and slant height 2.5 m

(iii) radius of base 21 cm and slant height 35 cm

(iv) height = 24 cm and slant height = 25 cm

(v) perimeter of base = 88 cm and height = 48 cm.

14. The curved surface area of a cone of slant height 50 cm is 2200 cm^2. Find the volume of the cone.

15. The volume of a cone of height 24 cm is 1232 cm^3. Find the slant height and total surface area of the cone.

Example 2 : How many spheres each of radius 1 cm will have their total surface area equal to that of a single sphere of radius 5 cm ?

Solution : Surface area of a sphere of radius 1 cm = 4π cm^2

Surface area of a sphere of radius 5 cm = $4\pi.5^2 = 100\pi$ cm^2

Required number of spheres = $\frac{100\pi}{4\pi} = 25$.

Example 3 : Assuming both the planets namely the Earth and the Saturn to be spheres, compare their surface areas and volumes if the diameter of the Saturn is 9 times that of the Earth.

Solution : Let r denote the radius of the Earth. The radius of the Saturn is then $9r$.

Surface area of the Earth = $4\pi r^2$

Surface area of the Saturn = $4\pi(9r)^2 = 81 \times 4\pi r^2$

Volume of the Earth = $\frac{4}{3}\pi r^3$

Volume of the Saturn = $\frac{4}{3}\pi(9r)^3 = 729 \times \frac{4}{3}\pi r^3$

Thus, the surface area of the Saturn is 81 times that of the Earth and its volume is 729 times that of the Earth.

Example 4 : A hemispherical bowl of internal diameter 60 cm is full of milk. Cylindrical bottles each of diameter 6 cm and height 25 cm are to be filled with milk from the bowl. How many bottles are necessary to empty the bowl ?

Solution : Volume of milk contained in the bowl = $\frac{2}{3}\pi \times 30^3$

$= 2\pi \times 10 \times 30 \times 30$ cm^3

Volume of one cylindrical bottle $= \pi \times 3^2 \times 25$

$= \pi \times 9 \times 25$ cm^3

Required number of bottles $= \frac{2\pi \times 10 \times 30 \times 30}{\pi \times 9 \times 25}$

$= 80.$

A section of the frustum by any plane containing its axis i.e., the line through the centres of the bases, is an isosceles trapezium (ABCD in Fig. 12.13). The length of any of the pair of non-parallel sides (AD and BC) is the slant height of the frustum.

Let h be the height, l the slant height and r_1, r_2 $(r_1 > r_2)$ the radii of the bases of a frustum of cone ABCD.

Then,

(i) volume of the frustum $= \frac{1}{3}\pi h\,(r_1^2 + r_1 r_2 + r_2^2)$

(ii) curved surface area of the frustum $= \pi l(r_1 + r_2)$

and (iii) total surface area of the frustum $= \pi(lr_1 + lr_2 + r_1^2 + r_2^2)$.

Example 1 : A conical vessel of height 24 cm and radius 10 cm is filled with water and then poured into a cylindrical flask of radius 5 cm. Find the height of water in the cylindrical flask.

Solution : Capacity of the conical vessel $= \frac{\pi}{3} \times 10^2 \times 24 = 800\pi$ cm^3.

Let the height of water in the flask be x cm.

Then volume of water in the flask $= \pi \times 5^2 \times x = 25\pi x$ cm^3.

$\therefore \quad 25\pi x = 800\pi$

$\therefore \quad x = \frac{800}{25} = 32$

The height of water in the cylindrical flask is 32 cm.

12.7 Surface Area and Volume of a Sphere

The formulae for finding the surface area S and volume V of a sphere of radius r are given by

(i) $S = 4\pi r^2$ (ii) $V = \frac{4}{3}\pi r^3$

These formulae will be proved in higher classes. However the formula for volume may be verified by way of an activity as follows :

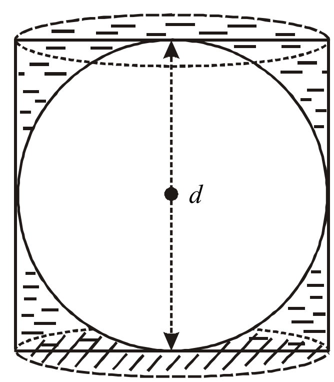


Fig. 12.14

Take a solid sphere of diameter, say d and take a cylindrical can of (inner) radius $\frac{d}{2}$ and height d. Fill the can with water to the brim. Immerse the sphere completely in water inside the can. Some quantity of water will spill out. Gently remove the sphere out of water. You will observe that the can is about one-third full with water.

If V be the vertex, O the centre of the base and A any point on the circular edge of the base of a right circular cone, then OV = h (say) is the height, OA = r is the base radius and AV = l is the slant height of the cone. Since OA lies in the base while OV is perpendicular to the base, therefore the ΔAOV is right angled at O so that

$$AV^2 = OA^2 + OV^2$$

or $$l^2 = r^2 + h^2$$

This is the relation between the slant height, the base radius and the height of a right circular cone.

In this unit, by a cone it shall be meant a right circular cone.

(a) Surface Area of a Right Circular Cone

Consider a hollow right circular cone of radius r and slant height l (Fig. 12.12). Cut it along a generator VA. On spreading out the cut out sheet on a plane surface, a sectorial region of radius l and arc length $2\pi r$ will be obtained. The area of this sectorial region is obviously $\pi l^2 \times \frac{2\pi r}{2\pi l}$ i.e. πrl (since the areas of sectors of the same circle are proportional to their arc lengths). Hence, the area of the curved surface of the cone = πrl.

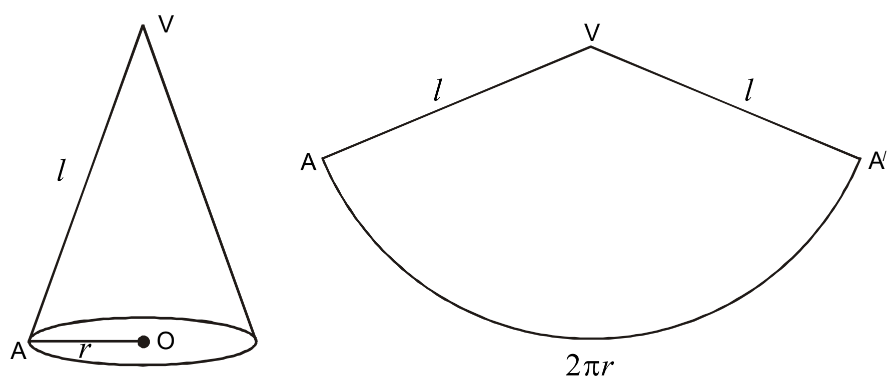


Fig. 12.12

Also, the total surface area of the cone

$$= \text{Area of the curved surface} + \text{Area of the base}$$

$$= \pi rl + \pi r^2$$

$$= \pi r(l + r)$$

Take a solid cylinder of height h and radius r. Take a jar of known capacity which is large enough to contain the solid cylinder inside it. Fill the jar with water upto the brim. Tie the cylinder by a fine thread and immerse it completely in water inside the jar. An amount of water of which the volume is the same as that of the cylinder will spill over. Carefully lift the cylinder out of the jar by means of the thread. Measure the volume of water left in the jar. The difference between the whole capacity of the jar and the volume of water left in the jar gives the volume of the cylinder. Thus the volume of the cylinder can be determined.

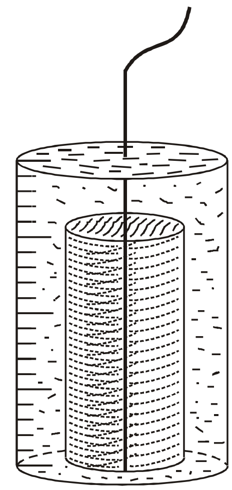


Fig. 12.10

Now calculate the value of $\pi r^2 h$. You will see that the value is almost the same as the volume of the cylinder as determined above. This verifies the formula $V = \pi r^2 h$ for the volume of a cylinder of radius r and height h.

Example 1 : Water is flowing at the rate of 10 km per hour through a pipe of internal diameter 14 cm into a rectangular tank, 55 m long and 40 m wide. By how much will the level of water in the tank rise in 5 hours?

Solution : Radius of the pipe = $\frac{14}{2}$ cm = $\frac{7}{100}$ m

Amount of water flowing through the pipe per hour

$$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 10000 \text{ m}^3 \qquad [\because 10 \text{ km} = 10000 \text{ m}]$$

$$= 154 \text{ m}^3$$

$\therefore$ Amount of water that flows into the tank in 5 hrs.

$$= 154 \times 5 \text{ m}^3$$

Surface area of water in the tank $= 55 \times 40 \text{ m}^2$

$\therefore$ Rise in water level in 5 hrs $= \frac{154 \times 5}{55 \times 40}$ m

$= \frac{7}{20}$ m $= 35$ cm.

Hence, in 5 hours the level of water in the tank will rise by 35 cm.

So, we can take

$$x = lb, \quad y = bh \text{ and } z = hl$$

The volume V of the cuboid is given by

$$V = lbh$$

$$\therefore \quad V^2 = l^2 b^2 h^2$$

$$= lb \times bh \times hl$$

$$= xyz$$

$$\therefore \quad V = \sqrt{xyz}$$

Example 3 : The dimensions of a rectangular field are 75 m by 50 m. A rectangular pool of length 35 m and breadth 15 m is dug inside the field and the earth dug out is spread evenly over the remaining portion of the field thereby raising the level by 42 cm. Find the depth of the pool.

Solution : Area of the field $= 75 \times 50 = 3750 \text{ m}^2$

Area covered by the pool $= 35 \times 15 = 525 \text{ m}^2$

Area of the remaining portion of the field $= 3750 - 525 = 3225 \text{ m}^2$

Volume of the earth dug out $= 3225 \times \frac{42}{100} \text{ m}^3$

$\therefore$ Depth of the pool $= 3225 \times \frac{42}{100} \div 525$

$= 2.58$ m

$= 2$ m 58 cm.

12.5 Right Circular Cylinder

You have studied about solid figures known as circular cylinders in Class VII. Gas cylinders, pipes, fish cans, rollers etc. are examples of circular cylinders. A solid circular cylinder has a curved surface called lateral surface, bounded by two identical circular plane faces. Either of the two circular plane faces is a base of the cylinder and the radius of a base is referred to as the radius of the cylinder. The line joining the centres of the two bases is the axis of the cylinder. If the axis is perpendicular to the bases, the cylinder is said to be a right circular cylinder. The distance between the two parallel bases is called the height or length of the right circular cylinder.

Cylinders considered in our discussion will be right circular cylinders only.

lengths customarily known as length, breadth and height (or thickness). The length, breadth and height of a cuboid are referred to as its dimensions and are usually denoted by the letters *l, b* and *h* respectively.

A point where three edges of a cuboid meet is called a vertex of the cuboid. A cuboid has eight vertices.

A cuboid having all its edges equal is called a *cube.* Each of the equal edges of a cube is called a side.

(a) Surface Area of a Cuboid

Consider the cuboid in Fig. 12.7. Here, the face ABCD is the base of the cuboid and its length AB and breadth AD are the length and breadth of the cuboid. The distance AA' between the base ABCD and the opposite face $A'B'C'D'$ is the height or thickness of the cuboid. Let *l, b* and *h* denote the length; breadth and height respectively for the cuboid. Then

$$AB = A'B' = D'C' = DC = l$$

$$AD = A'D' = B'C' = BC = b$$

$$AA' = BB' = CC' = DD' = h$$

Area of the face ABCD = area of the face $A'B'C'D'$ = $AB \times AD = lb$

Area of the face $ADD'A'$ = area of the face $BCC'B'$ = $AD \times AA' = bh$

Area of the face $AA'B'B$ = area of the face $DD'C'C$ = $AA' \times AB = hl$

$\therefore$ Total surface area of the cuboid = sum of areas of the six faces

$= 2(lb + bh + hl)$

Excluding the base and the top (opposite face to the box), the four other faces form the lateral surface of the cuboid. So the lateral surface area of the cuboid

$= 2(bh + hl)$

$= 2(l + b)h$

= perimeter of the base × height.

In case of a cube of side *a*, $l = b = h = a$ so that each of the six faces is a square of side *a*. Hence, for a cube of side *a*,

surface area $= 6a^2$ and lateral surface area $= 4a^2$.

So, area of $\Delta ABD = \sqrt{15(15-9)(15-13)(15-8)}$

$= \sqrt{15 \times 6 \times 2 \times 7}$

$= \sqrt{1260}$

$= 35.5 \text{ m}^2$ (approx.)

Also, area of $\Delta BCD = \frac{1}{2} \times BC \times CD$

$= \frac{1}{2} \times 12 \times 5$

$= 30 \text{ m}^2$

$\therefore$ area of the park $= 35.5 + 30$

$= 65.5 \text{ m}^2$ (approx.).

EXERCISE 12.2

1. Find the area of the quadrilateral ABCD in which

(i) AB = 5 cm, BC = 4.5 cm, CD = 3.5 cm, DA = 4 cm and AC = 6.5 cm

(ii) AB = 3 cm, BC = 5 cm, CD = 6 cm, DA = 6 cm and BD = 5 cm

(iii) AB = 3.5 cm, BC = 4.5 cm, CD = 6 cm, DA = 3 cm and BD = 5.5 cm

(iv) AB = 6 cm, BC = 4 cm, CD = 4 cm, DA = 5 cm and AC = 6 cm.

2. Find the area of the quadrilateral ABCD in which

(i) AB = 3 cm, BC = 4 cm, CD = 5 cm, DA = 4 cm and $\angle B = 90^0$

(ii) AB = 6 cm, BC = 4 cm, CD = 3 cm, DA = 5 cm and $\angle C = 90^0$

3. Each side of a rhombus shaped field is 30 m and its longer diagonal is 48 m. Find the area of the field.

4. A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m, and non-parallel sides are 14 m and 13 m. Find the area of the field.

[*Hint :* Draw CE || DA.]

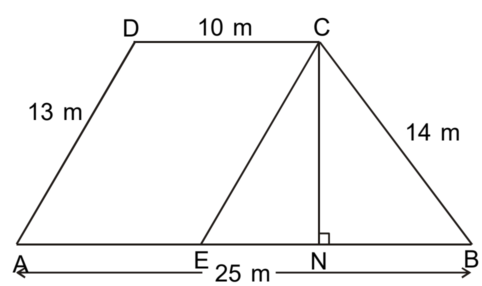


Fig. 12.6

ANSWERS

1.	(i) $8\sqrt{30}$ cm^2	**(ii)**	$\sqrt{6279}$ cm^2	**(iii)**	$2\sqrt{66}$ cm^2	**2.**	84 cm^2
3.	$2250\sqrt{3}$ cm^2	**4.**	$36\sqrt{3}$ cm^2	**5.**	$49\sqrt{3}$ cm^2	**6.**	60 cm^2
7.	$25\sqrt{5}$ cm^2	**8.**	$\frac{\sqrt{3}}{4}a^2$, $1225\sqrt{3}$ cm^2	**9.**	₹ 47,25,000	**10.**	12 cm^2

12.2 Application of Heron's Formula in Finding Area of Quadrilaterals

In the previous article you have learnt to find area of a triangle by Heron's formula. In this section we shall study about the calculation of area of a quadrilateral using Heron's formula. Let us take a quadrilateral ABCD. Suppose that we know all the sides and one diagonal, say AC, of the quadrilateral. We see that AC divides the quadrilateral into two triangles, viz. ABC and ACD (Fig. 12.3). We can find the areas of these two triangles by Heron's formula as we know the sides of these triangles. The sum of the areas of these two triangles gives the area of the quadrilateral ABCD.

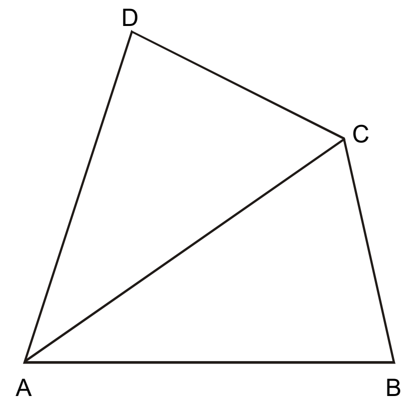


Fig. 12.3

Thus, we see that if the sides and one diagonal of a quadrilateral are known, then we can find its area by using Heron's formula.

Example 1 : Find the area of a quadrilateral ABCD in which AB = 3 cm, BC = 4 cm, CD = 4 cm, DA = 5 cm and AC = 5 cm.

Solution : The sides of the ΔABC are

AB = 3 cm,

BC = 4 cm and CA = 5 cm.

$\therefore \quad s = \frac{3+4+5}{2} = 6$ cm

Then by Heron's formula, we have

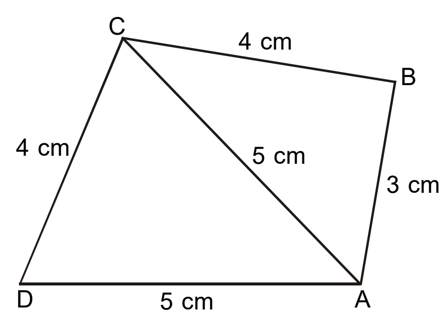


Fig. 12.4

$\therefore$ area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{35 \times 6 \times 14 \times 15}$

$= 210 \text{ cm}^2$.

Example 2 : Find the area of a triangle two sides of which are 18 cm and 10 cm and the perimeter is 42 cm.

Solution : Here, perimeter of the triangle = 42 cm

a = 18 cm and b = 10 cm

Therefore, third side c = 42 – (18 + 10) = 14 cm

Then we have $s = \frac{42}{2} = 21$ cm

$s - a = 21 - 18 = 3$ cm

$s - b = 21 - 10 = 11$ cm

$s - c = 21 - 14 = 7$ cm

$\therefore$ area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21 \times 3 \times 11 \times 7}$

$= 21\sqrt{11} \text{ cm}^2$.

Example 3 : The sides of a triangular plot are in the ratio 3 : 5 : 7 and its perimeter is 450 m. Find its area.

Solution : Let the sides of the triangular plot, in metres, be $3x$, $5x$ and $7x$.

Then by question, we have

$3x + 5x + 7x = 450$

i.e., $15x = 450$

$\therefore$ $x = 30$

So, the sides of the plot are 3 × 30 m, 5 × 30 m and 7 × 30 m i.e., 90 m, 150 m and 210 m.

Taking a = 90 m, b = 150 m and c = 210 m,

we have $s = \frac{450}{2} = 225$ m

$s - a = 225 - 90 = 135$ m

$s - b = 225 - 150 = 75$ m

$s - c = 225 - 210 = 15$ m

CHAPTER

12 MENSURATION

12.1 Heron's Formula

In the previous class you have learnt something about the calculation of the area of a triangle. You know that :

Area of a triangle = $\frac{1}{2}$ × base × altitude

Suppose that we know the three sides of a triangle. When the triangle is right angled, we can find the area of the triangle by directly applying the above formula taking on justifiable basis, the two sides containing the right angle as base and altitude. For example, if ABC is a triangle in which AB = 24 cm, BC = 7 cm and AC = 25 cm (Fig 12.1), then as $AB^2 + BC^2 = 24^2 + 7^2 = 25^2 = AC^2$, it is right-angled at B. On taking AB as the base, BC becomes the corresponding altitude and we have

$$\text{area of } \Delta ABC = \frac{1}{2} \times AB \times BC$$
$$= \frac{1}{2} \times 24 \times 7$$
$$= 84 \text{ cm}^2$$

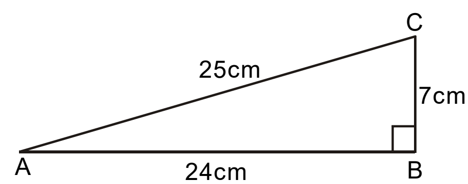


Fig. 12.1

We can also take BC as the base and AB as the corresponding altitude. By doing so we obtain the same area.

Again, suppose that the triangle is equilateral. In this case also we can find the area of the triangle by the above formula. For example, let us take an equilateral ΔDEF with side 12 cm (Fig12.2). To find its area we need its height. If DN be the perpendicular from D upon EF, then N will be the mid-point of EF. So, EN = $\frac{1}{2}$EF = 6 cm. As DEN is a right triangle, using Pythagoras Theorem, we have

ANSWER

1. $66^g\ 66'\ 66\frac{2}{3}''$, $60^g\ 55'\ 55\frac{5}{9}''$, $120^g\ 34'\ 16\frac{2}{3}''$

2. $\frac{\pi}{4}^{c}$, $\frac{\pi}{3}^{c}$, $\frac{2\pi}{3}^{c}$, $\frac{27\pi}{80}^{c}$

3. $\frac{360^0}{\pi}$, $\frac{180^0}{\pi}$, 45^0, 135^0, 120^0

4. $7\frac{1}{2}^0$, $8\frac{1}{3}^g$, $\frac{\pi}{24}^c$

5. $\frac{2\pi}{9}$, $\frac{\pi}{3}$, $\frac{4}{9}\pi$

6. 162^0, $\frac{9\pi}{10}^c$

7. 11 m.

8. 8:5

9. 44 cm.

****

3. The data of occurrence of a particular disease in different age groups of a locality is given below :

Age group	*No. of people*
15 – 25	25
26 – 35	30
36 – 40	27
41 – 50	18
51 – 55	47
56 – 60	20
61 – 75	6
76 – 90	2

Draw the histogram of the above data and also draw the frequency polygon of the same.

4. Draw the frequency polygon of the following data.

Marks	*No. of students*
0 – 10	6
10 – 20	14
20 – 30	25
30 – 40	35
40 – 50	46
50 – 60	51
60 – 70	32
70 – 80	24
80 – 90	7
90 – 100	2

5. Draw the ogive of the data (from Q. no. 1 to 4).

****

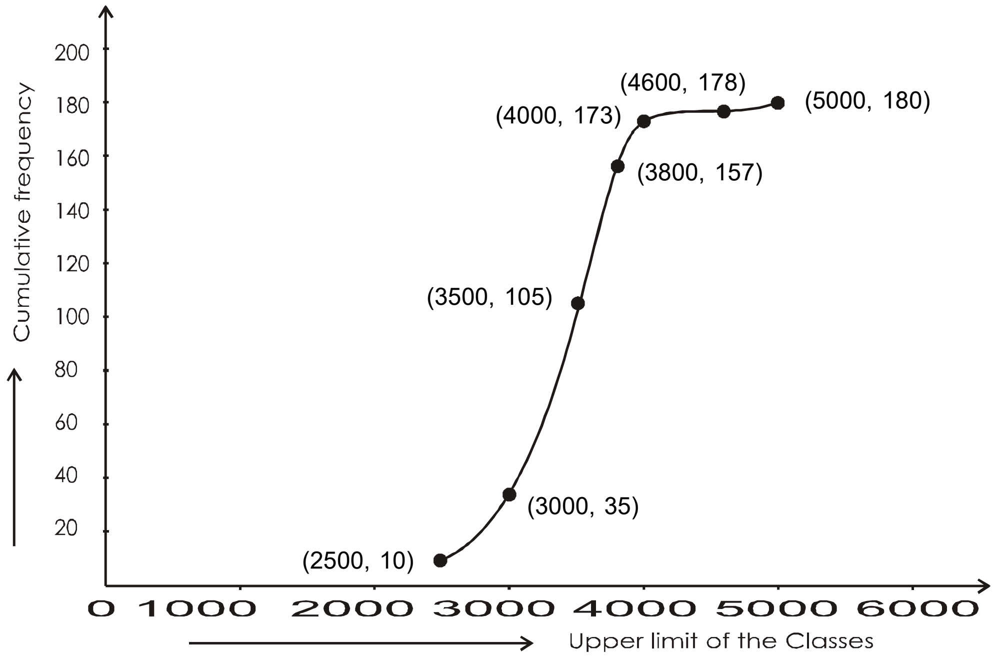


Fig. 14.5

A free-hand curve is drawn through these points and we get smooth rising curve which is the ogive of the data.

Note : We can extend the ogive both ways by assuming that there are classes of zero frequencies on both ways lower than the last class and higher than the upper class.

Solution :

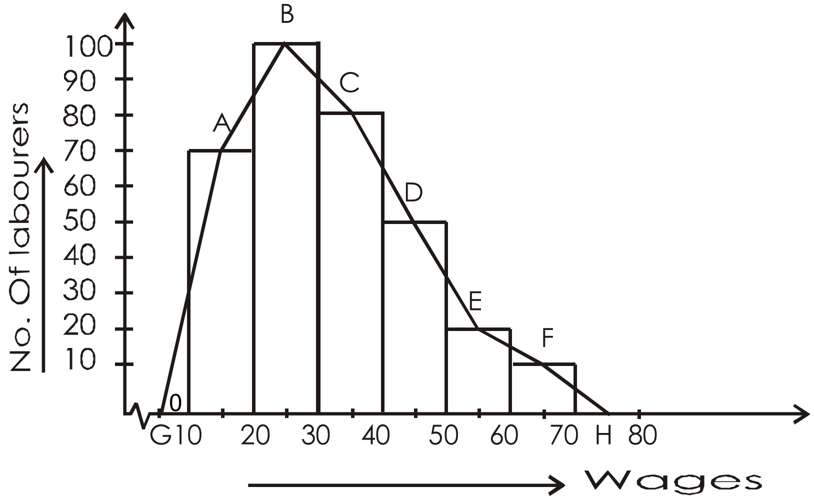


Fig. 14.4

In this case, we first draw the histogram.

The mid-points of the various ractangles are A, B, C, D, E, F. Two assumed classes 0 to 10 and 70 to 80 with frequency 0 are considered. Their corresponding rectangles will lie on the X-axis.

The mid-points of these rectangles are G and H. Joining A to G and F to H we get the closed frequency polygon.

The relation between the histogram and the frequency polygon is that areas enclosed by the histogram and the closed frequency polygon must be the same.

14.7 Cumulative Frequency Curve or the Ogive

For a classified data the free-hand smooth curve obtained by joining the points which are plotted by taking the upper limits of the various classes as the abscissae and the cumulative frequencies of the corresponding classes as the ordinates is called the *cumulative frequency curve or ogive* of the data.

An ogive is a never descending curve.

We consider the following example.

14.6 Frequency Polygon

The frequency polygon of a classified data is obtained first by plotting the points whose abscissae are the class marks of the various classes and the ordinates are the frequencies of the corresponding classes and then joining them by line segments.

However, if we are to draw both histogram and the frequency polygon we get the latter by joining the mid-points of the tops of all rectangles by line segments.

Example 3: Draw the frequency polygon of the data given below :

Classes (Marks)	*Frequency (No. of students)*
0 – 10	5
10 – 20	12
20 – 30	8
30 – 40	7
40 – 50	17
50 – 60	10
60 – 70	8
70 – 80	6
80 – 90	5
90 –100	1

Solution : For drawing the frequency-polygon we rewrite the data as follows :

Class Marks	*Frequency*
5	5
15	12
25	8
35	7
45	17
55	10
65	8
75	6
85	5
95	1

Thus, for equal width the height of the rectangle will be proportional to the frequency of the class.

To illustrate the idea consider the following example.

Example 1 : The following is the data of marks secured by 100 students in an examination of 50 marks.

Class :	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50
No. of students :	6	10	14	24	20	12	7	5	2

Represent the data by a histogram.

Solution : Here the classes are of equal width.

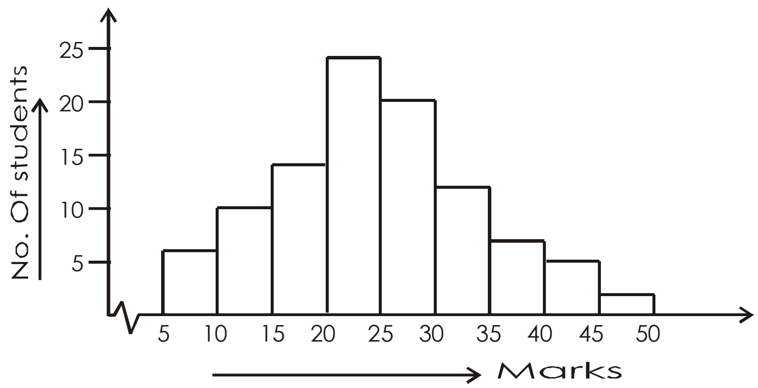


Fig. 14.1

There is no class below 5. So we make a kink along the X-axis near the origin indicating that the graph is drawn to scale from 5 and not from 0. Similar method can be used to case of the Y-axis if situation demands.

Example 2 : The following is the data of income groups and income tax payers of a certain region.

Monthly income (in ₹) group	*No. of tax payers*
1501 – 2000	150
2001 – 2500	200
2501 – 3500	300
3501 – 4400	100
4401 – 5303	55

1000 and above	–	560
1500 and above	–	320
2000 and above	–	150
2500 and above	–	75
3000 and above	–	50
3500 and above	–	24
4000 and above	–	12
4500 and above	–	6

From the above data construct a continuous frequency table and also find the cumulative frequency of each class. Further, find the number of families whose weekly income is at the most ₹ 1000.

ANSWERS

2. Secondary

3. Maximum number of students get B^+

Minimum number of students get D

Median grade is B

Mean is not determinable

4. 6, 6; yes. It is 6

5. Mean = 33.45

Median = 32

Mode = 32

7. 267,361

8. 11490

EXERCISE 14.1

1. State a common experience in which you collect data to get an information.

2. Suppose you visited a meteorological centre and collected the record of the maximum temperature on all days of the month July 2007. Is this a primary or a secondary data ?

3. **The performance given in grades of 45 students of a class are as follows (A^+, A, B^+, B, C^+, C and D are the 7 grades).**

A, B, C^+, C, C, D, A^+, B, B^+, B, B^+, C, C^+, D, A^+

B^+, A, A^+, C, C^+, A, B, B^+, C, D, A^+, B, B^+, B^+, C

B, B^+, B, B^+, B^+, A^+, C, C, C^+, A, A^+, B, B^+, C, A.

Construct a frequency distribution to represent the data. Determine the grades which are obtained by the maximum and minimum number of students. Find also the median grade. Which one of the measures of central tendency is not determinable for this non-numerical data ?

4. What are the mean and median of the first eleven natural numbers ? Can you estimate the mode ? If so what is it ?

5. **The following are the weight in kg of 20 pupils in a class**

30, 37, 45, 32, 26, 40, 26, 31, 30, 35,

28, 32, 35, 41, 28, 27, 42, 36, 32, 36.

Using tally marks form an ungrouped frequency table thereby showing the frequency and cumulative frequency columns. Also find the three measures of central tendency of the data.

6. Taking 2 as the width of each class construct a continuous grouped frequency table for the sample in article 7.2. Also show the columns of mid-values of the classes and the cumulative frequencies.

7. **For a certain locality, the data of monthly household consumption of electricity measured in units of energy is given below :**

45	–	I	–	1	65	–	I	–	1	85	–	I	–	1
46	–	II	–	2	66	–	II	–	2	86	–	I	–	1
47	–	II	–	2	67	–	II	–	2	87	–	I	–	1
48	–	II	–	2	68	–	I	–	1	88	–	I	–	1
49	–	II	–	2	69	–	II	–	2	89	–	I	–	1
50	–	I	–	1	70	–	I	–	1	90	–	I	–	1
51	–	I	–	1	71	–	II	–	2	91	–	I	–	1
52	–	I	–	1	72	–	I	–	1	92	–	I	–	1
53	–	I	–	1	73	–	I	–	1	93	–	I	–	1
54	–	I	–	1	74	–	I	–	1	94	–	I	–	1
55	–	I	–	1	75	–	I	–	1	95	–	I	–	1

Now, the ungrouped frequency table is :

Marks (x_i)	Frequencies (f_i)	Cumulative Freq.	Marks (x_i)	Frequencies (f_i)	Cumulative Freq.
36	2	2	62	1	24
37	2	4	64	1	25
38	2	6	65	1	26
45	1	7	68	1	27
47	2	9	70	1	28
48	2	11	71	2	30
49	2	13	72	1	31
50	1	14	73	1	32
51	1	15	74	1	33
52	1	16	75	1	34
53	1	17	76	2	36
55	1	18	83	1	37
57	1	19	84	1	38
58	1	20	85	1	39
60	1	21	95	1	40
61	2	23			

14.4 : Mid – Values (Class Marks) :

Consider the following grouped frequency distribution :

Classes :	0 – 6	6 – 12	12 – 18	18 – 24	24 – 30
Frequency :	4	7	5	3	2

The interpretation of this distribution is that in the first class there are 4 observations each of whose values in either 0 or more but less than 6 ; in the second class there are 7 observations each of whose values is either 6 or more but less than 12 and so on. In the last class there are 2 observations each of whose values is either 24 or more but less than 30.

We make the following assumption. Each of the 4 observations in the class 0 – 6 is considered to have the value $\frac{\text{lower limit of the class + upper limit of the class}}{2}$ i.e., $\frac{0+6}{2}$ i.e., 3.

In other words each of the 4 observations of the class is taken to assume the value 3. This 3 is the mid value or class mark of the class 0 - 6.

Thus, class mark or mid value of the class = $\frac{\text{lower limit + upper limit}}{2}$.

The mid values are symbolically denoted by $x_1, x_2, ..., x_n$ or x_i, $i = 1, 2,..., n$.

The corresponding frequencies are denoted by $f_1, f_2, ..., f_n$ or f_i $(i = 1, 2, ...,n)$.

The sum of all the frequencies i.e. $f_1 + f_2 + ... + f_n = \sum_{i=1}^{n} f_i$ = N is called the size of the sample or population as the case may be.

Thus, the modified grouped frequency distribution is as follows :

Classes	*Mid-Values*	*Frequencies*	*Cumulative Frequencies*
0 – 6	3	4	4
6 – 12	9	7	11
12 – 18	15	5	16
18 – 24	21	3	19
24 – 30	27	2	21

N = 21

CLASSES	TALLY MARKS	FREQUENCY	CUMULATIVE FREQ.
15 - 20	~~IIII~~ ~~IIII~~ I	11	11
20 - 25	~~IIII~~ I	6	17
25 - 30	~~IIII~~ ~~IIII~~ ~~IIII~~	15	32
30 - 35	~~IIII~~ ~~IIII~~ III	13	45
35 - 40		0	45
40 - 45	II	2	47
45 - 50	II	2	49
50 - 55	I	1	50

Total = 50

Here, lower limit of the first class (15 – 20) is 15. The width of the class = 20 – 15 = 5 and the above data in the tabular form is a grouped data. In constructing this table on observation like 30 is not costumarily counted in the class (25 – 30), it is counted in the class (30 –35).

In making this grouped frequency distribution from the raw data the following questions arise.

(i) From which group should we start construction of the classes ?

(ii) What should be the width of each class ?

(iii) Should the classes be of equal width ?

(iv) Can we take discontinuous classes like 15 – 19, 20 – 24, 25 – 29 etc.,

(v) How to convert a discontinuous type to a continuous type of classification.

Note : A better classification could have been 15 – 17, 17 – 19, 19 – 21,....etc. save for its lengthyness as the width of each class is 2 $\left(<\frac{1}{4}a\right)$.

We give the answers to the above questions as follows :

(i) We identify the smallest observation or observed value and the lowest class can be formed by taking the smallest observation as the lower limit and adding the width of the class in case of continuous type of grouping or classification we get the upper limit of the lowest class. In case of discontinuous type, to the smallest observation we add one less than the width of the class to get the upper limit of the lowest class.

Thus, A.M $= \bar{x} = \frac{1}{50}$ [15 × 1 + 17 × 1 + 18 × 5 + 19 × 4 + 20 × 2 + 23 × 2
+ 25 × 1 + 26 × 6 + 27 × 2 + 28 × 3 + 29 × 3 + 30 × 2
+ 31 × 2 + 32 × 4 + 33 × 1 + 34 × 4 + 40 × 1 + 42 × 1
+ 45 × 1 + 46 × 1 + 52 × 1]

$= \frac{1}{50}$ [1378]

= 27.56.

(iv) To find the measure (weight) such that half of the sample i.e, 25 are less than (or equal to) and the other half i.e, the other 25 are more than (or equal to) it, we imagine that all the observations are put in ascending order of magnitude and find out the 25th and 26th. We can easily find them from the cumulative frequency columm. Both have the same value i.e, 27 gm. This is the median of the sample.

When the size N of a sample i.e, the total number of observations of the data is odd, say $N = 2n + 1$ then arranging the observations in ascending or descending order of magnitude in a row or a columm, the $(n + 1)^{th}$ observation either from the beginning or the end is the median or the median value of the observations.

However, if N is even say $N = 2n$ then, the two observations namely, the $\left(\frac{2n-2}{2}+1\right)^{th}$ and the $\left(\frac{2n-2}{2}+2\right)^{th}$ i.e, the n^{th} and the $(n + 1)^{th}$ observations in row wise or columm wise aranged position are the two median values. The average of these two is the median.

Thus, we observe that the three entities viz, the mean, median and mode of a sample are closely located.

These three are the measures of central tendency.

We can examine the near validity of Karl Pearson's empirical formula. viz.

Mean – Mode = 3(Mean – Median)

In our case, Mean – Mode = (27.56 – 26)
= 1.56

and 3(Mean – Median) = 3(27.56 – 27)
= 3 × 0.56
= 1.68.

Weight		*Tally mark*	*Weight*		*Tally mark*	*Weight*		*Tally mark*
15	–	I	28	–	III	41	–	
16	–		29	–	III	42	–	I
17	–	I	30	–	II	43	–	
18	–	~~IIII~~	31	–	II	44	–	
19	–	IIII	32	–	IIII	45	–	I
20	–	II	33	–	I	46	–	I
21	–		34	–	IIII	47	–	
22	–	II	35	–		48	–	
23	–	II	36	–		49	–	
24	–		37	–		50	–	
25	–	I	38	–		51	–	
26	–	~~IIII~~ I	39	–		52	–	I
27	–	II	40	–	I			

The number of observations corresponding to a weight is the frequency of that weight. The ungrouped frequency distribution is given below :

A columm called cumulative frequency is also constructed to show the total frequency upto and below that weight.

Weight in grams	*freq.*	*cu. freq.*	*Wt. in grams*	*freq.*	*cu. freq.*
15	1	1	29	3	32
17	1	2	30	2	34
18	5	7	31	2	36
19	4	11	32	4	40
20	2	13	33	1	41
22	2	15	34	4	45

CHAPTER

14 STATISTICS

14.1 Introduction

Although statistics as a subject is defined in a variety of ways when looked upon from different standpoints, it deals mainly with collection, organisation, analysis and interpretation of data. In earlier classes we have seen that the data that is yet to be processed is a raw data while a processed data is a grouped data.

A statistical data may be numerical or otherwise. We shall be more concerned with numerical data. From a raw numerical data we get only a few information. However, after due process a number of important information can be extracted from the same data. An investigator may collect data in two ways. If he/she personally collects the data with a specific purpose then, that is a primary data. On the other hand an investigator can get a data indirectly from some other source. Such a data is a secondary data.

For example a person who is interested to get the information of the maximum and minimum temperatures of his/her locality on the different days of a week may himself or herself take the thermometer readings and collect the data of maximum and minimum temperatures during the week. This is a primary data. On the other hand he/she can get the same information from the weather bulletin of the AIR or DOORDARSHAN or any other source. The data so collected is a secondary data.

An investigator collecting a secondary data must keep in mind two relevent aspects. One is the objectives of the primary imustigator of the data which he/she is taking as a secondary data. It must be ensured that the objectives of the investigators taking the same data as primary by one and secondary by the other must be compatible. The other aspect is the reliability of the secondary data.

One salient feature of a statistical study is that such a study gives less importance to individual values. It gives stress to the representative nature of certain parameters of the sample or the population. For example, the study of the income of a country, rather than the income of the richest person of the country helps to a large extent in characterising the economic status of the people of the country as a whole.

ANSWER

1. 0.016 , 51 – 60

2. X , Probability of getting A^+ > Probability of getting C^+

3. 0.35

4. Yes.

****

In fact,

$$x_1 = \frac{x'_0 + x'_1}{2}$$

$$x_2 = \frac{x'_1 + x'_2}{2} \quad \text{etc.,}$$

$$x_i = \frac{x'_{i-1} + x'_i}{2}; \qquad i = 1, 2, 3, ..., n.$$

If we look upon the above data as an ungrouped one with observations $x_1, x_2, ..., x_n$ having respective frequencies $f_1, f_2, ..., f_n$, then, the A.M. $\bar{x}$ is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ...+f_nx_n}{f_1 + f_2 + ...+ f_n} \left(= \frac{1}{N}\sum_{i=1}^{n} f_i x_i \right)$$

$$\Rightarrow \quad \bar{x} = \frac{f_1}{N}x_1 + \frac{f_2}{N}x_2 + ...+ \frac{f_n}{N}x_n \qquad \text{—— (i)}$$

Now, the probability p_i that any observation chosen at random may assume the value x_i is given by

$$p_i = \frac{f_1}{N}$$

Thus, $\bar{x} = p_1x_1 + p_2x_2 + ... + p_nx_n$

$$\Rightarrow \quad \bar{x} = \sum_{i=1}^{n} p_i x_i \qquad \text{... (ii)}$$

Equation (ii) gives the A.M. in terms of probability.

The expression on the right side of (ii) is known as the expected value or expectation of the variate ✷ x assuming the values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$.

We denote it by E(x).

Thus, E(x) = $\bar{x}$, the A.M.

✷ A variate or a stochastic variable x is a variable assuming the values $x_1, x_2, ..., x_n$ with a specitive probabilities $p_1, p_2, ..., p_n$.

Example 3 : The following is the grouped frequency table of a number of workers belonging to different weekly income groups.

Weekly Income in Rupees	*Number of Workers*
130 – 150	7
150 – 170	15
170 – 190	30
190 – 210	35
210 – 230	18
230 – 250	10
250 – 270	5

If a worker is chosen at random find the probability that his weekly income is in the group ₹ 170 – ₹ 190. Also find the probability that he belongs to the group whose weekly income is ₹ 190 or above. If the events of a worker chosen at random to belong separately to the above two groups are respectively denoted by E_1 and E_2, find $P(E_1) + P(E_2)$, and give reason why their sum is not 1. How can you modify event E_1 so that $P(E_1) + P(E_2) = 1$?

Solution : The frequency table is reproduced with the cumulative frequency table.

Income Group (Classes)	*No. of Workers (Frequency)*	*Cumulative Frequency*
130 – 150	7	7
150 – 170	15	22
170 – 190	30	52
190 – 210	35	87
210 – 230	18	105
230 – 250	10	115
250 – 270	5	120
	Total = 120	

Out of the 120 workers there are 30 in the income group of ₹ 170 – ₹ 190. Hence, the probability of the event E_1 in which the worker chosen at random may belong to this group is given by

$$P(E_1) = \frac{30}{120} = \frac{1}{4}.$$

If a student is chosen at random from the class, find the probability that he is one of the students who got the minimum marks. Also, find the probability that a student chosen at random from the class gets 75% or above.

Solution : We construct the frequency and cumulative frequency table as follows :

Marks	*Frequencies*	*Cumulative Frequencies*
12	4	4
15	6	10
17	7	17
18	7	24
20	6	30
22	8	38
24	7	45
25	3	48
29	2	50

If E_1 be the event in which the chosen student is one of the 4 who got minimum marks i.e. 12.

$$\text{Then, } P(E_1) = \frac{\text{No. of students who get 12 marks}}{\text{Total number of students}}$$

$$= \frac{4}{50}$$

$$= \frac{2}{25}$$

$$\text{Now, } 75\% \text{ of } 30 = \frac{75}{100} \times 30 = 22.5$$

From the cumulative frequency table we see that out of the 50 students 38 of them get below 22.5 marks and (50 – 38) i.e. 12 of them get above 22.5 .

Thus, if E_2 be the event in which a student chosen at random gets 75% or above.

$$\text{Then, } P(E_2) = \frac{\text{No. of students who get above 22.5 marks}}{\text{Total number of students}}$$

$$= \frac{12}{50}$$

$$= \frac{6}{25}.$$

We shall not solve the problem. But the theoritical answer is $\frac{2\delta}{\pi d}$ which involves the important irrational number π being the ratio of the circumference of any circle to its diameter. By taking different pairs of values of δ and d and also by throwing the needle a large number of times the empirical value of the probability has been determined experimentally. Equating the two probabilities the value of π has been estimated. It has been found that the value so determined agrees to a very fair degree to the known value of π.

After A.N. Kolmogorov enunciated his set theoritic definition of probability in 1933, this branch of Mathematics has become a confluence of the two branches viz. Measure theory and Probability theory with a wide range of applications in scientific and industrial fields.

15.3 Terms Related to Probability

Probability is defined in three apparently different but reconcilable ways. One of the definitions is the empirical or statistical definition. In lower classes we considered this definition and solved problems. Here also, we shall use the same definition but from another point of view. However, some terms associated with probability will be discussed briefly in some way of recapitulation of what we discussed in lower classes.

Trial :

In statistics a trial is an experiment whose outcomes can be labelled either a success or a failure. For example, in the tossing of a coin if getting a head is taken as a success and that of a tail a failure, then, the outcomes of a toss is either a success or a failure. Thus, tossing a coin is a trial.

Event :

An event is an outcome or a collection of outcomes of a trial. E.g., In tossing a coin once, getting a head is an event.

Randomness :

A trial is said to be random if there is no pre-arrangement to get a particular outcome. For example, while drawing a lottery the selection of a winner is a random choice.

PROOF IN MATHEMATICS

APPENDIX 1

Introduction

In our daily social life there are some intuitive notions which are taken to be true or false or uncertain. Whenever we assert something to be true or false it is not unexpected if somebody demands the proof of the assertion.

For example, when a person claiming to be a voter enters a polling booth to cast his/her vote, the polling officials demand documents to prove the identity of the voter. He/she has to produce his/her identity card or some other document or at least there must be some respectable person to prove the genuineness of the person in support of the claim.

Thus, to prove some assertion a reference is always necessary.

A mathematical proof, in a similar way depends on some accepted premise. The premise may be an axiom or postulate or some already accepted principles. Against the backdrop of the premise arguments are built up using logic which is the science of reasoning to prove or disprove the assertion. Sir Isaac Newton, postulated that light travels along straight lines. Basing on this postulate many phenomena on light are explained satisfactorily.

We shall discuss a little on the types of logic that we use to prove or diprove a certain assertion.

Logic and Reasoning

In logic two types of approach of reasoning are used. One is inductive and the other is deductive. In the inductive approach it is from individual to general and in the deductive it is the other way round.

***For example*:** Last Sunday night was cloudy and it was humid and hot. Yesternight was also cloudy and it was humid and hot. To night it is cloudy and it is humid and hot. Thus cloudy night are humid and hot. This type of reasoning is inductive. But such conclusion drawn by induction may suffer from pit falls.

***For example*:** Lal is from Rajasthan and he is a vegetarian. Jadu is from Rajasthan, he is a vegetarian. Rani is also from Rajasthan and she is also a vegetarian.

Therefore, all persons from Rajasthan are vegetarians. This conclusion may not be correct. Some people from Rajasthan may be non vegetarians.

Let us also see the pattern :

All are strings of 1(one) $\begin{bmatrix} 1^2 = 1 \\ 11^2 = 121 \\ 111^2 = 12321 \\ 1111^2 = 1234321 \\ 11111^2 = 123454321 \\ ----------- \end{bmatrix}$ All are palindrome numbers.

An observation of the above pattern makes us so tempting to say that this pattern of forming palindrome numbers by squaring repunit numbers which are formed by strings of ones (1) holds for all strings of ones.

But $1111111111^2 = 1234567900987654321$

(ten ones) which is not a palindrome number.

Again consider the Fermat numbers F_n given by

$F_n = 2^{2^n} + 1$; $n = 0,1,2,3, \ldots$

Now, $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, ...

These are all prime numbers. Fermat himself thought that F_n might generate prime numbers. But, Euler in 1732, more than half a century after the death of Fermat proved that

$F_5 = 4294967297$

$= 641 \times 6700417$ showing that F_5 is not a prime number.

Thus, inductive logic has a probable weak point which may lead to a pitfall.

However, in employing inductive method to prove a mathematical proposition due care is taken to overcome such a weak point. This you will discuss in the principles of Mathematical Induction.

The other approach used in reasoning where there are no such weak points is the deductive way. We shall discuss a few examples.

***Example* :**

All metals are conductors of heat and electricity. Iron is a metal. Iron is a conductor of heat and electricity.

Here the approach is from general to indivisual. Again let us examine the following :

The present age of a man is three times that of his son. When the son's age is double of his present age the father's age will be only double of the son's age.

To examine the truth of the statement let us think that the son's present age is 10 years. Then the fathers present age is 30 years. When the son's age is double of this present age, it is 20 years which is 10 years hence. By that time the father's age is (30 + 10) is 40 years. Thus, the statement is true.

Again if we take the son's present age as 15 years, then the father's present age is 45 years. When the son becomes 30 years old the father becomes (45 + 15) is 60 yrs. old. Here also the statement is true.

But these two verifications are not sufficient to say that the statement is true for all cases.

However, if we take x yrs. as the present age of the son then, the father's present age is $3x$ years.

The son's age will be double that of his present age after x years. By that time the father's age is $(3x + x)$ is $4x$ years.

Now,
$$\begin{aligned} 4x &= 2 \times 2x \\ &= 2 \times (x + x) \\ &= 2 \times \text{age of the son after } x \text{ years} \end{aligned}$$

But $4x = 2 \times 2x$ being an identity, holds for all values of x.

Thus, the statement is true in all cases.

The first approach of taking different present ages of the son in verifying the truth is indivisual and inductive while the second is general and deductive. The second method is a mathematically valid proof while the first is not.

Mathematical Logic

In Mathematical Logic a statement is one which is either true or false. A statement which is true to somebody and false to some other, an uncertain statement, an interrogation or an exclamation, a wish etc. are not considered as logical statements. Thus statement like, "It is a nice song", "It may rain today", "Who are you ?" "What a nice goal !" "Kindly help me" etc. are not logical statements.

Statements like "Imphal is the Capital of Manipur". "2 + 7 = 10" are logical statement in which the first is true while the second is false. They are also said to be mathematically valid.

Concept of Mathematical Proof.

As stated earlier, a mathematical proof is a set of logical arguments based on axioms or postulates or accepted principles.

We shall consider some terms first.

An axiom

An axiom is a self evident statement. The truth of an axiom is such that everybody is ready to accept it without a proof.

For example, one of Euclid's axioms

"Things which are equal to the same thing are equal to each other".

Symbolically, If $x = y$ and $x = z$

Then, $y = z$ because y and z are equal to the same quantity x.

is taken or accepted without a proof.

A Postulate

A postulate is an assumption which is taken to be true without a proof. Euclid classified axioms related to geometry as postulates. For example one of Euclid's postulates, "A straight line can be drawn from one point to another point" is accepted as it states. But now a days mathematically, axioms and postulates are considered the same.

The concept of postulates is also prevelent in other branches of science. For example Newton's postulate on the linear propagation of light is well known and accepted in Geometrical optics.

Mathematical Theorems

A mathematical theorem is a proposition whose truth is established by giving necessary arguments based on an axiom or axioms or some accepted principles.

For instance

Using the following two Euclid's axioms

viz (i) Things equal to the same thing are equal,

and (ii) If equals are subtracted from equals the differences are the same.

We can establish the following Euclid's theorem.

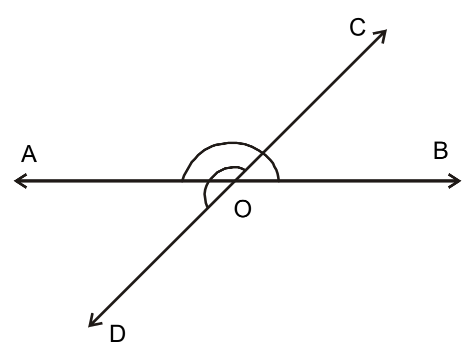
viz, If two straight lines intersect each other then, the vertically opposite angles are equal.

The truth of this proposition can be established by using the above two axioms.

Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be two straight lines intersecting at the point O. We are to prove that the vertically opposite angles $\angle$ AOD and $\angle$ BOC are equal.

Similarly,

$\angle$AOC and $\angle$BOD are equal.



Now, $\angle$AOB = A straight angle = two right angles and $\angle$COD = A straight angle = two right angles.

$\therefore$ By axiom (i) $\angle AOB = \angle COD$

$\Rightarrow \angle AOC + \angle AOD$

$= \angle AOC + \angle BOC$

Subtracting the same angle AOC from both the sides by axiom (ii) $\angle AOD = \angle BOC$.

In a similar way we can prove that

$$\angle AOC = \angle BOD.$$

Thus, from self evident axioms we can arrive at not so apparent and more complicated results some of which are so important in the study of the subject that they are taken as theorems or principles.

Some of Euclid's theorems whose proofs were also given by himself are now taken as postulates.

For example, the SAS postulate was earlier taken as a theorem.

Validity of a Statement

The correctness of a mathematical argument is dependent on the ambit of the definition and validity of the connotations of the terms involved therein.

For example, when we say "The equation $x^2 - 2 = 0$ has no solution", the statement is correct if the solutions are to be rational numbers.

However, if the roots are real numbers the statement is wrong.

Verification, General Proof and Counter Example

A set of examples of the validity of a proposition verified individually for specific cases does not constitute a proof of the proposition.

On the other hand a single instance of violation of the same proposition negates the validity of the proposition. Such an example is called a counter example.

Let us see the following examples.

Example 1 : Any square number when divided by 3 leaves either 0 or 1 as remainder.

A few square numbers are

1, 4, 9, 16, 25, 36, 49, ...

Each one when divided by 3 leaves 1 or 0 as the remainder.

Because

$$1 = 0 \times 3 + 1$$
$$4 = 1 \times 3 + 1$$
$$9 = 3 \times 3 + 0$$
$$16 = 5 \times 3 + 1$$
$$25 = 8 \times 3 + 1$$
$$36 = 12 \times 3 + 0$$
$$49 = 16 \times 3 + 1 \text{ etc.}$$

But these are verifications and they do not constitute a proof of the proposition.

On the other hand let us proceed in the following way :

Any whole number when divided by 3 will leave one of the numbers 0, 1, 2 as the remainder.

Thus, any whole number is in one of the three forms

viz, $3n$, $3n + 1$ and $3n + 2$ where, $n = 0, 1, 2, 3, ...$

Hence, any square number is in one of the three forms namely,

$(3n)^2$, $(3n + 1)^2$ and $(3n + 2)^2$.

When the square number is of the form $(3n)^2$ i.e. $9n^2$, on dividing by 3 the remainder is 0

as $9n^2 = 3 \times 3n^2$

When the square number is of the form

$$(3n + 1)^2 = 9n^2 + 6n + 1$$
$$= 3(3n^2 + 2n) + 1,$$ the remainder is 1 on dividing by 3.

Again, when the square number is of the form

$$(3n + 2)^2 = 9n^2 + 12n + 4$$
$$= 9n^2 + 12n + 3 + 1$$
$$= 3(3n^2 + 4n + 1) + 1$$

the remainder is 1 on dividing by 3.

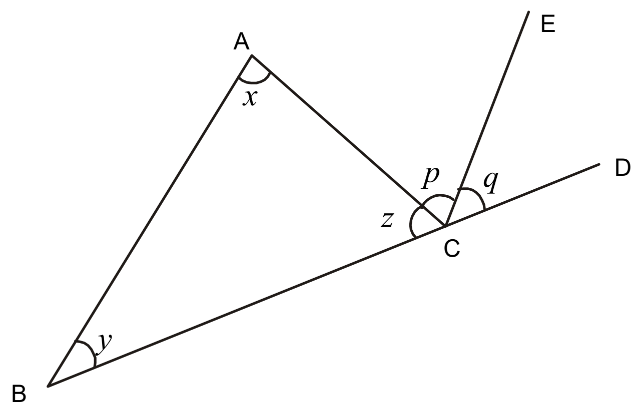
Thus in all the cases, the remainder is either 0 or 1.

This is a general proof.

Example 2 : The sum of the three angles of a triangle is 180^0.

We can draw any number of triangles and by measuring the three angles in each case and adding the three angles we can show that the sum is always 180^0. But these verifications do not constitute a proof.

On the other hand, let ABC be any arbitrary triangle.



Produce the side BC to D and draw the line CE parallel to BA. Then, by the properties of parallel lines and their transversals we see that $\angle x = \angle p$

$$\angle y = \angle q$$

So that $\angle x + \angle y + \angle z = \angle p + \angle q + \angle z$

$= 180^0$

Thus, $\angle A + \angle B + \angle C = 180^0$.

Since, the result holds for any aribtrary triangle the proposition is true for all triangles.

This is one of the basic theorems related to the angles of a triangle.

Example 3 : Any quadratic expression $ax^2 + bx + c$ with integral co-efficients can be factorized into linear factors with integral co-efficients.

We consider the expressions

$$x^2 + 7x + 12, \quad 2x^2 + x - 1 \text{ and } 6x^2 - 11x + 3$$

We observe that

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

$$2x^2 + x - 1 = (2x - 1)(x + 1)$$

and $$6x^2 - 11x + 3 = (3x - 1)(2x - 3).$$

In these few cases the proposition is true.

But taking the expression $x^2 + x + 2$ we see that it cannot be factorized into linear factors with even rational co-efficients.

This simple instance is enough to negate the truth of the proposition.

Thus, to disprove a proposition one example, called a counter example is sufficient.

On the other hand a number of verificiations or examples of the validity of a proposition do not constitute a proof. Only a general proof is sufficient to establish the validity of truth.

Conjecture

A conjecture is a statement whose truth is observable in any particular case but eludes not only a general proof but also a counter example so far.

One very interesting conjecture is the famous Goldbach conjecture after the Prussian born mathematician Christian Goldbach (1690 – 1764).

It states as follow : "Every even number except 2 is the sum of two prime numbers."

We can verify its validity by taking any even number greater than 2.

e.g.
$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 3 + 5$$
$$10 = 3 + 7$$
$$12 = 5 + 7 \quad \text{etc.}$$

We observe that the right side all the equations are prime numbers.

So far mathematicians are not successful either in giving a general proof of the validity or a counter example of the proposition.

Thus a statement which neither proved nor disproved is a conjecture.

There are many such conjectures in Mathematics.

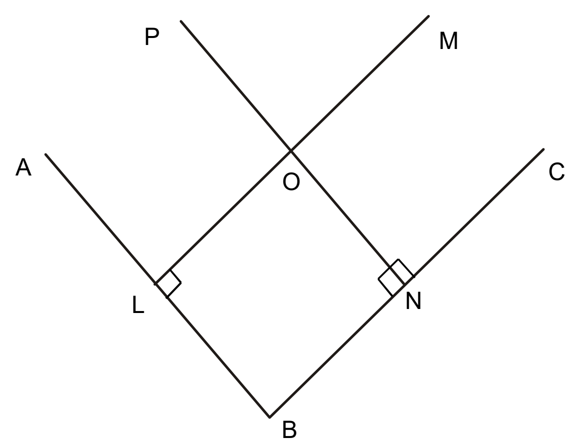
Had a conjecture been proved it would have been taken as a theorem. On the other hand had it been disproved it would have been rejected.

Hypothesis

A hypothesis is an assumed proposition used as a promise to prove some other proposition.

Let us consider the following examples.

Example 1: Show that one and only one circle can pass through three non-collinear points.



A, B, C are the three non-collinear points. LM and PN are the perpendicular bisctors of the line segments AB and BC respectively. They intersect at O. By the properties of perpendicular bisectors, O is equidistant from A, B, C. As such it is the centre of the circle whose radius is OA (= OB = OC).

Thus, it is possible to draw a circle with centre O and passing through the three non-collinear points A, B and C. That is the first part of the proposition. The second part is to show that there is no other circle which can pass through these three points. We make the following hypothesis.

If possible, let there be another circle passing through these three non-collinear points. When we say another circle, three cases arise

(i) A circle with centre O but of different radius.

(ii) A circle of the same radius but of different centre.

(iii) A circle with different radius and different centre.

Now, a hypothesis is either wrong or right. It cannot be both. Thus, if the hypothesis is wrong then it cannot be right.

In case (i) OA (= OB = OC) must have a value different from the earlier one i.e. OA should have two values, which is impossible.

In (ii) and (iii), the two perpendicular bisectors intersect at more than one point which is impossible in view of the principle that two distinct lines cannot have more than one point in common which in turn is derived from the axiom that there is only one unique line passing through two distinct points.

Thus, in all the cases there cannot be another circle.

In other words the hypothesis is wrong concluding thereby that it is not possible to have another circle that passes through the three points A, B and C. Hence, there is one and only one circle that passes through three non-collinear points.

In statistical experiments, we make a number of hypotheses. After due testings the hypothesis is either accepted or rejected. We shall not discuss that type of hypothesis here.

A set of axioms is said to be incositent if they lead a statement to results which are both true and false. An axiom is said to be independent of the others in a set of axioms if it is not a consequence of the others.

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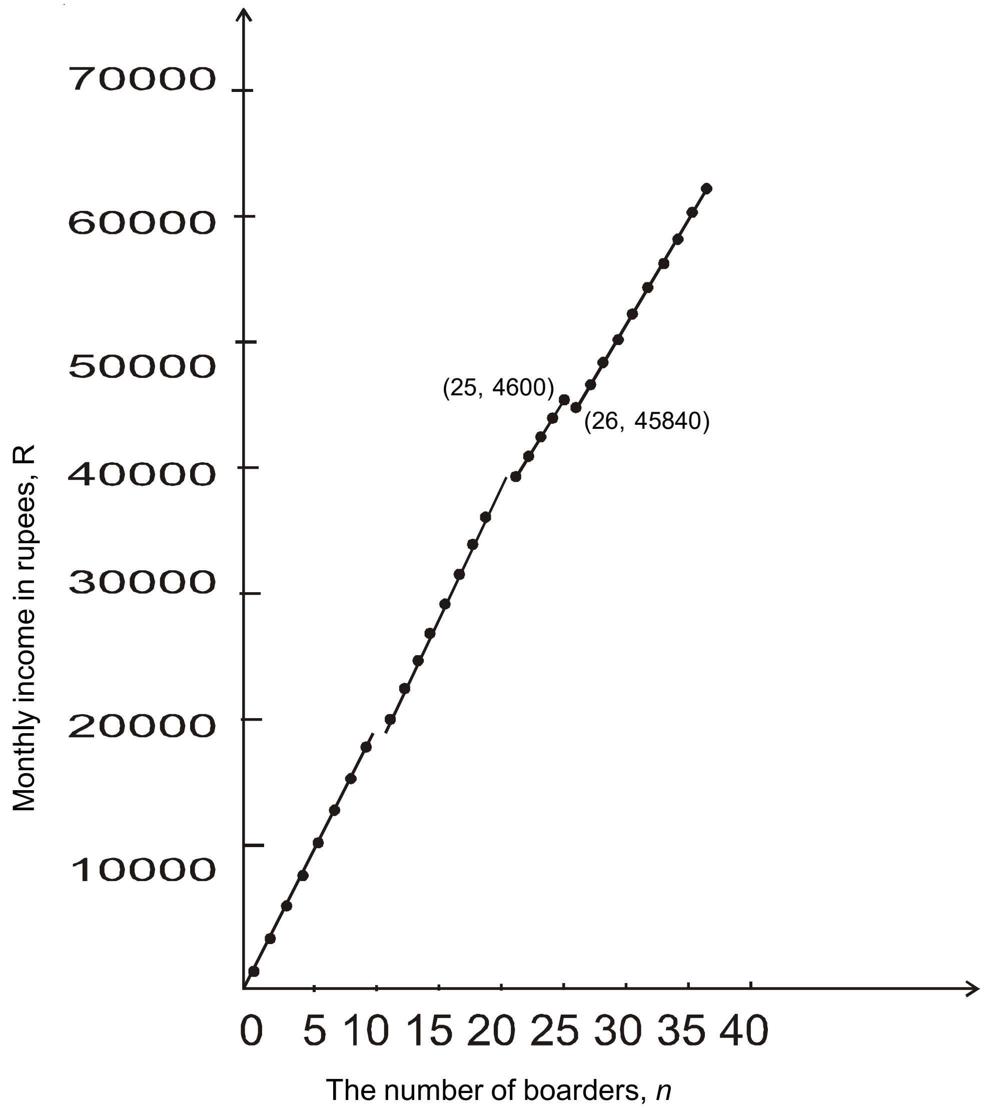

We do not draw continuous straight line graphs as the points are isolated from one another. We observe that the monthly income when there are 26 boarders is Rs 45840 while it is Rs 46000 when there are only 25 boarders. Thus there is a reduction in income even if the number of boarders is more at this juncture. The situation is indicated by the sudden break in the pattern at this juncture corresponding to $n = 25$ and $n = 26$.

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Solution : From the similar Δ^s OAB and OPQ.

$$\frac{h}{H} = \frac{s}{s+x} \Rightarrow Hs = hs + hx$$

$$\Rightarrow s\,(H - h) = hx$$

$$\Rightarrow s = \frac{s}{H-h}.x \qquad \text{—— (i)}$$

Now, equation (i) gives the relation between s and x. Since H and h are known we can find the length of the shadow of the man when his distance from the foot of the post is known.

When H = 3h equation (i) gives

$$s = \frac{h}{2h}.x \Rightarrow s = \frac{1}{2}x.$$

This shows that, in this special case the length of the shadow is half of his distance from the foot of the post.

Interpretation under special cases :

When H = h i.e. the post is as high as the man, then from (i)

$$s = \frac{h}{0}.x \Rightarrow s = \infty$$

This case is interpreted as the total obstruction of light by a horizontal shadow band of height h parallel to the ground and the shadow is endless.

Again when H < h, let H – h = – k (say).

Then from (i)

$$S = \frac{h}{H-h}x$$
$$= \frac{h}{-k}x = -\left(\frac{h}{k}\right)x$$

i.e. s is –ve. But length cannot be negative. This shows that in this case there will be not shadow.

Again, when the man is at the foot of the post, $x = 0$

Then, from (i) $s = \frac{h}{H-h}\times 0 = 0$

In this case the shadow is reduced to a point. In actual practice there will be a shapless smudge at the foot of the post.

Thus from this simple mathematical model we can predict a number of special cases.