CONTENTS

CHAPTER

1 NUMBER SYSTEM

1.1 Introduction

We introduce this chapter with a brief discussion of the integers and close the same with a brief discussion of the field properties and absolute values of the real numbers. Integers as you know, look very simple and very unlikely to hide any interesting properties. However, this impression is far from reality. Based on the properties of integers, so far explored by Mathematicians both ancient and modern, a fascinating branch of Mathematics under the name " Number Theory" comes into existence. Almost every great mathematician has made contribution to the development of this branch which goes on expanding. Here in this chapter, two interesting topics from the elementary number theory and some fundamental properties of the real numbers will be discussed.

1.2 Euclid's Division Lemma

About arithmetic, Carl Friedrich Gauss (1777-1855) a German mathematician, considered to be one of the greatest mathematicians in history once said, " Mathematics is the Queen of all Sciences and Arithmetic is the Queen of Mathematics". And integers are introduced very early in the elementary arithmetic and thereupon you have studied many interesting properties of integers. You have learnt that the sum , difference and product of two integers are all integers. In the language of Modern Algebra these properties are expressed by saying that the set of integers Z is closed under addition, subtraction and multiplication. Can you now tell whether Z is closed under division or not ? Certainly Z is not closed under division, for the result of division of one integer by another is not always an integer. For example, 36 divided by 7 yields $\frac{36}{7}$ which is not an integer. Further, division of an integer by the integer zero is undefined. This may appear to be an obstacle in the discussion of properties of integers, but it is not so. Had Z been closed under division, Mathematics would have been deprived of one of its fascinating branches namely Number Theory.

Let us consider the division of 36 by 7 . As usual it is carried out as follows:

$$\begin{array}{l} 7)36(5 \\ \underline{35} \\ 1 \end{array}$$

For such a long division sum you have seen that

Dividend = Divisor × Quotient + Remainder

In the present case,

$$36 = 7 \times 5 + 1$$

Observe that 36 and 7 are two given integers and 5 and 1 are determined in the process of long division. Further observe that 7×5 i.e. 35 is the largest multiple of 7 that does not exceed 36 so that the remainder 1=36–35, is less than the divisor 7. Besides, the quotient 5 and the remainder 1 in this division sum, are unique i.e no integer other than 5 can be the quotient and no integer other than 1 can be the remainder of this division. Thus, given two integers 36 and 7 (where the divisor 7 is non-zero) we find two unique integers 5 and 1 such that $36=7\times 5+1$ and $0 \le 1 < 7$. This result can be generalised in the following theorem.

Theorem 1.1 Let a and b be any two integers and $b>0$. Then there exist unique integers q and r such that $a = bq + r$ and $o \le r < b$.

This theorem being found recorded in Book VII of Euclid's Element, is known as Euclid's Division Lemma" and also as "Euclid's Division Algorithm" Here, q is called the quotient of a with respect to b and r, the remainder of a with respect to b.

- ✶ **A lemma is a provable statement used in proving another statement.**
- ✶ **An algorithm is a well defined sequence of steps forming a process of solving a given problem.**

Based on the division lemma, Euclid propounded an algorithm for finding HCF of two given positive integers. The process involved may be illustrated by means of an example. To find the HCF of two given integers, say 264 and 192, we divide the greater number by the smaller and obtain $264 = 192 \times 1 + 72$.

Next, we take the non-zero remainder 72 as the new divisor and the previous divisor 192 as the new dividend and obtain on division

$$192 = 72 \times 2 + 48$$

As in the previous step, we take the non-zero remainder 48 as the new divisor and the previous divisor 72 as the new dividend and obtain

$$72 = 48 \times 1 + 24$$

Again taking 24 as divisor and 48 as dividend, we get

$$48 = 24 \times 2 + 0$$

Now, the remainder is zero and the process ends here. The last divisor 24 is the HCF of the given numbers 264 and 192. It can be verified actually as

$264 = 2 \times 2 \times 2 \times 3 \times 11$ and $192=2\times 2\times 2\times 2\times 2\times 2\times 3$ so that their HCF $= 2\times 2\times 2\times 3 = 24$.

We now state Euclid's algorithm for finding HCF of two given positive integers, stepwise as follows:

Step 1. Find the quotient and remainder of the division of the greater number by the smaller.

Step 2. If the remainder is zero, then the divisor is the HCF.

Step 3. Else, taking the previous remainder as the new divisor and the previous divisor as the new dividend, find the quotient and remainder.

Step 4. Continue the process till the remainder is zero. The last divisor is the required HCF.

Let a, b be two given positive integers where $a>b$. Then applying the successive steps of the above algorithm, we obtain a series of relations as follows :

$$a = bq + r_1 \quad , \quad 0<r_1<b \quad (1)$$

$$b = r_1q_1 + r_2 \quad , \quad 0<r_2<r_1 \quad (2)$$

$$r_1 = r_2q_2 + r_3 \quad , \quad 0<r_3<r_2 \quad (3)$$

$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots$$

$$r_{n-2} = r_{n-1}q_{n-1} + r_n \quad , \quad 0<r_n<r_{n-1} \quad (n)$$

$$r_{n-1} = r_nq_n + 0 \quad (n+1)$$

Here, $a>b>r_1>r_2> \ldots >r_n$ and as shown in the $(n+1)$th relation, the last remainder is zero. The divisor at this stage i.e. the last divisor r_n is the HCF of a and b.

The set of equations (1) to $(n+1)$ is usually referred to as "Euclid's algorithm for a and b". To find the HCF of two given positive integers, we first form Euclid's algorithm for the integers and then conclude that the last divisor is the HCF. We shall use the symbol (x,y) to denote the HCF of the two integers x and y.

An important result from number theory, is given here in the form of the following theorem, the proof of which goes beyond the scope of this book.

Theorem: If $a = bq + r$

then $(a,b) = (b, r)$.

Applying this result successively to the relations (1) to $(n+1)$, it may be verified that

$(a,b) = (b, r_1) = (r_1, r_2) = \ldots = (r_{n-1}, r_n) = r_n$.

Example 1. Using Euclid's algorithm find the HCF of 378 and 735.

Solution : Euclid's algorithm for the two integers comprises of the following three equalities:

$$735 = 378 \times 1 + 357$$
$$378 = 357 \times 1 + 21$$
$$357 = 21 \times 17 + 0$$

Hence, the required HCF = 21

Example 2. Find the HCF of 682 and 297 by Euclid's algorithm.

Solution: We have

$$682 = 297 \times 2 + 88$$
$$297 = 88 \times 3 + 33$$
$$88 = 33 \times 2 + 22$$
$$33 = 22 \times 1 + 11$$
$$22 = 11 \times 2 + 0$$

It follows that, (682, 297) = 11.

Example 3. Show that every integer is of the form 2*q* or 2*q* + 1.

Solution: Let a be any integer. Taking $b = 2$ and applying Euclid's division lemma we get

$$a = 2q + r\ ,\ 0 \le r < 2$$

Here, $r = 0$ or 1. Hence a is of the form $2q$ or $2q + 1$.

Note : Every integer is either even or odd. Every even integer is of the form 2*q* and every odd integer is of the form 2*q* + 1, for some integer *q*.

Example 4. Show that every integer is of the form 3*q*, 3*q* + 1 or 3*q* – 1.

Solution: Let a be any integer. Taking $b = 3$ we get by Euclid's division lemma,

$a = 3q + r\ ,\ 0 \le r < 3$.

Here, $r = 0$, 1 or 2 so that a is of the form $3q$, $3q+1$ or $3q+2$.

Also,

$$3q+2 = 3q + 3 - 1$$
$$= 3(q+1) - 1$$
$$= 3k - 1 \quad \text{where } k = q + 1 \text{ is an integer.}$$

Thus, the form $3q+2$ may also be put in the form $3q-1$. Hence any integer a is of the form $3q$, $3q+1$ or $3q-1$.

Example 5. Show that every odd integer is of the form $4k + 1$ or $4k-1$.

Solution : Let a be any odd integer

Then , $a = 2q + 1$ for some integer q, even or odd.

If q is even, then $q = 2k$ for some integer k and $a = 2(2k) + 1 = 4k + 1$.

If q is odd, then $q = 2k-1$ for some integer k and $a = 2(2k-1) + 1 = 4k - 1$.

Thus, every odd integer a is of the form $4k + 1$ or $4k - 1$.

Example 6. Show that every square integer is of the form $4k$ or $4k + 1$.

Solution : Let a^2 be a given square integer. The integer a is of the form $2q$ or $2q + 1$. Also

$$(2q)^2 = 4q^2 = 4k, \text{ where } k = q^2 \text{ is an integer,}$$

and $$(2q+1)^2 = 4q^2+4q+1$$

$$= 4k + 1, \quad \text{where } k = q^2 + q \text{ is an integer.}$$

$\therefore$ a^2 is of the form $4k$ or $4k + 1$.

Example 7. Prove that one of every three consecutive integers is divisible by 3.

Solution : Let a, $a+1$, $a+2$ be three consecutive integers. The integer a is of the form $3q$, $3q+1$ or $3q-1$.

If $a = 3q$, the result holds trivially

If $a = 3q+1$, then $a+2 = 3q+3 = 3(q+1)$ which is divisible by 3.

If $a = 3q-1$, then $a+1 = 3q$ is divisible by 3

Hence, one of a, $a+1$ and $a+2$ is divisible by 3. This proves the result.

Example 8. If a is an odd integer, show that $a^2+(a+2)^2+(a+4)^2+1$ is divisible by 12.

Solution : The odd integer a is of the form $4k+1$ or $4k-1$ for some integer k.

If $a = 4k+1$, then

$$a^2+(a+2)^2+(a+4)^2 +1 = (4k+1)^2+(4k+3)^2+(4k+5)^2+1$$

$$= 48k^2 +72k + 36$$

$$= 12(4k^2+6k +3) \quad \text{which is divisible by 12.}$$

If $a = 4k-1$, then

$$a^2+(a+2)^2+(a+4)^2+1 = (4k-1)^2+(4k+1)^2+(4k+3)^2+1$$
$$= 48k^2 + 24k + 12$$
$$= 12(4k^2+2k+1) \quad \text{which is divisible by 12}$$

Thus, for any odd integer a , $a^2+(a+2)^2+(a+4)^2+1$ is divisible by 12.

EXERCISE 1.1

1. Using Euclid's algorithm find the HCF of

(i) 1240 and 1984 **(ii)** 348 and 504 **(iii)** 986 and 899
(iv) 4216 and 1240 **(v)** 10605 and 5256 **(vi)** 10005 and 9269

2. Show that the product of two consecutive integers is divisible by 2.

3. Show that the product of two consecutive even integers is divisible by 8.

4. Show that every integer is of the form $4q$, $4q+1$, $4q+2$ or $4q-1$.

5. Show that the product of three consecutive integers is divisible by 6.

6. Show that the square of an odd integer is of the form 8k + 1

7. If a is divisible by neither 2 nor 3, show that a^2-1 is divisible by 24.

8. Show that any square number cannot be put in the form $4k + 2$.

9. Show that any square number is of the form $3n$ or $3n+1$.

10. Show that one of three consecutive odd integers is a multiple of 3.

11. Show that the product of any three consecutive even integers is divisible by 48.

ANSWER

1. (i) 248	(ii) 12	(iii) 29	(iv) 248	(v) 3	(vi) 23

1.3 The Fundamental Theorem of Arithmetic

You are familiar with the concept of prime numbers which had been introduced in the primary stage. Recall that an integer $p>1$ is prime if it has exactly two (positive) factors viz1 and the number itself and that every positive integer having more than two distinct factors is a composite number. Thus a composite number is one which has at least one factor greater than 1 but less than the number itself. In lower classes, you have also studied factorisation of composite numbers by using factor tree or successive division. Let us consider some large number, say 291060 and factorise it into prime factors by means of successive division as follows:

$$\begin{array}{r|l}
2 & 291060 \\
2 & 145530 \\
3 & 72765 \\
3 & 24255 \\
3 & 8085 \\
5 & 2695 \\
7 & 539 \\
7 & 77 \\
 & 11
\end{array}$$

$$\therefore\ 291060 = 2\times 2\times 3\times 3\times 3\times 5\times 7\times 7\times 11$$

$$= 2^2\times 3^3\times 5\times 7^2\times 11$$

There are in all 9 prime factors in the factorisation and the distinct prime factors are 2,3,5,7 and 11. Also we see that, in the factorisation the factor 3 is repeated thrice, the factors 2 and 7 are each repeated twice while the factors 5 and 11 are non-repeating.

This factorisation of the composite number 291060 is unique except for the order in which the factors occur. Will such a factorisation exist for each composite number ? This question is answered in the affirmative by the following theorem which looks so simple inspite of its basic importance.

Theorem 1.2

[Fundamental Theorem of Arithmetic or Unique Factorisation Theorem]

Every composite number can be expressed as a product of primes uniquely except for the order of the factors.

This theorem is often stated in the following alternative form also.

Every integer $n>1$ can be expressed uniquely in the form

$$n = p_1^{a_1} p_2^{a_2} ....p_k^{a_k} \qquad (1)$$

where $p_1,p_2,..........p_k$ are primes such that $p_1<p_2<.......<p_k$ and $a_1,a_2,\ ....a_k$ are all positive integers.

This theorem has many applications in several branches of Mathematics. The factorisation of the number n as given by (1) is called the **standard** or **canonical decomposition** of n. The canonical decomposition of integers enable us to find the HCF and LCM of two or more given integers easily without using Euclid's algorithm. The following examples will make the process clear. The symbols $(a_1, a_2, ..., a_k)$ and $[a_1, a_2, ..., a_k]$ will denote the HCF and LCM respectively for the numbers $a_1, a_2, ... a_k$. You may recall that for two integers a and b, $(a,b)\ [a,b] = ab$ i.e. the product of the HCF and LCM of two integers is equal to the product of the integers.

Example 9. Find the HCF and LCM of 360 and 588. Verify that (360, 588) × [360, 588] = 360 x 588.

Solution : The canonical decomposition of 360 and 588 gives

$360 = 2^3 \times 3^2 \times 5$, $588 = 2^2 \times 3 \times 7^2$ (1)

The primes common to 360 and 588 are 2 and 3. The minimum powers to which they are raised are 2 and 1 respectively.

Hence

(360, 588) = Product of the smallest power of each common factor in the numbers

$= 2^2 \times 3$

$= 12$

The primes which appear in the two numbers are 2, 3, 5, 7 and the maximum powers to which they are raised are 3, 2, 1, 2 respectively.

Hence,

[360, 588] = Product of the greatest power of each prime factor involved in the numbers

$= 2^3 \times 3^2 \times 5 \times 7^2$

$= 17640$

Now using (1) we find that

$$(360, 588) \times [360, 588] = 2^2 \times 3 \times 2^3 \times 3^2 \times 5 \times 7^2$$
$$= (2^3 \times 3^2 \times 5) \times (2^2 \times 3 \times 7^2)$$
$$= 360 \times 588$$

Example 10. Using the prime factorisation method find the HCF and LCM of 308, 336 and 420.

Solution : Canonical decomposition of the numbers gives

$$308 = 2^2 \times 7 \times 11$$
$$336 = 2^4 \times 3 \times 7$$
$$420 = 2^2 \times 3 \times 5 \times 7$$

The primes common to all the numbers are 2, 7 and the minimum powers to which they are raised are 2, 1 respectively. Hence

$(308, 336, 420) = 2^2 \times 7 = 28.$

The primes which appear in the three numbers are 2, 3, 5, 7, 11 and the maximum powers to which they are raised are 4, 1, 1, 1, 1 respectively. Hence

$[308, 336, 420] = 2^4 \times 3 \times 5 \times 7 \times 11 = 18480.$

Some other applications, direct or indirect of the unique factorisation theorem are illustrated in the following examples.

Example 11. For any natural number n, prove that the digital root of 5^n cannot be a multiple of 3.

Solution : Since the only prime involved in the canonical form 5^n is 5, therefore the only prime that divides 5^n is 5. It means that 5^n is not divisible by any prime other than 5. In particular 5^n is not divisible by 3.

We also know that any number whose digital root is a multiple of 3, is divisible by 3.

It therefore follows that the digital root of 5^n cannot be a multiple of 3 for any natural number n.

Example12. Find any six consecutive composite numbers less than 300.

Solution : Consider the primes 2, 3, 5, 7 whose product is 210.

Clearly the six consecutive numbers

$$2 \times 3 \times 5 \times 7 + 2 = 212$$
$$2 \times 3 \times 5 \times 7 + 3 = 213$$
$$2 \times 3 \times 5 \times 7 + 4 = 214$$
$$2 \times 3 \times 5 \times 7 + 5 = 215$$
$$2 \times 3 \times 5 \times 7 + 6 = 216$$
$$2 \times 3 \times 5 \times 7 + 7 = 217$$

are composite, the first, third and fifth being divisible by 2 and the second, fourth and sixth by 3, 5 and 7 respectively.

Note : There is no hard and fast rule to find a given number of consecutive composite numbers. However, it can be observed that for any positive integer n and for any integer r, $2 \le r \le n$, the number $2 \times 3 \times 4 \times .... \times n + r$ is a composite number being divisible by r. Using this fact you can construct as many consecutive composite numbers as you want by choosing n suitably.

Example 13. Find the least multiple of 11 which when divided by 6, 7 and 10 leaves the remainder 4 in each case.

Solution : The LCM of 6, 7 and 10 = 210

By Euclid's division lemma

$$210 = 11 \times 19 + 1$$

The required number will be of the form

$$210k + 4 = (11 \times 19 + 1)k + 4$$

$$= 11 \times 19k + (k+4)$$

We are therefore, to find the least positive integer k which makes $210k+4$

i.e. $11 \times 19k + (k+4)$ divisible by 11. By inspection, such value of k is found to be 7.

Hence, the required number $= 210 \times 7 + 4$

$$= 1474$$

Example 14. By what numbers may 472 be divided so that the remainder is 17 ?

Solution : A required number will be greater than 17 and will be a factor of $472 - 17$ i.e. 455.

We have $455 = 5 \times 7 \times 13$

Hence, the factors of 455 are

$1, 5, 7, 13, 5 \times 7, 5 \times 13, 7 \times 13$ and $5 \times 7 \times 13$

i.e. 1, 5, 7, 13, 35 , 65, 91 and 455

The required numbers are the factors of 455 greater than 17 i.e. 35, 65, 91 and 455.

EXERCISE 1.2

1. Find the canonical decomposition of the numbers :

   (i) 1914 **(ii)** 2332 **(iii)** 4284

   (iv) 190575 **(v)** 133848 **(vi)** 217350

2. Find (a, b) , $[a,b]$ and verify that $(a,b)\,[a,b] = ab$ for each of the following pairs of integers :

   (i) $a = 429$, $b = 715$ **(ii)** $a = 756$, $b = 1044$ **(iii)** $a = 576$, $b = 2520$

3. Find the HCF and LCM of the following integers by prime factorisation method

   (i) 204, 1020, 1190 **(ii)** 126, 882, 1617

   (iii) 504, 2394, 4725 **(iv)** 1260, 1800, 3780, 7560

4. The HCF and LCM of two numbers are 27 and 29295 respectively.If one number is 837, find the other.
5. Show that for any natural number n, the digit in units place of 3^n cannot be even.
6. Find any five consecutive composite numbers.
7. Find any four consecutive odd composite numbers.
8. Find the least number which when divided by 24, 36 and 60 will leave in each case the same remainder 7.
9. Find the least number which when divided by 7, 8, and 12 leaves the same remainder 5 in each case.
10. Find the least multiple of 13 which when divided by 5,8 and 12 leaves the same remainder 2 in each case.
11. By what prime numbers may 319 be divided so that the remainder is 4?
12. By what numbers may 27 be divided so that the remainder is 3.?

ANSWER

1.	**(i)** $2\times3\times11\times29$	**(ii)** $2^2\times 11\times 53$	**(iii)** $2^2\times 3^2\times7\times17$
	(iv) $3^2\times5^2\times7\times11^2$	**(v)** $2^3\times3^2\times11\times13^2$	**(vi)** $2\times3^3\times5^2\times7\times23$
2.	**(i) 143, 2145**	**(ii) 36, 21924**	**(iii) 72, 20160**
3.	**(i) 34, 7140**	**(ii) 21, 9702**	**(iii) 63, 718200**
	(iv) 180, 37800		
4.	**945**	**6. 32,33, 34, 35, 36**	
7.	**213, 215, 217, 219**	**8. 367**	
9.	**173**	**10. 962**	
11.	**5, 7**	**12. 4, 6, 8, 12, 24**	

1.4 Field Properties of Real Numbers

The rational numbers and irrational numbers together form the system of real numbers. The collection of all these real numbers is denoted by R and the symbol $x \in R$ (read x belongs to R) denotes that x is a member of R i.e. x is a real number. Here we state a set of basic properties of the real numbers associated with the operations of addition and multiplication, known by the name 'field properties' and take them as axioms. Although these properties are of fundamental importance, they are quite familiar to you and you have been using them very often while dealing with numerical calculations.

1. Closure under addition: The sum of two real numbers is a real number

i.e. $x+y \in R$ whenever $x, y \in R$.

2. Associativity of addition : For every $x, y, z \in R$, $(x+y)+z = x+(y+z)$

3. Commutativity of addition : $x+y = y+x$ for every $x, y \in R$.

4. Existence of additive identity : There exists a real number 0 (zero)called the additive identity such that $x + 0 = x$ for every $x \in R$.

5. Existence of additive inverse : For each $x \in R$, there exists $-x \in R$ called the additive inverse or negative of x such that $x + (-x) = 0$ (additive identity).

6. Closure under multiplication : The product of two real numbers is a real number i.e. $xy \in R$ whenever $x, y \in R$.

7. Associativity of multiplication : For every $x, y, z \in R$, $(xy)z = x(yz)$ [The product $(xy)z$ or $x(yz)$ is denoted by xyz].

8. Commutativity of multiplication : $xy = yx$ for every $x, y \in R$.

9. Existence of multiplicative identity : There exists a real number 1, called the multiplicative identity such that $x \times 1 = x$ for any $x \in R$.

10. Existence of multiplicative inverse : For each non - zero real number x, there exists $\frac{1}{x} \in R$ called the multiplicative inverse or reciprocal of x such that $x \times \frac{1}{x} = 1$ (multiplicative identity).

11. Multiplication distributes over addition : For any real numbers x, y, z,
$x(y+z) = xy + xz$
From the field properties of real numbers cited above, the following corollaries which we use very often, are deduced.

Corollary 1. (Cancellation law for addition) If $x, y, z, \in R$ and $x+y = x+z$, then $y = z$

Proof : $x + y = x + z$ ....(1)

Since, $x \in R$, $-x \in R$(existence of additive inverse). Now adding $-x$ to both sides of (1) we get

$$(-x)+(x+y)=(-x)+(x+z)$$

$\Rightarrow (-x+x)+y=(-x+x)+z$ (by associativity of addition)

$\Rightarrow 0+y=0+z$ (by property of additive inverse)

$\Rightarrow y=z$ (by property of additive identity)

Corollary 2. (Cancellation law for multiplication)

If $x,y,z \in R$, $x \neq 0$ and

$xy = xz$, then $y = z$.

Proof : Since $x \neq 0$, by existence of multiplicative inverse, $\frac{1}{x} \in R$

Thus $xy = xz$

$\Rightarrow \frac{1}{x}(xy)=\frac{1}{x}(xz)$

$\Rightarrow \left(\frac{1}{x}\times x\right)y=\left(\frac{1}{x}\times x\right)z$ (by associativity of multiplication)

$\Rightarrow 1.y=1.z$ (by property of multiplicative inverse)

$\Rightarrow y=z$ (by property of multiplicative identity)

Corollary 3. For any $x \in R, x.0 = 0$

Proof : $0+0=0$ (by property of additive identity)

$\Rightarrow x.(0+0)=x.0$

$\Rightarrow x.0+x.0=x.0$ (by distributive law)

$\Rightarrow x.0+x.0=x.0+0$ (by property of additive identity)

$\Rightarrow x.0=0$ (by cancellation law)

Corollary 4. For $x, y \in R$, $x(-y)=-xy$

Proof : We have

$x(-y)+xy = x(-y+y)$ (by distributive law)

$= x.0$ $(\because -y+y=0)$

$= 0$ (by corollary 3)

$\therefore x(-y)$ is the additive inverse of *xy*. In other words, $x(-y) = -xy$

Corollary 5. For $x, y \in R, (-x)(-y) = xy$

Proof: $(-x)(-y) = -\{(-x)y\}$ (by corollary 4)

$= -\{y(-x)\}$

$= -(-yx)$ (by corollary 4)

$= -(-xy)$

$= xy$ (since additive inverse of $-xy$ is xy)

Corollary 6. If $x, y \in R$ and $xy = 0$, then $x = 0$ or $y = 0$.

Proof : If $x \neq 0$, then $\frac{1}{x} \in R$. Then

$$xy = 0 \Rightarrow \frac{1}{x}(xy) = \frac{1}{x}.0$$

$$\Rightarrow \left(\frac{1}{x}.x\right)y = 0$$

$$\Rightarrow 1.y = 0$$

$$\Rightarrow y = 0$$

Again if $y \neq 0$, then $\frac{1}{y} \in R$ and

$$xy = 0 \Rightarrow (xy)\frac{1}{y} = 0.\frac{1}{y}$$

$$\Rightarrow x\left(y.\frac{1}{y}\right) = 0$$

$$\Rightarrow x.1 = 0$$

$$\Rightarrow x = 0$$

It now follows that

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

1.5 Absolute Value or Modulus of a Real Number

Definition : The absolute value or modulus of a real number *x*, denoted by $|x|$ is defined by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Thus $|0| = 0$, $|3| = 3$, $|-3| = 3$, $|\sqrt{5}| = \sqrt{5}$, $|-\sqrt{5}| = \sqrt{5}$ etc.

Observe the following facts :

(i) If $x = 0$, then $x = -x = 0$

(ii) If $x > 0$, then $-x < 0$ and $x > -x$

(iii) If $x < 0$, then $-x > 0$ and $-x > x$

Keeping these facts in mind, we can also define $|x|$ as follows:

$$|x| = \begin{cases} 0, \text{ if } x = 0 \\ \text{the greater of } x \text{ and } -x, \text{ if } x \neq 0 \end{cases}$$

It is readily seen from the last definition that, for any $x \in R$

$$|x| \geq x \text{ and } |x| \geq -x$$

Some fundamental properties concerning absolute values of real numbers are given below. In the following, *x, y* are any real numbers and δ, a positive real number.

(1) $|x| \geq 0$

(2) $|-x| = |x|$

(3) $|xy| = |x||y|$

(4) $|x + y| \leq |x| + |y|$

(5) $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$

(6) $|x - y| < \delta$ if and only if $y - \delta < x < y + \delta$

Proof: **(1)** $|x| \geq 0$

Case I. If $x = 0$, then $|x| = 0$

Case II. If $x > 0$, then $|x| = x > 0$

Case III. If $x < 0$, then $-x > 0$ and $|x| = -x > 0$

Thus in all cases, $|x| \geq 0$ holds.

(2) $|-x| = |x|$

Clearly the equality holds for $x = 0$. If $x \neq 0$, then

$|-x|$ = the greater of $-x$ and $-(-x)$

= the greater of $-x$ and x

= the greater of x and $-x$

= $|x|$

Thus, $|-x| = |x|$ for any $x \in R$.

(3) $|xy| = |x||y|$

Case I. $xy = 0$, then $x = 0$ or $y = 0$

$\therefore$ $|x| = 0$ or $|y| = 0$

$\therefore$ $|x||y| = 0$

Also, $|xy| = |0| = 0$

Hence, $|xy| = |x||y|$

Case II. $xy > 0$, then either $x > 0$, $y > 0$ or else $x < 0$, $y < 0$

If $x > 0$, $y > 0$, then $|x||y| = xy = |xy|$

Again if $x < 0$ and $y < 0$, then $|x||y| = (-x)\,(-y) = xy = |xy|$

Case III. $xy < 0$. Then either $x > 0$, $y < 0$ or else $x < 0,\ y > 0$.

If $x > 0,\ y < 0$, then $|x||y| = x(-y) = -xy = |xy|$ and if $x < 0$, $y > 0$, then

$|x||y| = (-x)y = -xy = |xy|$.

Hence, $|xy| = |x||y|$ in any case.

(4) $|x + y| \leq |x| + |y|$

Case I. $x+y=0$. Then $|x + y| = 0 \leq |x| + |y|$ since both $|x|$ and $|y|$ are non-negative.

Case II. $x+y>0$. Then $|x + y| = x + y \leq |x| + |y|$ since $x \leq |x|$ and $y \leq |y|$

Case III. $x + y < 0$. Then $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$ since $-x \leq |x|$ and $-y \leq |y|$.

Thus in any case $|x + y| \leq |x| + |y|$

(5) $|x-y| \geq |x|-|y|$ and $|x-y| \geq |y|-|x|$

We have $|x| = |(x-y)+y| \leq |x-y|+|y|$, by (4)

$\Rightarrow |x|-|y| \leq |x-y|$

i.e. $|x-y| \geq |x|-|y|$

Similarly, $|y-x| \geq |y|-|x|$

$\Rightarrow |x-y| \geq |y|-|x| \qquad \because |y-x| = |x-y|$ by (2)

(6) $|x-y| < \delta$ if and only if $y-\delta < x < y+\delta$

Suppose $|x-y| < \delta$

Then $x-y \leq |x-y| < \delta$

$\Rightarrow x-y < \delta$

$\Rightarrow x < y+\delta$ ----------(i)

and $-(x-y) \leq |x-y| < \delta$

$\Rightarrow -x+y < \delta$

$\Rightarrow y < x+\delta$

$\Rightarrow y-\delta < x$ ---------(ii)

Combining (i) and (ii), $y-\delta < x < y+\delta$

Conversely suppose $y-\delta < x < y+\delta$

Then $y-\delta < x \Rightarrow y-x < \delta$

$\Rightarrow -(x-y) < \delta$

And $x < y+\delta \Rightarrow x-y < \delta$

Hence $|x-y|$ = the greater of $x-y$ and $-(x-y) < \delta$

as both $x-y$ and $-(x-y)$ are less than δ

i.e. $|x-y| < \delta$

Example 15. Is the system of irrational numbers closed under (i) addition (ii) multiplication? Justify your answer by means of examples.

Solution: (i) The system of irrational numbers is not closed under addition for the sum of two irrational numbers need not be an irrational number. For example, $2+\sqrt{3}$ and $2-\sqrt{3}$ are irrational numbers whereas their sum $(2+\sqrt{3})+(2-\sqrt{3})=4$, is not an irrational number.

(ii) Similarly, the system is not closed under multiplication.

For example, $(2+\sqrt{3})(2-\sqrt{3})=1$ which is a rational number.

Example 16. State a field property which is not satisfied by the set Z of all integers.

Solution : In *Z*, the existence of multiplicative inverse should have been stated as follows:

'For any non-zero integer *x*, there exists $\frac{1}{x}\in Z$ such that $x.\frac{1}{x}=1$'

But it is false, for instance 2 is a non-zero integer but its reciprocal $\frac{1}{2}$ is not an integer.

Thus Z does not possess the field property called 'existence of multiplicative inverse'.

Example 17. Prove that $|a|^2 = a^2$ for any $a \in R$

Solution : Since $|xy|=|x||y|$ we have

$|a^2|=|a||a|=|a|^2$ ....(1)

Also, $a^2 \geq 0$ for any $a \in \mathrm{R}$, so that

$|a^2|=a^2$ ....(2)

From (1) and (2), it follows that

$|a|^2=a^2$

Example 18. If $a^2 > b^2$, prove that $|a|>|b|$.

Solution : $a^2>b^2 \Rightarrow |a^2|>|b^2|$ $(\because |a^2|=a^2$ etc.)

$\Rightarrow |a^2|-|b^2|>0$

$\Rightarrow (|a|+|b|)(|a|-|b|)>0$

$\therefore\ |a|+|b|$ and $|a|-|b|$ are either both positive or both negative.

But $|a|+|b|$ being the sum of non-negative numbers cannot be negative.

Hence both $|a|+|b|$ and $|a|-|b|$ should be positive. In particular

$$|a|-|b|>0$$

$$\therefore\ |a|>|b|$$

EXERCISE 1.3

1. **Examine whether the following statements are true or false:**

(i) The reciprocal of an irrational number is irrational.

(ii) The set of natural numbers contains additive identity.

(iii) The set of integers has multiplicative identity.

(iv) The reciprocal of a non-zero rational number is rational.

(v) The operation of subtraction in R is commutative.

(vi) The operation of division in R is associative.

2. **(a)** Is there any real number x such that $\frac{1}{x} \notin$ R?

(b) Is there any $x \in$ R such that $-x \notin$ R ?

(c) Is there any $x \in$ R such that x^2 is not positive ?

(d) Is there any $x \in$ R such that x^2 is negative?

(e) Can two different real numbers have the same absolute value?

3. If $|a|=|b|$, find all possible relations between a and b.

4. Give any three values of x satisfying $|x-3|<1$.

5. Find x if

(i) $|x-2|=0$ **(ii)** $|x-2|=3$ **(iii)** $|x-3|=\sqrt{2}$ **(iv)** $|x-2|=x$

6. If $a^2+b^2=0$, prove that $a=0$ and $b=0$.

7. Identify on the number line, the points x satisfying :

(i) $|x|\le 3$ **(ii)** $|x|<3$ **(iii)** $|x|\ge 3$

(iv) $|x|>3$ **(v)** $|x-2|\le 3$

ANSWER

1. **(a) T (b) F (c) T (d) T (e) F**

2. **(a) Yes, namely 0 (b) No (c) Yes, 0 (d) No**

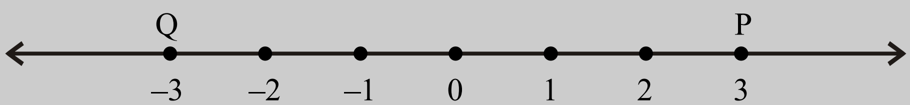
(e) Yes, e.g 3 and –3

3. $a=b$ **or** $a=-b$

4. **2.5, 3.1, 4.5 etc.**

5. **(i) 2 (ii) 5, –1 (iii)** $3+\sqrt{2}$, $3-\sqrt{2}$ **(iv) 1**

7. **(i) The points belong to the segment joining P(3) and Q(–3)**



(ii) The points belong to the segment from P(3) to Q(–3) excluding the end points: P and Q etc.

SUMMARY

In this chapter, you have studied the folowing points :

1. Euclid's Division Lemma :

Let a and b be any two integers and $b>0$. Then there exist unique integers q and r such that $a = bq+r$ and $0\le r<b$.

2. Euclid's algorithm for finding HCF of two given integers which states stepwise as follows:

Step 1. Find the quotient and remainder of the division of the greater number by the smaller.

Step 2. If the remainder is zero, then the divisor is the HCF.

Step 3. Else, taking the previous remainder as the new divisor and the previous divisor as the new dividend, find the quotient and remainder.

Step 4. Continue the process till the remainder is zero. The last divisor is the required HCF.

3. The Fundamental Theorem of Arithmetic or Unique Factorisation Theorem:

Every composite number can be expressed as a product of primes, uniquely except for the order of the factors or equivalently

Every integer $n > 1$ can be expressed uniquely in the form

$n = p_1^{a_1} p_2^{a_2} \dots\dots\dots\dots p_k^{a_k}$

where $p_1, p_2, \dots\dots, p_k$ are primes such that $p_1 < p_2 < \dots\dots < p_k$ and $a_1, a_2, \dots\dots, a_k$ are all positive integers.

4. Eleven properties of real numbers associated with the operations of addition and multiplication, known by the name 'Field properties of real numbers'.

5. The properties concerning absolute values of real numbers as listed below:

(1) $|x| \geq 0$

(2) $|-x| = |x|$

(3) $|xy| = |x||y|$

(4) $|x + y| \leq |x| + |y|$

(5) $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$

(6) $|x - y| < \delta$ if and only if $y - \delta < x < y + \delta$.

****

To start with, let us try when the divisor is a monomial. For example, let us divide the polynomial $5x^3 – 2x^2 + x$ by the monomial x.

We have ,

$$(5x^3–2x^2+x) \div x = \frac{5x^3 - 2x^2 + x}{x}$$

$$= \frac{5x^3}{x} - \frac{2x^2}{x} + \frac{x}{x}$$

$$= 5x^2 - 2x + 1$$

Then we express this fact as $5x^3–2x^2+x = x(5x^2–2x+1) + 0$ and we say that x and $5x^2–2x+1$ are factors of $5x^3–2x^2+x$ and $5x^3–2x^2+x$ is a multiple of x as well as a multiple of $5x^2–2x+1$.

Again, let us divide $2x^3+4x+5$ by $2x$

Here, $(2x^3+4x+5) \div 2x = x^2+2+\frac{5}{2x}$

We see that $\frac{5}{2x}$ cannot be a term of any polynomial. So, in this case we stop the process of division keeping 5 as the remainder.

Therefore,

$$2x^3 + 4x + 5 = 2x(x^2 + 2) + 5.$$

i.e. Dividend = Divisor × Quoteint + Remainder.

Observe that the degree of the remainder 5 is 0, which is less than that of the divisor $2x$, which is 1.

Again consider the division of $7x+3x^2 – 4$ by $3 + x$.

We carry out the division by means of the following steps.

Step 1: We write the dividend and the divisor after arranging the terms in the descending order of their degrees. Then the dividend is $3x^2+7x-4$ and divisor is $x + 3$.

Step 2 : We divide the first term of the dividend by the first term of the divisor i.e., we divide $3x^2$ by x and we get $3x$. This gives the first term of the quotient.

Step 3 : We multiply the divisor $x+3$ by $3x$ (the first term of the quotient) and obtain the product as $3x^2+9x$. Substract this product from the dividend $3x^2+7x–4$ and we get the remainder as $–2x–4$.

$$\begin{array}{r} 3x-2 \\ x+3\,\overline{)\,3x^2+7-4} \\ \underline{3x^2+9x} \\ -2x-4 \\ \underline{-2x-6} \\ 2 \end{array}$$

First term of quotient $= \frac{3x^2}{x} = 3x$

Second term of quotient $= \frac{-2x}{x} = -2$

$$p(x) = d(x) \times q(x) + r(x)$$

where either $r(x) = 0$ or degree of $r(x) <$ degree of $d(x)$.

This result is known as the Division Algorithm for polynomials.

Example 1. Divide $3x^2-x^3+2x+1$ by $2x+1-x^2$, and verify the division algorithm.

Solution: To carry out the division process, we first write the terms of both the dividend and divisor in descending order of their degrees. So, dividend $= -x^3+3x^2+2x+1$ and divisor $= -x^2+2x+1$. Division process is shown below:

$$\begin{array}{r|l} & x-1 \\ -x^2+2x+1 & -x^3+3x^2+2x+1 \\ & \underline{-x^3+2x^2+x} \\ & \quad x^2+x+1 \\ & \quad \underline{x^2-2x-1} \\ & \qquad 3x+2 \end{array}$$

We see that the degree of $3x+2$ is less than the degree of the divisor $-x^2+2x+1$. So, we cannot continue the division any further.

Then, quotient $= x-1$ and remainder $= 3x+2$

Now, Divisor $\times$ Quotient + Remainder

$$= (-x^2+2x+1)(x-1)+(3x+2)$$
$$= -x^3+2x^2+x+x^2-2x-1+3x+2$$
$$= -x^3+3x^2+2x+1$$
$$= \text{Dividend}$$

Hence verified.

EXERCISE 2.1

1. Divide the polynomial p(x) by the polynomial d(x) and find the quotient and remainder, and verify division algorithm in each of the following:

(i) $p(x) = 2x^3+x^2-x,\quad d(x) = x$

(ii) $p(x) = 3x^2-x-1,\quad d(x) = -x$

(iii) $p(x) = 3x^2+2x+1,\quad d(x) = x+1$

(iv) $p(x) = x^3-1,\quad d(x) = x-1$

(v) $p(x) = 2x^2+3x+1,\quad d(x) = 2+x$

(vi) $p(x) = x^3-3x^2+5x-3,\quad d(x) = x^2-2$

(vii) $p(x) = 4x^3+4x^2-x+1,\quad d(x) = x^2+2x$

(viii) $p(x) = 3x^2-x^3-3x+5,\quad d(x) = x-1-x^2$

2.3 Remainder Theorem

From the previous article, we know that if p(x) and d(x) are two polynomials with $d(x) \neq 0$, then we can divide p(x) by d(x) to get a polynomial q(x) as quotient and a remainder r(x). The degree of r(x), if it is non-zero, is definitely less than the degree of the divisor d(x). Further,

Dividend = Divisor $\times$ Quotient + Remainder

i.e. $p(x) = d(x) \times q(x) + r(x)$ where r(x) =0 or degree of r(x) is less than that of d(x).

This relation is true for all values of x.

Note: In case degree of the dividend p(x) is less than that of the divisor, then we take q(x) = 0 and r(x) = p(x).

If the divisor d(x) is a linear polynomial i.e. degree of d(x) is 1, then r(x) will be a constant. Let it be denoted by R.

Then the relation takes the form :

$p(x) = d(x) \times q(x) + R$

To illustrate the above discussion, let us divide the polynomial p(x) $= 5x^2 + 2x - 3$ by the linear polynomial $x - 2$. By long division, we have :

$$
\begin{array}{r|l}
 & 5x + 12 \\
\hline
x-2 & 5x^2 + 2x - 3 \\
 & 5x^2 - 10x \\
\hline
 & \quad 12x - 3 \\
 & \quad 12x - 24 \\
\hline
 & \qquad\quad 21
\end{array}
$$

Hence, we see that the remainder is 21, which is a constant. Now, let us investigate whether there is any link between the constant remainder and value of the dividend corresponding to certain value of x.

The zero of the divisor $x - 2$ is 2. If we replace x by 2 in $p(x) = 5x^2 + 2x - 3$, we have

$$p(2) = 5 \times 2^2 + 2 \times 2 - 3$$

$$= 20 + 4 - 3$$

$= 21$, which is the remainder.

Example 2. **Find the remainder when $x^4 + 2x^3 - 3x^2 - 5x + 4$ is divided by x–3, without actual division.**

Solution : Let p(x) = $x^4 + 2x^3 - 3x^2 - 5x + 4$

The zero of x–3 is 3.

Hence, by Remainder Theorem, the remainder when p(x) is divided by x–3

$$= p(3)$$
$$= 3^4 + 2 \times 3^3 - 3 \times 3^2 - 5 \times 3 + 4$$
$$= 81 + 54 - 27 - 15 + 4$$
$$= 97.$$

Example 3. **Check whether the polynomial $4x^3+4x^2-x-1$ is a multiple of $2x + 1$.**

Solution : Let p(x)= $4x^3 + 4x^2 - x - 1$

The zero of $2x + 1$ is $-\frac{1}{2}$

So, the remainder when p(x) is divided by $2x + 1$

$$= p(-\tfrac{1}{2})$$
$$= 4 \times (-\tfrac{1}{2})^3 + 4 \times (-\tfrac{1}{2})^2 - (-\tfrac{1}{2}) - 1$$
$$= -\frac{1}{2} + 1 + \frac{1}{2} - 1 = 0$$

Hence, $4x^3 + 4x^2 - x - 1$ is a multiple of $2x + 1$.

2.4 Factor Theorem

Factor Theorem can be deduced from the Remainder Theorem as a particular case.

Statement : **If p(x) is a polynomial of degree ≥ 1 and a is any real number, then $x - a$ is a factor of p(x) if and only if p(a) = 0.**

Proof : Let us suppose that p(a) = 0.

By Remainder Theorem, we know that p(a) is the remainder when p(x) is divided by x–a.

$$\therefore \quad p(x) = (x - a)q(x) + p(a)$$
$$= (x - a)q(x) \quad [\because p(a) = 0]$$

Hence $x - a$ is a factor of p(x).

Conversely, let us suppose that $x - a$ is a factor of p(x).

EXERCISE 2.2

1. Find without actual division, the remainder when **x^3+3x^2+3x+1 is divided by**

   (i) $x - 1$ **(ii)** $x+1$ **(iii)** $x-\frac{1}{2}$ **(iv)** $2x+1$

2. **Determine whether x + 1 is a factor of :**

   (i) x^3+x^2+x+1 **(ii)** $x^4+x^3+x^2+x+1$

   (iii) $x^4+3x^3+3x^2+x+1$ **(iv)** $x^3+3x^2+\left(2+\sqrt{2}\right)x+\sqrt{2}$

3. **Use Factor Theorem to determine whether q(x) is a factor of p(x) in each of the following cases:**

   (i) p(x) = x^4-81, q(x) = $x+3$

   (ii) p(x)= $2x^3+x^2-2x-1$, $q(x)=x+1$

   (iii) p(x)= x^3-3x^2-3x+1, $q(x)=x-1$

   (iv) p(x)= x^3-4x^2+x+6, $q(x)=x-4$

4. **Using Factor Theorem, show that**

   (i) $x-1$ is a factor of $3x^5-2x^2-6x+5$

   (ii) $x+1$ is a factor of $2x^4-3x^2+6x+7$

   (iii) $x-2$ is a factor of $x^3-9x^2+26x-24$

   (iv) $x+y, y+z, z+x$ are factors of $(x+y+z)^3-(x^3+y^3+z^3)$.

   (v) $x-y$, $y-z$, $z-x$ are factors of $x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$.

5. **Find the value of k, if x–1 is a factor of p(x) in each of the following cases :**

   (i) $p(x)= x^2-2x+k$ **(ii)** $p(x)= \sqrt{2}x^2+kx-1$

   (iii) $p(x)= kx^2-\sqrt{2}x+1$ **(iv)** $p(x)= kx^2-3x+2k$

6. **If x^2+px+q and x^2+lx+m are both divisible by $x+a$, show that**

$$a=\frac{m-q}{l-p}.$$

(ii) Let $p(x) = x^3 - 2x^2 - x + 2$.

The factors of 2 (the constant term in $p(x)$) are $\pm 1, \pm 2$.

Now, $p(1) = 1^3 - 2 \times 1^2 - 1 + 2 = 0$

So, $x - 1$ is a factor of p(x).

Again, $p(-1) = (-1)^3 - 2 \times (-1)^2 - (-1) + 2 = 0$

$\therefore$ $x + 1$ is also a factor of p(x)

Dividing p(x) by the product of the two factors i.e. $(x-1)\ (x+1)$ or $x^2 - 1$, we have :

$$
\begin{array}{r|l}
 & x - 2 \\
\hline
x^2 - 1 & x^3 - 2x^2 - x + 2 \\
 & x^3 \qquad\quad - x \\
\hline
 & \qquad - 2x^2 \qquad + 2 \\
 & \qquad - 2x^2 \qquad + 2 \\
\hline
\end{array}
$$

Hence, $x^3 - 2x^2 - x + 2 = (x^2 - 1)\ (x-2)$

$$= (x-1)\ (x+1)\ (x-2).$$

(iii) Let $p(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$.

The factors of 4 are $\pm 1, \pm 2, \pm 4$.

By trial, we find that p(1)= 0. So, $x - 1$ is a factor of p(x). Similarly, we can find that $x - 2$ is also a factor of p(x).

Then, we have two factors of p(x), namely $x - 1$ and $x - 2$. Dividing p(x) by the product of these two factors i.e. $(x-1)\ (x-2) = x^2 - 3x + 2$, we have :

$$
\begin{array}{r|l}
 & x^2 - 3x + 2 \\
\hline
x^2 - 3x + 2 & x^4 - 6x^3 + 13x^2 - 2x + 4 \\
 & x^4 - 3x^3 + 2x^2 \\
\hline
 & \quad -3x^3 + 11x^2 - 12x \\
 & \quad -3x^3 + 9x^2 - 6x \\
\hline
 & \qquad\qquad 2x^2 - 6x + 4 \\
 & \qquad\qquad 2x^2 - 6x + 4 \\
\hline
\end{array}
$$

$\therefore$ $x^4 - 6x^3 + 13x^2 - 12x + 4 = (x^2 - 3x + 2)\ (x^2 - 3x + 2)$

ANSWER

1.	**(i)** $(x-1)(x-3)$	**(ii)** $(x-2)(x-4)$	**(iii)** $(x+3)(x+5)$
	(iv) $(x+1)(x-7)$	**(v)** $(x+5)(x-2)$	**(vi)** $(x-5)(x+1)^2$
	(vii) $(x+1)(x+2)(x+10)$	**(viii)** $(x-1)(x-2)(x+3)$	
	(ix) $(x+1)(x+2)(x+4)$	**(x)** $(x+1)(x+3)(x-4)$	**(xi)** $(x+1)^2(x+2)^2$
	(xii) $(x-1)(x+1)(x^2+1)$		
2.	$(x+y)(y+z)(z+x)$		

SUMMARY

Main points studied in this chapter are given below :

1. The division algorithm for polynomials states that if p(x) and d(x) are any two polynomials with $d(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that $p(x) = d(x) \times q(x) + r(x)$, where either $r(x) = 0$ or degree of $r(x) <$ degree of $d(x)$.

2. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to one and p(x) is divided by the linear polynomial $x-a$, then the remainder is p(a).

3. Factor Theorem : If p(x) is a polynomial of degree ≥ 1 and a is any real number, then $x-a$ is a factor of p(x) if and only if p(a) = 0.

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CHAPTER

3

FACTORISATION

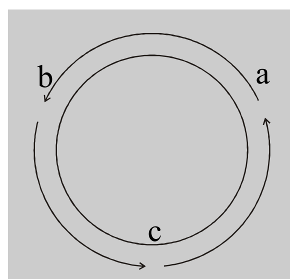
3.1 Introduction

In the previous classes, you have studied about factors and factorisation of polynomials. Recall that a factor (of a polynomial) which has no further factor other than itself, its negative and ± 1 is called a prime factor of the polynomial and factorisation is the process of expressing a given polynomial as the product of its prime factors. You have already learnt factorisation of polynomials of the types x^2-y^2, $x^3 \pm y^3$, ax^2+bx+c, ax^3+bx^2+cx+d, $ax^4+bx^3+cx^2+dx+c$ etc. In this chapter we shall consider factorisation of some cyclic expressions.

3.2 Cyclic Expressions and Cyclic Factors

By cyclical replacement of the letters *a, b, c* (i.e. by replacing *a* by *b, b* by *c* and *c* by *a*) the expression $a+b-c$ gives rise to the following three forms:

$b+c-a,\ c+a-b,\ a+b-c$



Similarly, $a-b$ gives rise to $b-c,\ c-a,\ a-b$;

$a+b$ gives rise to $b+c,\ c+a,\ a+b$;

ab gives rise to $bc,\ ca,\ ab$.

An algebraic expression which remains unchanged under cyclical replacement of the letters involved is called a cyclic expression. For example, $a+b+c$, $xy+yz+zx$, $a(b-c)+b(c-a)+c(a-b)$ etc. are cyclic expressions.

An algebraic expression is said to have cyclic factors if it has as its factors all the expressions obtained by cyclical replacement in any one of the factors. For example, if we factorise the expression $a^2(b+c) + b^2(c+a)+c^2(a+b)+ 2abc$, we get $(a+b)\ (b+c)\ (c+a)$ (Art. 3.4). Observe that all the factors i.e. $a+b$, $b+c$, $c+a$ are obtained by cyclical replacement of the letters *a, b, c* in any one of them say, $a+b$. So, the expression $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$ has cyclic factors.

3.3 Factorisation of Cyclic Expressions

In many cases, cyclic expressions can be factorised by using the following steps:

(i) Write the terms of the expression according to the ascending or descending powers of one of the letters involved in the expression.

(ii) Take out the factor(s) common to each coefficient.

(iii) Write the terms of the other factor according to the ascending or descending powers of any letters other than the previous.

(iv) Repeat the process till the factorisation is complete.

Now, let us illustrate the process by considering some cyclic expressions.

3.4 To Factorise $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$

$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$

$= a^2(b+c)+b^2c+ab^2+c^2a+bc^2+2abc$

$= a^2(b+c)+a(b^2+c^2+2bc)+(b^2c+bc^2)$ [arranging in descending powers of a]

$= a^2(b+c)+a(b+c)^2+bc(b+c)$

$= (b+c)\ [a^2+a(b+c)+bc]$

$= (b+c)[b(c+a)+a(c+a)]$ [arranging in descending powers of b]

$= (b+c)(c+a)(b+a)$

$= (a+b)(b+c)(c+a)$

3.5 To Factorise $a^2(b-c)+b^2(c-a)+c^2(a-b)$

$a^2(b-c)+b^2(c-a)+c^2(a-b)$

$=a^2(b-c)+b^2c-ab^2+c^2a-bc^2$

$= a^2(b-c)-a(b^2-c^2)+(b^2c-bc^2)$ [arranging in descending powers of a]

$= a^2(b-c)-a(b+c)(b-c)+bc(b-c)$

$= (b-c)[a^2-a(b+c)+bc]$

$= (b-c)[b(c-a)-a(c-a)]$ [arranging in descending powers of b]

$= (b-c)(c-a)(b-a)$

$= -(a-b)(b-c)(c-a)$

3.6 To Factorise $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

$= a^3(b^2-c^2)+b^3c^2-a^2b^3+c^3a^2-b^2c^3$

$= a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$ [arranging according to descending powers of a]

$= a^3(b+c)(b-c)-a^2(b-c)(b^2+bc+c^2)+b^2c^2(b-c)$

$= (b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}$

$= (b-c)(a^3b+ca^3-a^2b^2-a^2bc-c^2a^2+b^2c^2)$

$= (b-c)\{b^2(c^2-a^2)-a^2b(c-a)-ca^2(c-a)\}$ [arranging according to descending powers of b]

$= (b-c)(c-a)\{b^2(c+a)-a^2b-ca^2\}$

$= (b-c)(c-a)(b^2c+ab^2-a^2b-ca^2)$

$= (b-c)(c-a)(b^2c-ca^2+ab^2-a^2b)$ [arranging according to descending powers of c]

$= (b-c)(c-a)\{c(b^2-a^2)+ab(b-a)\}$

$= (b-c)(c-a)(b-a)\{c(b+a)+ab\}$

$= (b-c)(c-a)(b-a)(bc+ca+ab)$

$= -(a-b)(b-c)(c-a)(ab+bc+ca)$

Note : There are cyclic expressions which cannot be factorised by the above method. See the following articles.

3.7 To Factorise $(a+b+c)^3-a^3-b^3-c^3$

$(a+b+c)^3 = \{(a+b)+c\}^3$

$= (a+b)^3+c^3+3(a+b).c(a+b+c)$

$[\because (x+y)^3 = x^3+y^3+3xy(x+y)]$

$= a^3+b^3+3ab(a+b)+c^3+3c(a+b)(a+b+c)$

$= a^3+b^3+c^3+3(a+b)[ab+c(a+b+c)]$

$\therefore \quad (a+b+c)^3-a^3-b^3-c^3 = 3(a+b)(ab+ca+bc+c^2)$

$= 3(a+b)[(bc+ab)+(c^2+ca)]$

$= 3(a+b)[b(c+a)+c(c+a)]$

$= 3(a+b)(b+c)(c+a)$

3.8 To Factorise $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc.$

$$\begin{aligned}
&a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc\\
&=a^2b+ca^2+b^2c+ab^2+c^2a+bc^2+3abc\\
&=(a^2b+ab^2+abc)+(abc+b^2c+bc^2)+(ca^2+abc+c^2a)\\
&=ab(a+b+c)+bc(a+b+c)+ca(a+b+c)\\
&=(a+b+c)(ab+bc+ca)
\end{aligned}$$

Note : $a^2(b+c)+b^2(c+a)+c^2(a+b)$

$=bc(b+c)+ca(c+a)+ab(a+b)$............(1)

$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$........(2)

Therefore, $a^2(b+c)+b^2(c+a)+c^2(a+b)$ can be replaced by either (1) or (2).

3.9 To Factorise $a^3+b^3+c^3-3abc.$

$a^3+b^3+c^3-3abc$

$=(a+b)^3-3ab(a+b)+c^3-3abc$ $[\because a^3+b^3=(a+b)^3-3ab(a+b)]$

$=(a+b)^3+c^3-\{3ab(a+b)+3abc\}$

$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c)$

$[\because x^3+y^3=(x+y)(x^2-xy+y^2)]$

$=(a+b+c)(a^2+2ab+b^2-ca-bc+c^2-3ab)$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.....................(1)

$=\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$

$=\frac{1}{2}(a+b+c)\{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)\}$

$=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$.....................(2)

Hence, $a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$

3.10 To Factorise $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

We have, $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca.$

$\therefore$ The given expression

$$= 4b^2c^2 - (a^4 + b^4 + c^4 + 2b^2c^2 - 2c^2a^2 - 2a^2b^2)$$
$$= (2bc)^2 - (a^2 - b^2 - c^2)^2$$
$$= (2bc + a^2 - b^2 - c^2)(2bc - a^2 + b^2 + c^2)$$
$$= \{a^2 - (b^2 - 2bc + c^2)\}\{(b^2 + c^2 + 2bc) - a^2\}$$
$$= \{a^2 - (b - c)^2\}\{(b + c)^2 - a^2\}$$
$$= (a + b - c)(a - b + c)(b + c + a)(b + c - a)$$
$$= (a + b + c)(a + b - c)(b + c - a)(c + a - b)$$

EXERCISE 3.1

1. **Factorise the following:**

(i) $x^3 + y^3 - z^3 + 3xyz$ **(ii)** $a^3 - b^3 + 9ab + 27$

(iii) $8a^3 + 27b^3 + 64c^3 - 72abc$ **(iv)** $x^3 - y^3 - 125z^3 - 15xyz$

(v) $a^6 + 5a^3 + 8$ **(vi)** $x^6 + 8x^3 + 27.$

2. **Factorise the following :**

(i) $yz(y - z) + zx(z - x) + xy(x - y)$

(ii) $yz(y + z) + zx(z + x) + xy(x + y) + 2xyz$

(iii) $bc(b + c) + ca(c + a) + ab(a + b) + 3abc$

(iv) $x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2) + 2xyz$

(v) $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3abc$

(vi) $x^4(y - z) + y^4(z - x) + z^4(x - y)$

(vii) $yz(y^3 - z^3) + zx(z^3 - x^3) + xy(x^3 - y^3)$

(viii) $b^2c^2(b^2 - c^2) + c^2a^2(c^2 - a^2) + a^2b^2(a^2 - b^2)$

(ix) $x^2y^2(x - y) + y^2z^2(y - z) + z^2x^2(z - x)$

(x) $8z^3 - (x - y)^3 - (y + z)^3 - (z - x)^3$

(xi) $x^6(y^4 - z^4) + y^6(z^4 - x^4) + z^6(x^4 - y^4)$

(xii) $(a + b + c)(bc + ca + ab) - abc$

(xiii) $a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 3abc$

(xiv) $8(a + b + c)^3 - (b + c)^3 - (c + a)^3 - (a + b)^3$

3. Prove that $(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)=0$

4. If $a^3+b^3+c^3=3abc$, prove that either $a+b+c=0$ or $a=b=c$.

5. If $x+y+z=9$, $xy+yz+zx=26$ and $xyz=24$, find the value of $x^2(y+z)+y^2(z+x)+z^2(x+y)$.

6. If $x+y-z=2$, $y+z-x=4$ and $z+x-y=6$, find the value of $2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$.

7. If $x+y+z=12$ and $x^2+y^2+z^2=44$, find the value of $(x+y+z)^3-x^3-y^3-z^3+3xyz$.

8. Find the value of $xy(x+y)+yz(y+z)+zx(z+x)+3xyz$, when $x=a(b-c), y=b(c-a), z=c(a-b)$.

ANSWER

1. **(i)** $(x+y-z)(x^2+y^2+z^2-xy+yz+zx)$

(ii) $(a-b+3)(a^2+b^2+9+ab+3b-3a)$

(iii) $(2a+3b+4c)(4a^2+9b^2+16c^2-6ab-12bc-8ca)$

(iv) $(x-y-5z)(x^2+y^2+25z^2+xy-5yz+5zx)$

(v) $(a^2-a+2)(a^4+a^3-a^2+2a+4)$

(vi) $(x^2-x+3)(x^4+x^3-2x^2+3x+9)$

2. **(i)** $-(x-y)(y-z)(z-x)$

(ii) $(x+y)(y+z)(z+x)$

(iii) $(a+b+c)(ab+bc+ca)$

(iv) $(x+y)(y+z)(z+x)$

(v) $(a+b+c)(ab+bc+ca)$

(vi) $-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$

(vii) $-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$

(viii) $-(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)$

(ix) $-(x-y)(y-z)(z-x)(xy+yz+zx)$

(x) $3(x+z)(y+2z-x)(z-y)$

(xi) $-(x-y)(y-z)(z-x)(x+y)(y+z)(z+x)(x^2y^2+y^2z^2+z^2x^2)$

(xii) $(a+b)(b+c)(c+a)$

(xiii) $(a+b+c)(ab+bc+ca)$

(xiv) $3(2a+b+c)(a+2b+c)(a+b+2c)$

5. **162** **6.** **576** **7.** **1800** **8.** **0**

SUMMARY

Main points studied in this chapter are given below :

1. A cyclic expression is an algebraic expression which remains unchanged under cyclical replacement of the letters involved in the expression.

2. An algebraic expression is said to have cyclic factors if it has as its factors all the expressions obtained by cyclical replacement in any one of the factors.

3. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc=(a+b)(b+c)(c+a)$

4. $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$

5. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=-(a-b)(b-c)(c-a)(ab+bc+ca)$

6. $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$

7. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=(a+b+c)(ab+bc+ca)$

8. $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$$=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

9. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

$$=(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$

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CHAPTER

4 PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

4.1 Introduction

In class IX, you have learnt about linear equations in two variables and their solutions. Recall that the general form of a linear equation in two variables is $ax + by + c = 0$, where a , b, c are constants and $a^2 + b^2 \neq 0$, i.e. a and b are not zero simultaneously. An ordered pair (x_1, y_1) which satisfies this equation is a solution of the equation and a linear equation in two variables has infinitely many solutions. You also know that the graph of a linear equation in two variables is a straight line whose points make up the aggregate of solutions of the equation. In this chapter, we shall discuss about the solution of a pair of linear equations in two variables.

4.2 Solution of a Pair of Linear Equations in Two Variables

Let us consider the statement "The sum of two numbers is 9 and their difference is 5". If we denote the two numbers by x and y ($x>y$), then the statement can be expressed as $x+y = 9$ and $x - y = 5$.

Thus, the statement gives a pair of linear equations in two variables (x and y). Some other examples of pair of linear equations in two variables are

$3x + 2y - 1 = 0$ and $2x + 3y + 4 = 0$;

$x = 5y$ and $-2x + 4y + 7 = 0$;

$x - y = 3$ and $3x - y - 11 = 0$ etc.

The general form of a pair of linear equations in two variables x and y is

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are all real numbers and $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$. If $x = x_1$, $y = y_1$ satisfy both the linear equations, then the ordered pair (x_1, y_1) is called a solution of the pair of linear equations.

For example, let us consider the pair of linear equations

$x + y - 5 = 0$ ....(i)

and $x - 7y + 3 = 0$ ....(ii)

Each of these equations has infinite numbers of solutions. For instance, equation (i) is satisfied by the ordered pairs (0,5), (5,0),(4,1), (3,2) etc. and equation (ii) is satisfied by the ordered pairs (–3,0), (4,1), (–10, –1) etc. Now we observe that only (4,1) satisfies the equations (i) and (ii) simultaneously.

So, the ordered pair (4,1) i.e. $x = 4$, $y = 1$ is the only solution of the pair of linear equations (i) and (ii).

We have seen that the pair of linear equations in the above example has a unique solution. However, there are pairs of linear equations which have no solution and which have infinite number of solutions.

A pair of linear equations in two variables having at least one solution is called a **consistent** pair and a pair having no solution is called an **inconsistent** pair. A pair of linear equations is said to be a dependent pair if one equation is obtained from the other on multiplying by a constant. We shall discuss about the solutions of all such pairs graphically (geometrically) in the next section.

4.3 Graphical Method of Solution of a Pair of Linear Equations

You have already known that a linear equation in two variables is geometrically represented by a straight line. So, a pair of linear equations in two variables will be geometrically represented by two straight lines. Now, the following three possibilities arise :

(i) The two lines intersect at one point.

(ii) The two lines do not intersect i.e. they are parallel.

(iii) The two lines are coincident.

We shall try to find the solutions of the pair of linear equations in each case from the geometrical point of view. The process involved is illustrated in the following examples.

Example 1. **Solve graphically the pair of equations** $x + y = 5$ and $3x + 2y = 12$.

Solution :

$x + y = 5$ ....(1)

$3x + 2y = 12$ ....(2)

Some of the values of x and y satisfying equations (1) are given in the Table (I) below.

x	0	5	–1
y	5	0	6

Table (I)

Similarly , some of the values of x and y satisfying equation (2) are given in the Table (II) below.

x	0	4	–2
y	6	0	9

Table (II)

The points given by Table (I) namely (0,5), (5,0) and (–1, 6) are plotted in the Cartesian plane.

Joining these points, we get the straight line $\overleftrightarrow{AB}$. This is the graph of the equation $x + y = 5$ (Fig. 4.1).

Again, in the same Cartesian plane and taking the same units the points given by Table (II) i.e., (0,6), (4,0) and (-2, 9) are plotted. Joining these points we get the straight line $\overleftrightarrow{CD}$. This is the graph of the equation $3x + 2y = 12$ (Fig. 4.1).

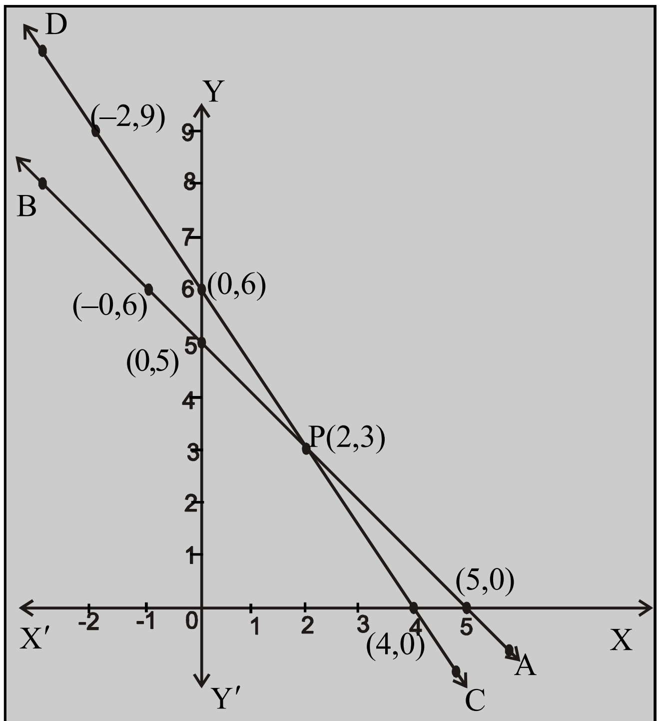


Fig. 4.1

It is seen from the Fig. 4.1 that the two straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at the point P(2,3). This is a point common to both the straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Therefore, the co-ordinates of P will satisfy both the equations (1) and (2).

Hence, the solution of the given pair of equations is given by

$$x = 2 \text{ and } y = 3.$$

Example 2. **Solve graphically the pair of equations**

$$3x + 2y = 12$$

and $$6x + 4y = 24.$$

Solution : $3x + 2y = 12$ ....(1)

$6x + 4y = 24$ ....(2)

For the equation (1), some of the values of x and y satisfying the equation are given in the Table (1).

x	0	4	– 4
y	6	0	12

Table (I)

Similarly , for the equation (2) we get

x	0	4	– 4
y	6	0	12

Table (II)

When we draw the graphs of equations (1) and(2) by plotting the points given by Tables (I) and (II) in the same Cartesian plane, we get a single straight line AB [instead of two] (Fig 4.2).

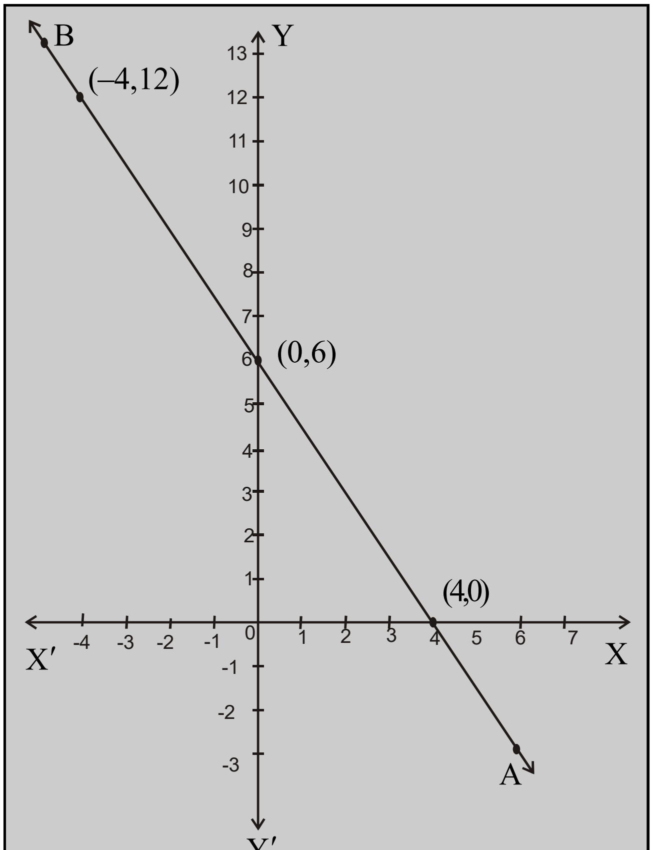


Fig. 4.2

This shows that every point on the graph of equation (1) is also a point on the graph of equation (2). Therefore, there is an infinite number of points common to both the graphs.

Hence, there are an infinite number of solutions of the given pair of equations.

Observe that the above pair of linear equations is dependent. In general, a dependent pair of linear equations has two coincident lines as the graph and has infinitely many solutions. So, a dependent pair of equations is also a consistent pair.

Example 3. **Solve graphically $2x + 3y = 6$**

and $2x + 3y = 12$.

Solution : $2x + 3y = 6$ ....(1)

$2x + 3y = 12$ ....(2)

Some values of x and y satisfying equation (1) are given in Table (I).

x	0	3	–3
y	2	0	4

Table (I)

Some values of x and y satisfying equation (2) are given in Table (II)

x	6	0	3
y	0	4	2

Table (II)

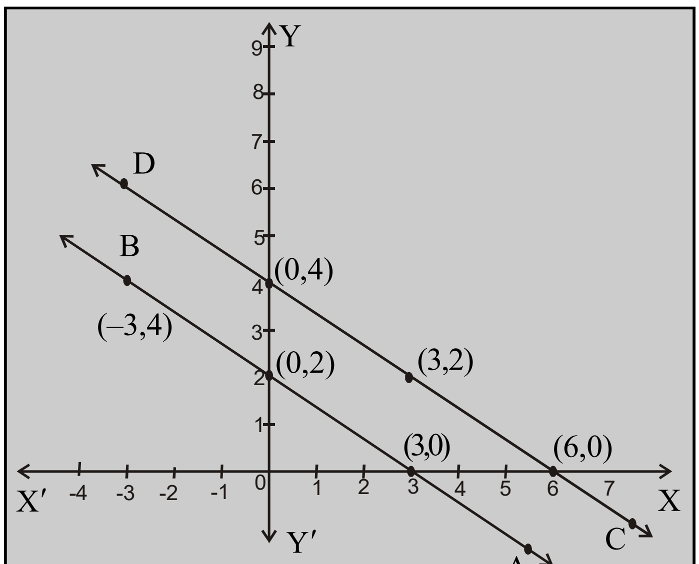


Fig. 4.3

When we draw the graph of equations (1) and (2) with the points given by Table (I) and (II) we get two parallel straight lines AB and CD (Fig. 4.3). That is, the points lying on AB are completely different from those lying on CD. In other words, the two lines AB and CD have no point in common. Hence, the given pair of equations has no solution.

Let us now write the pair of linear equations in Examples 1, 2, and 3 in the general form

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

and compare the values of $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ in each.

Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Comparison of the ratios	Geometrical representation	Algebraic interpretation
$x + y - 5 = 0$ $3x + 2y - 12 = 0$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{-5}{-12}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (i.e. unique solution)
$3x + 2y - 12 = 0$ $6x + 4y - 24 = 0$	$\frac{3}{6}$	$\frac{2}{4}$	$\frac{-12}{-24}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
$2x + 3y - 6 = 0$ $2x + 3y - 12 = 0$	$\frac{2}{2}$	$\frac{3}{3}$	$\frac{-6}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

From the above table, you can observe that if the lines represented by the equations

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

are **(i)** intersecting , then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

In fact, the converse is also true in each case.

Thus, the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

(i) has a unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(ii) is dependent if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) is inconsistent if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

Example 4. Examine whether the lines representing the pair of linear equations $5x - 4y + 8 = 0$ and

$7x + 6y - 9 = 0$, intersect at a point, are parallel or coincident.

Solution : Comparing the given pair of equations with the general form, we get

$a_1 = 5$, $b_1 = -4$, $c_1 = 8$

and $a_2 = 7$, $b_2 = 6$, $c_2 = -9$.

$\therefore$ $\frac{a_1}{a_2} = \frac{5}{7}$, $\frac{b_1}{b_2} = \frac{-4}{6}$, $\frac{c_1}{c_2} = \frac{8}{-9}$

We see that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Hence, the two lines representing the given pair of equations intersect at a point.

Example 5. Examine whether the pair of linear equations

$\frac{4}{3}x + 2y = 9$

and $2x + 3y = 12$ is consistent or not.

Solution : Writing the given pair of equations in the general form, we get

$\frac{4}{3}x + 2y - 9 = 0$

$2x + 3y - 12 = 0$

Here, $\frac{a_1}{a_2} = \frac{4/3}{2} = \frac{2}{3}$, $\frac{b_1}{b_2} = \frac{2}{3}$ and $\frac{c_1}{c_2} = \frac{-9}{-12} = \frac{3}{4}$.

We see that, $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

Hence the given pair of equations is inconsistent.

Example 6. For what values of *k* does the pair of equations given below have a unique solution?

$$2x + ky + 3 = 0$$

and $$x + y + 2 = 0$$

Solution : Here $a_1 = 2$, $b_1 = k$, $a_2 = 1$, $b_2 = 1$

For the given pair to have a unique solution,

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

i.e. $\frac{2}{1} \neq \frac{k}{1}$

i.e. $k \neq 2$

Hence for all values of *k*, except 2, the given pair of equations will have a unique solution.

Example 7. For what values of *k* will the following pair of linear equations have infinitely many solutions ?

$$kx + 3y - (k-3) = 0$$

and $12x + ky - k = 0$

Solution : Here, $\frac{a_1}{a_2} = \frac{k}{12}$, $\frac{b_1}{b_2} = \frac{3}{k}$, $\frac{c_1}{c_2} = \frac{k-3}{k}$

For the given pair to have infinitely many solutions,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \text{ i.e. } \frac{k}{12} = \frac{3}{k} = \frac{k-3}{k}$$

Now, $\frac{k}{12} = \frac{3}{k} \Rightarrow k^2 = \pm 36$

$$\Rightarrow k = \pm 6$$

By actual substitution we see that $k = 6$ satisfies the remaining equation $\frac{3}{k} = \frac{k-3}{k}$ whereas $k = -6$ does not.

Hence the given pair of equations has infinitely many solutions when $k = 6$.

EXERCISE 4.1

1. By comparing the ratios of the coefficients, find out whether the lines representing the following pair of linear equations intersect at a point, are parallel or coincident:

(i) $3x+2y-5=0$
$4x-3y+2=0$

(ii) $9x+3y+4=0$
$18x+6y+8=0$

(iii) $2x+y=3$
$3x-2y+6=0$

(iv) $6x-2y+5=0$
$3x-y+9=0$

(v) $\frac{3}{2}x+\frac{5}{3}y=7$
$9x+10y=14$

(vi) $x+y=5$
$3x-5y=-9$

2. By comparing the ratios of the coefficients, find out whether the following pairs of linear equations are consistent or not:

(i) $2x+y-3=0$
$3x+4y+6=0$

(ii) $3x+4y=12$
$6x+8y=24$

(iii) $3x-y+5=0$
$4x+3y=11$

(iv) $2x-3y=8$
$4x-6y=9$

(v) $\frac{2}{3}x+2y=8$
$x+3y=2$

3. Solve the following pair of equations graphically:

(i) $x-3y+6=0$
$x-3y-12=0$

(ii) $2x-y=2$
$3x+2y=17$

(iii) $2x+3y=5$
$5x-4y+22=0$

(iv) $y=x$
$5x-2y=9$

(v) $x+4y=2$
$3x+12y=6$

(vi) $4x+6y=18$
$9-2x-3y=0$

(vii) $2x+y=7$
$3x+2y=11$

(viii) $x+2y=8$
$3x-y=3$

(ix) $3x+y=9$
$2x-3y+16=0$

(x) $\frac{x}{2}+\frac{y}{5}=1$
$5x+2y=10$

(xi) $2x+3y=12$
$2x=3y$

(xii) $3x+2y=4$
$6x+4y=13$

(xiii) $2x+y=6$ **(xiv)** $2x+y-8=0$ **(xv)** $\frac{x}{3}+\frac{y}{4}=1$

$x-2y=8$ $x-y-1=0$ $5x-3y=15$

4. **Find the values of *p* for which the following pair of equations has unique solution :**

(i) $4x+py+5=0$ **(ii)** $px+2y=5$ **(iii)** $7x-5y-4=0$

$2x+3y+7=0$ $3x+4y=1$ $14x+py+4=0$

5. **For what value of *a* does the pair of equations**

$2x+3y=7$

and $(a-1)x+(a+1)y=3a-1$

have infinitely many solutions ?

6. **Find the value of k for which the pair of equations**

$3x+y=1$

and $(2k-1)x+(k-1)y=2k+1$

has no solution.

ANSWER

1. **(i) intersect at a point** **(ii) are coincident** **(iii) intersect at a point**
(iv) are parallel **(v) are parallel** **(vi) intersect at a point**

2. **(i) consistent** **(ii) consistent** **(iii) consistent**
(vi) inconsistent **(v) inconsistent**

3. **(i) No solution** **(ii)** $x=3, y=4$ **(iii)** $x=-2, y=3$
(iv) $x=3, y=3$ **(v) Infinitely many solutions**
(vi) Infinitely many solutions **(vii)** $x=3, y=1$ **(viii)** $x=2, y=3$
(ix) $x=1, y=6$ **(x) Infinitely many solutions** **(xi)** $x=3, y=2$
(xii) No solution **(xiii)** $x=4, y=-2$ **(xiv)** $x=3, y=2$
(xv) $x=3, y=0$

4. **(i)** $p\neq 6$ **(ii)** $p\neq\frac{3}{2}$ **(iii)** $p\neq -10$

5. ***a* = 5** **6. 2**

4.4 Algebraic Method of Solving a Pair of Linear Equations

In the previous section, we learnt how to solve a pair of linear equations in two variables graphically. This method is not convenient in cases when the point representing the solution of the pair of equations has non-integral co-ordinates like $\left(\frac{1}{11}, \frac{2}{13}\right)$, (2.73, –3.41), $\left(-\sqrt{2}, \sqrt{5}\right)$ etc. as there is difficulty in reading such co-ordinates accurately from the graph. To avoid such disadvantages of graphical method, we shall now discuss algebraic methods.

4.4.1 Substitution Method

To find the solution of a pair of linear equations in two variables by substitution method, we are to follow the following steps:

Step 1. From either equation, whichever is convenient, find the value of one variable, say y in terms of the other variable i.e. x.

Step 2. Substitute the value of y thus obtained in Step 1 in the other equation and reduce it to an equation in only one variable x, which can be solved; and hence obtain the value of x.

Sometimes after substitution you may get an equality relation with no variable. If this relation is true, you can conclude that the pair of linear equations has infinitely many solutions. If the relation is false, then the pair of linear equation is inconsistent.

Step 3. Substitute the value of x obtained in Step 2 in the equations of Step 1 and obtain the value of y.

This method is illustrated in the following examples.

Example 8. Solve the following pair of equations by substitution method:

$$2x + 3y = 32$$
$$11y - 9x = 3$$

Solution : Rewriting the given equations, we have

$$2x + 3y = 32 \quad \text{..................(1)}$$
$$11y - 9x = 3 \quad \text{..................(2)}$$

Step 1. From (1) , we have

$$y = \frac{32 - 2x}{3} \quad \text{..................(3)}$$

Step 2. Substituting the value of y in (2), we have

$$11\left(\frac{32-2x}{3}\right)-9x=3$$

$$\Rightarrow 11(32-2x)-27x=9$$

$$\Rightarrow 352-22x-27x=9$$

$$\Rightarrow -49x=-343$$

$$\Rightarrow x=\frac{-343}{-49}=7.$$

Step 3. Substituting this value of x i.e. $x = 7$ in (3), we have

$$y=\frac{32-2\times 7}{3}=\frac{18}{3}=6$$

Hence , the solution of the given pair of equations is $x = 7$ and $y = 6$.

Alternatively, we can find the solution by expressing x in terms of y from either of the equations as follows:

From (1), we have

$$x=\frac{32-3y}{2}$$

Substituting this value of x in (2), we get

$$11y-9\left(\frac{32-3y}{2}\right)=3$$

$$\Rightarrow 22y-9(32-3y)=6$$

$$\Rightarrow 22y-288+27y=6$$

$$\Rightarrow 49y=294$$

$$\Rightarrow y=\frac{294}{49}=6$$

$$\therefore\ x=\frac{32-3\times 6}{2}=\frac{14}{2}=7.$$

Example 9. Solve $2x - 3y = 5$
$4x - 6y = 10$

Solution : Rewriting the given equations, we have

$2x-3y=5$ ....(1)

$4x-6y=10$ ....(2)

From (1), we have

$$x = \frac{3y+5}{2}$$

Substituting this value of x in (2) , we have

$$4\left(\frac{3y+5}{2}\right) - 6y = 10$$

$\Rightarrow 2(3y+5) - 6y = 10$

$\Rightarrow 6y + 10 - 6y = 10$

$\Rightarrow 10 = 10$, which is true.

Hence the given pair of equations has infinitely many solutions.

4.4.2 Elimination Method

To solve a pair of linear equations in two variables by elimination method, we are to follow the following steps:

Step 1. Multiply or divide both the equations by suitable non-zero constants so that the coefficients of one variable (either x or y) become numerically equal.

Step 2. Then add one equation to the other or subtract one from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If we obtain a true equality relation involving no variable, then the original pair of equations has infinitely many solutions.

If we obtain a false relation involving no variable, then the original pair of equations has no solution.

Step 3. Solve the equation obtained in step 2 and obtain the value of the variable which is not eliminated.

Step 4. Substitute the value of the variable obtained in Step 3 in any of the given equations to get the value of the other variable.

The following examples will illucidate the process.

Example 10. Solve the following pair of equations by elimination method:

$3x + 2y = 13$

$7x - 5y = 11$

Solution : Rewriting the given equations , we have

$3x + 2y = 13$ ....(1)

$7x - 5y = 11$ ....(2)

Step 1. To eliminate the variable y, we multiply (1) by 5 and (2) by 2. Then we get the following equations :

$15x + 10y = 65$ ....(3)

$14x - 10y = 22$ ....(4)

Step 2. Adding (3) and (4) , we get

$29x = 87$

$\Rightarrow x = \frac{87}{29} = 3$

Step 3. Substituting the value of x in (1),we have

$3 \times 3 + 2y = 13$

$\Rightarrow 2y = 4$

$\Rightarrow y = 2$

Hence the solution is $x = 3$, $y = 2$.

Example 11. Solve $3x + 4y = 7$
$6x + 8y = 9$

Solution : Rewriting the given equations, we have

$3x + 4y = 7$ ....(1)

$6x + 8y = 9$ ....(2)

Multiplying (1) by 2, we get

$6x + 8y = 14$ ....(3)

Subtracting (2) from (3), we get

$(6x + 8y) - (6x + 8y) = 14 - 9$

$\Rightarrow 0 = 5$, which is false.

Hence the given pair of equations has no solution.

4.4.3 Cross-Multiplication Method

In the previous sections, you have learnt how to solve a pair of linear equations in two variables by graphical, substitution and elimination methods. Here we shall discuss another algebraic method known as 'cross-multiplication method'.

Let us consider the linear equations

$a_1x + b_1y + c_1 = 0$ ....(1)

$a_2x + b_2y + c_2 = 0$ ....(2)

Multiplying (1) by b_2 and (2) by b_1, and then subtracting, we get

$$(a_1b_2 - a_2b_1)x + (c_1b_2 - c_2b_1) = 0$$

$$\Rightarrow (a_1b_2 - a_2b_1)x = b_1c_2 - b_2c_1$$

$$\Rightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \text{provided } a_1b_2 - a_2b_1 \neq 0 \quad ....(3)$$

Again, multiplying (1) by a_2 and (2) by a_1, and then subtracting, we get

$$(b_1a_2 - b_2a_1)y + (c_1a_2 - c_2a_1) = 0$$

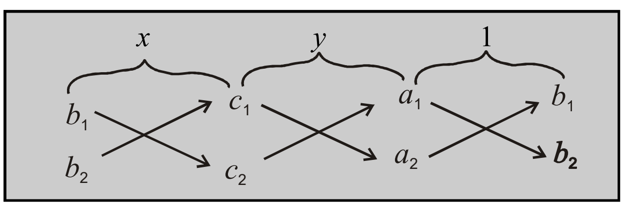
$$\Rightarrow (a_1b_2 - a_2b_1)y = c_1a_2 - c_2a_1$$

$$\Rightarrow y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}, \text{provided } a_1b_2 - a_2b_1 \neq 0 \quad ....(4)$$

Hence, from (3) and (4), we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \quad ....(5)$$

The above result can be remembered from the following diagram :



The arrow between two numbers indicates that they are to be multiplied. In finding the corresponding denominator, the numbers with downward arrow are multiplied first, and then from their product, the product of the numbers with upward arrow is to be subtracted.

To find the solution of a pair of linear equations in two variables, we will follow the following steps :

Step 1. Write the given equations in the general form.

Step 2. Taking the help of the above diagram, write equations as given in (5).

Step 3. From the equations obtained in Step 2, find the values of x and y, provided

$$a_1b_2 - a_2b_1 \neq 0.$$

Note : When $a_1b_2 - a_2b_1 \neq 0$ i.e. when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the pair of equations (1) and (2) has a unique solution. In the case $a_1b_2 - a_2b_1 = 0$ i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$ (say), the pair has infinitely many solutions or no solution according as $\frac{c_1}{c_2} = k$ or $\frac{c_1}{c_2} \neq k$ (as we have already discussed in Art. 4.3)

Example 12. Solve the following pair of linear equations by cross-multiplication method:

$$8x + 5y = 9$$

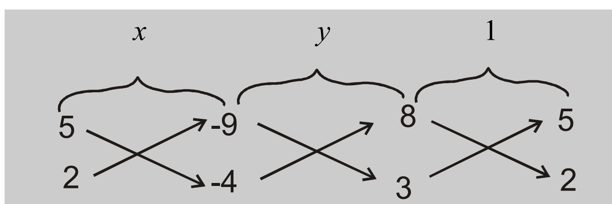
$$3x + 2y = 4$$

Solution : The given equations are written as

$$8x + 5y - 9 = 0$$

$$3x + 2y - 4 = 0$$

To solve the equations by cross-multiplication method, we draw the diagram as given below :



Then $\frac{x}{5.(-4) - 2.(-9)} = \frac{y}{(-9).3 - (-4).8} = \frac{1}{8.2 - 3.5}$

$$\Rightarrow \frac{x}{-20 + 18} = \frac{y}{-27 + 32} = \frac{1}{16 - 15}$$

$$\Rightarrow \frac{x}{-2} = \frac{y}{5} = \frac{1}{1}$$

$\therefore$ $x = -2$ and $y = 5$.

Example 13. Solve by cross-multiplication method

$$ax + by = c$$

$a^2x + b^2y = c^2$ **where** $a \neq b \neq 0$**.**

Solution : The given equations are written as

$$ax+by-c=0$$

$$ax^2+b^2y-c^2=0$$

Then by cross -multiplication , we have

$$\frac{x}{b(-c^2)-b^2(-c)}=\frac{y}{(-c)a^2-(-c^2)a}=\frac{1}{ab^2-a^2b}$$

$$\Rightarrow \frac{x}{bc(b-c)}=\frac{y}{ca(c-a)}=\frac{1}{ab(b-a)}$$

$$\Rightarrow x=\frac{c(b-c)}{a(b-a)}, y=\frac{c(c-a)}{b(b-a)}$$

Example 14. Solve the following pair of equations by reducing them to a pair of linear equations:

$$\frac{14}{x+y}+\frac{3}{x-y}=5$$

$$\frac{21}{x+y}-\frac{1}{x-y}=2$$

Solution : Writing $u=\frac{1}{x+y}$ and $v=\frac{1}{x-y}$, the given equations are

$$\begin{cases} 14u+3v=5 \\ 21u-v\ =2 \end{cases}$$

i.e. $$\begin{cases} 14u+3v-5=0 \\ 21u-v-2=0 \end{cases}$$

Then by cross-multiplication , we get

$$\frac{u}{3.(-2)-(-1).(-5)}=\frac{v}{(-5).21-(-2).14}=\frac{1}{14.(-1)-21.3}$$

$$\Rightarrow \frac{u}{-6-5}=\frac{v}{-105+28}=\frac{1}{-14-63}$$

$$\Rightarrow \frac{u}{-11}=\frac{v}{-77}=\frac{1}{-77}$$

$$\therefore\ u=\frac{1}{7}$$

$\Rightarrow \frac{1}{x+y} = \frac{1}{7}$

$\Rightarrow x + y = 7$ ....(1)

and $v = 1$

$\Rightarrow \frac{1}{x-y} = 1$

$\Rightarrow x - y = 1$ ....(2)

Adding (1) and (2) , we get,

$2x = 8$

$\Rightarrow x = 4$

Substituting the value of x in (1), we have

$4 + y = 7$

$\Rightarrow y = 3$

Hence the solution of the given pair of equations is $x = 4$ and $y = 3$.

EXERCISE 4.2

1. Solve the following pair of linear equations by the substitution method:

(i) $x + y = 5$, $x - y = 1$ **(ii)** $x + y = 7$, $2x + 3y = 18$ **(iii)** $3x + 5y = 4$, $4x - 3y = 15$

(iv) $3x - 4y - 11 = 0$, $5x + 3y + 1 = 0$ **(v)** $2x + 3y = 0$, $3x - 2y = 13$ **(vi)** $x + 2y = 3$, $2x + y = 0$

(vii) $x - y = 3$, $\frac{x}{3} + \frac{y}{2} = 6$ **(viii)** $3x - y = 3$, $6x - 2y = 6$ **(ix)** $x + 2y = 5$, $3x + 6y = 7$

(x) $\sqrt{2}x + \sqrt{3}y = 0$, $\sqrt{3}x - 2\sqrt{2}y = 0$ **(xi)** $17x + 12y = 27$, $12x + 17y = 2$ **(xii)** $2x - 3y = 11$, $3x + 4y = 8$

2. Solve the following pair of linear equations by the elimination method :

(i) $x + y = 14$, $x - y = 4$ **(ii)** $x + y = 3$, $7x - 3y = 41$ **(iii)** $x + 2y = 0$, $3x - y = 7$

(iv) $8u - 9v = 20$, $7u - 10v = 9$ **(v)** $5x + 2y = 4$, $7x + y = 5$ **(vi)** $4u + 7v = 21$, $21u - 13v = 160$

(vii) $3x+5y=7$, $12x-13y=-5$ (viii) $4x-3y=10$, $-5x+4y=-13$ (ix) $5x+2y=4$, $10x+4y=8$

(x) $2x-5y=6$, $4x-10y=9$ (xi) $\frac{x}{2}+\frac{2y}{3}=-1$, $\frac{x}{3}-\frac{y}{9}=1$ (xii) $3x-5y+2=0$, $9x=2y+7$

3. **Examine whether the following pairs of linear equations have unique solution, no solution or infinitely many solutions or Not? In case there is a unique solution, find it by using cross-multiplication method :**

(i) $2x-3y-4=0$, $4x-6y+5=0$ (ii) $4x-3y=5$, $3x-5y=1$ (iii) $2x+y-5=0$, $3x+2y-8=0$

(iv) $x-2y+3=0$, $3x-6y+9=0$ (v) $x-3y=7$, $x-y=5$ (vi) $ax+by=c^2$, $\frac{x+a}{b}-\frac{y+b}{a}=0$

(vii) $7x-5y=11$, $3x+2y=13$ (viii) $2x+3y-8=0$, $3x-4y+5=0$ (ix) $\frac{x}{a}+\frac{y}{b}=2$, $ax-by=a^2-b^2$

(x) $ax+by=a-b$, $bx-ay=a+b$ (xi) $\frac{x}{a}+\frac{y}{b}=a+b$, $\frac{x}{a^2}+\frac{y}{b^2}=2$ (xii) $\frac{x}{a}-\frac{y}{b}=0$, $ax+by=a^2+b^2$

4. **Solve the following pair of equations by reducing them to a pair of linear equations (by any algebraic method):**

(i) $\frac{6}{x}+\frac{8}{y}=5$, $\frac{8}{x}+\frac{12}{y}=7$ (ii) $\frac{6}{x}+\frac{10}{y}=7$, $\frac{2}{x}+\frac{3}{y}=\frac{13}{6}$ (iii) $\frac{3}{y}-\frac{1}{x}=1$, $\frac{2}{5x}+\frac{5}{2y}=7$

(iv) $\frac{a}{x}+\frac{b}{y}=p$, $\frac{b}{x}+\frac{a}{y}=q$, where $\frac{a}{b}\neq\frac{p}{q}\neq\frac{q}{p}$ (v) $\frac{2}{\sqrt{x}}+\frac{3}{\sqrt{y}}=2$, $\frac{4}{\sqrt{x}}-\frac{9}{\sqrt{y}}=-1$

(vi) $\frac{7x-2y}{xy}=5$, $\frac{8x+7y}{xy}=15$

(vii) $\frac{5}{x-1}+\frac{1}{y-2}=2$, $\frac{6}{x-1}-\frac{3}{y-2}=1$

(viii) $2x+y=2xy$, $2x+4y=5xy$

(ix) $\frac{5}{x+y}+\frac{1}{x-y}=2$, $\frac{15}{x+y}-\frac{5}{x-y}=-2$

(x) $\frac{1}{3x+y}+\frac{1}{3x-y}=\frac{3}{4}$, $\frac{1}{3x+y}-\frac{1}{3x-y}=\frac{-1}{4}$

(xi) $\frac{8}{x+y}+\frac{4}{x-y}=1$, $\frac{4}{x+y}+\frac{8}{x-y}=\frac{5}{4}$

(xii) $\frac{6}{x+y}+\frac{4}{x-y}=3$, $\frac{9}{x+y}-\frac{4}{x-y}=\frac{-1}{2}$

ANSWER

1. **(i)** $x = 3, y = 2$ **(ii)** $x = 3, y = 4$ **(iii)** $x = 3, y = -1$
(iv) $x = 1, y = -2$ **(v)** $x=3, y=-2$ **(vi)** $x=-1, y=2$
(vii) $x=9, y=6$ **(viii)** Infinitely many solutions **(ix)** No solutions
(x) $x=0, y=0$ **(xi)** $x=3, y=-2$ **(xii)** $x=4, y=-1$

2. **(i)** $x=9, y=5$ **(ii)** $x=5, y=-2$ **(iii)** $x=2, y=-1$
(iv) $u=7, v=4$ **(v)** $x=\frac{2}{3}, y=\frac{1}{3}$ **(vi)** $u=7, v=-1$
(vii) $x=\frac{2}{3}, y=1$ **(viii)** $x=1, y=-2$ **(ix)** Infinitely many solution
(x) No solution **(xi)** $x=2, y=-3$ **(xii)** $x=1, y=1$

3. **(i)** No solution **(ii)** Unique solution ; $x=2, y=1$
(iii) Unique solution; $x=2, y=1$ **(iv)** Infinitely many solutions
(v) Unique solution ; $x=4, y=-1$
(vi) Unique solution ; $x=\frac{b^2+c^2-a^2}{2a}, y=\frac{c^2+a^2-b^2}{2b}$
(vii) Unique solution ; $x=3, y=2$ **(viii)** Unique solution ; $x=1, y=2$
(ix) Unique solution ; $x=a, y=b$ **(x)** Unique solution; $x=1, y=-1$
(xi) Unique solution ; $x=a^2, y=b^2$ **(xii)** Unique solution; $x=a, y=b$

4. **(i)** $x=2, y=4$ **(ii)** $x=3, y=2$ **(iii)** $x=\frac{1}{5}, y=\frac{1}{2}$

(iv) $x=\frac{a^2-b^2}{ap-bq}, y=\frac{a^2-b^2}{aq-bp}$ **(v)** $x=4, y=9$

(vi) $x=1, y=1$ **(vii)** $x=4, y=5$ **(viii)** $x=1, y=2$

(ix) $x=3, y=2$ **(x)** $x=1, y=1$ **(xi)** $x=12, y=4$

(xii) $x=4, y=2$

4.5 Problems Involving Pair of Linear Equations in Two Variables

Pair of linear equations in two variables can be used to solve a wide variety of problems. We discuss a few problems below.

Example 15. Two numbers are in the ratio 2:3. If 6 is added to each number, the sums are in the ratio 3:4. Find the numbers.

Solution : Let x and y be the two numbers.

Now, from the given conditions of the problem, we get

$$\frac{x}{y}=\frac{2}{3}$$

$$\Rightarrow 3x=2y \quad ....(1)$$

and $$\frac{x+6}{y+6}=\frac{3}{4}$$

$$\Rightarrow 4x-3y+6=0 \quad ....(2)$$

Solving for x and y , we get

$x = 12$

and $y = 18$

Hence, the required numbers are 12 and 18.

Example 16. A number of two digits is equal to four times the sum of its digits. If 18 is added to the number, the digits are reversed; find the number.

Solution : Let x be the digit in the unit's place and y the digit in the ten's place of the number.

Then the number $=10y+x$ ....(1)

The number formed by interchanging the places of the two digits $=10x+y$.

The sum of the digits $=x+y$

From the given conditions of the problem, we get

$$10y + x = 4(x + y)$$

$$\Rightarrow x = 2y \qquad ...(2)$$

and $$10y + x + 18 = 10x + y$$

$$\Rightarrow x - y = 2 \qquad ...(3)$$

Solving for x and y, we get $x = 4$

$y = 2$

So, substituting the values of x and y in equation (1), we get

the required number $= 10 \times 2 + 4 = 24$.

Example 17. The sum of a two-digit number and the number obtained by reversing the digits is 77. If the difference of the digits of the number is 3, find the number. How many such numbers are there ?

Solution : Let x be the digit in the ten's place and y the digit in the unit's place.

Then the number = $10x + y$

The number formed by reversing the digits = $10y + x$

According to the given condition, we have

$$(10x + y) + (10y + x) = 77$$

$$\Rightarrow 11(x + y) = 77$$

$$\Rightarrow x + y = 7 \qquad ...(1)$$

Also we are given that the difference of the digits is 3, therefore,

either $x - y = 3$ ...(2)

or $y - x = 3$ ...(3)

If $x - y = 3$, then solving (1) and (2), we get

$x = 5$ and $y = 2$.

In this case, we get the number 52.

If $y - x = 3$, then solving (1) and (3), we get

$x = 2$ and $y = 5$.

In this case, we get the number 25.

Thus, there are two such numbers, 52 and 25.

Example 18. Twenty years ago a father was five times as old as his son and 4 years hence he will be twice as old as his son. Find their present ages.

Solution : Let x years and y years be the present ages of the father and the son respectively. From the given conditions of the problem, we get

$$x - 20 = 5(y - 20)$$

$$\Rightarrow x - 5y = -80 \qquad ...(1)$$

and $$x + 4 = 2(y + 4)$$

$$\Rightarrow x - 2y = 4 \qquad ...(2)$$

Solving (1) and (2) for x and y , we get $x = 60$ and $y = 28$
Hence, the present age of the father is 60 years and that of the son is 28 years.

Example 19. A man purchased a cycle and a watch for ₹ 2,500. He then sold the cycle at a profit of 20 % and the watch at a loss of 10%. He then made a profit of 8% on the whole. What was the cost price of the cycle ?

Solution : Let ₹ x and ₹ y be the prices of the cycle and the watch respectively.

Then , $x + y = 2500$ ...(1)

Total selling price = (x+20% of x) + (y – 10% of y)

$$= \frac{6x}{5} + \frac{9y}{10} = \frac{12x + 9y}{10}$$

Actual gain = selling price – cost price

$$= \frac{12x + 9y}{10} - (x + y) = \frac{2x - y}{10}$$

But gain in the whole transaction is 8% of the total cost price.

$$\therefore \quad \frac{2x - y}{10} = 8\% \text{ of } (x + y) = \frac{8}{100}(x + y)$$

$$\Rightarrow 2x = 3y \qquad ...(2)$$

Solving (1) and (2) for x and y , we have

$x = 1500$

and $y = 1000$

Hence, the cost price of the cycle is Rs. 1500.

Example 20. Chaoba and Tomba are at two stations on a highway 27 km. apart. They meet each other in 9 hours, if they walk in the same direction, but in 3 hours, if they walk in opposite directions. Find their rates of walking if Chaoba can walk faster than Tomba.

Solution : Let x km/hr and y km/hr be the walking speeds of Chaoba and Tomba respectively so that $x>y$. When they walk in opposite directions their relative speed = (x+y) km/hr and the relative speed is (x–y) km/hr when they walk in the same direction. But, distance travelled with uniform speed = speed × time.

$$\therefore \quad 27 = 3(x + y)$$

$$\Rightarrow x + y = 9 \qquad ...(1)$$

and $27 = 9(x - y)$

$\Rightarrow x - y = 3$ ....(2)

Solving (1) and (2) for x and y, we get

$x = 6$ and $y = 3$.

Hence, rate of walking of Chaoba is 6 km/hr and that of Tomba is 3 Km/hr.

EXERCISE 4.3

1. Divide 250 into two parts so that 3 times the first part and 5 times the second part together make 950.

2. There are two numbers. When 1 is added to each of the numbers their ratio becomes 1:2 and when 5 is subtracted from each their ratio becomes 5:11. Find the numbers.

3. The sum of the digits of a two digit number is 12. If the digits are interchanged, the number is increased by 18. Find the number.

4. The sum of the two digits of a two digit number is 9. If 9 is added to the number, the digits are reversed, find the number.

5. In a given fraction, the denominator is greater than the numerator by 2. If 7 is added to the numerator, the resulting fraction becomes greater than the given fraction by 1. Find the fraction.

6. Four years ago a father was nine times as old as his son, and 8 years hence the father's age will be three times the son's age. Find their present ages.

7. The present age of a father exceeds that of his son by 20 years. Twenty years ago, the age of the father was five times that of his son. Find their present ages.

8. A chair and a table cost ₹1,200. By selling the chair at a profit of 20% and the table at a loss of 5%, there is a profit of 4% on the whole. Find the cost price of the table and the Chair.

9. Two tables and three chairs cost ₹ 3,500, and three tables and two chairs cost ₹ 4,000. What is the cost of a table and that of a chair?

10. A farmer sold a cow and a calf for ₹ 12750 thereby making a profit of 25% on the cow and 10% on the calf. By selling them for ₹ 11925, he would have realised a profit of 10% on the cow and 25% on the calf. Find the cost price of the cow and the calf.

11. If we buy 3 tickets for Imphal to Dimapur and 2 tickets for Imphal to Mao, the total cost is ₹ 633, but if we buy 2 tickets for Imphal to Dimapur and 5 tickets for Imphal to Mao, the total cost is ₹ 642. Find the bus fares from Imphal to Dimapur and Mao.

12. Two stations A and B on a highway are 90km. apart. A car starts from A and another car starts from B at the same time. If they travel in the same direction they meet in 9 hours, but if they travel towards each other they meet in 1 hour after start. Find the speeds of the two cars, the car from A moving faster.

13. 90% and 95% pure mustard oils are mixed to obtain 20 litres of a 92% pure mustard oil. How many litres of each kind of mustard oil are needed?

14. A steamer goes 50 km. downstream and 45 km. upstream in 5 hours. In 5 hours 8 minutes it can go 50 km. upstream and 45 km. downstrem. Find the speed of the stream and that of the steamer in still water.

15. In a rectangle, if the length is reduced by 5cm. and the breadth is increased by 2cm, the area is reduced by 40 sq. cm. If however, the length is increased by 2cm. and the breadth by 4cm, the area is increased by 92 sq. cm. Find the length and breadth of the rectangle.

16. The area of a rectangular garden is increased by 55 sq. m. if its length is reduced by 2 m. and the breadth is increased by 5 m. If the length is increased by 3m. and the breadth by 2 m. the area is increased by 70 sq. m. Find the area of the rectangular garden.

17. A rectangle is of the same area as another which is 6 m longer and 4 m narrower. It is also of the same area as a third rectangle which is 8 m longer and 5 m. narrower. Find the area of the rectangle.

18. An annual income of ₹ 1,200 is derived from two sums invested, one at 4% and the other at 6% per annum simple interest. If the rates of interest are changed to 5% and 7% per annum simple interest respectively, the annual income derived from the investments is ₹ 1450 ; find the sums invested.

19. A man invested ₹ 36,000, a part of it at 12% and the rest at 15% per annum simple interest. If he recieved a total annual interest of ₹ 4,890, how much did he invest at each rate?

20. In a classroom there are a number of benches. If 4 students sit on each bench, five benches are left vacant; and if 3 students sit on each bench, 4 students are left standing. Find the number of benches and students in the class room.

21. The sum of a two-digit number and the number obtained by reversing the digits is 110. If the difference of the digits of the number is 4, find the number. How many such numbers are there?

22. The ratio of incomes of two persons is 9:7 and the ratio of their expenditures is 4:3. If each of them saves ₹ 2000 per month, find their monthly incomes.

23. A two - digit number is obtained either by multiplying the sum of the digits by 8 and adding 1, or by multiplying the difference of the digits by 13 and adding 2. Find the number. How many such numbers are there?

ANSWER

1. 150 and 100   2. 35 and 71   3. 57
4. 45   5. $\frac{5}{7}$ 6. Present age of father = 40 and that of son = 8 years.
7. Present age of father = 45 years and that of son = 25 years
8. Cost of table = ₹ 768 ; cost of chair = ₹ 432
9. Cost of table = ₹ 1000 ; cost of chair = ₹ 500
10. Cost of the cow = ₹ 8000, cost of the calf = ₹ 2500
11. Bus fare from Imphal to Dimapur = ₹ 171
    Bus fare from Imphal to Mao = ₹ 60
12. 50km/hr. and 40km/hr.
13. Quantity of 90% pure mustard oil = 12 litres
    Quantity of 95% pure mustard oil = 8 litres
14. Speed of the stream = 5 km/hr.
    Speed of the steamer = 20 km/hr.
15. Length = 15 cm and breadth = 12 cm.
16. 170 sq. m.
17. 480 sq. m.
18. First sum = ₹ 15,000 ; second sum = ₹ 10,000
19. Rs. 17,000 at 12% and ₹ 19,000 at 15%
20. No. of benches = 24, No. of students = 76.
21. 73 and 37; two
22. ₹ 18,000 and ₹ 14,000
23. 41; one.

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CHAPTER

5

QUADRATIC EQUATIONS

5.1 Introduction

In class IX, you have studied different types of polynomials. One type is the quadratic polynomials like $2x^2+x+1, x^2-16, x^2-3x+5, \sqrt{3}\ x^2-x-2$ etc.(each of the form $ax^2+bx+c, a\neq 0$). Equating these polynomials to zero, we get the equations $2x^2+x+1=0, x^2-16=0, x^2-3x+5=0, \sqrt{3}\ x^2-x-2=0,$ etc. Each of these equations is a second degree equation. Such an equation is called a quadratic equation.

In general, an equation of the form $ax^2+bx+c=0$ where $a, b, c \in R$ and $a\neq 0$ is called a quadratic equation with real co-efficients in the variable *x*.

In fact, any equation of the form *p(x)=0* where *p(x)* is a polynomial of degree 2, is a quadratic equation. But when we write the terms of *p(x)* in descending order of their degrees, we get the form $ax^2+bx+c=0, a\neq 0$ which is called the standard form of a quadratic equation.

5.2 Roots of a quadratic equation

A real number α is called a root of the quadratic equation $ax^2+bx+c=0, a\neq 0$ if $a\alpha^2+b\alpha+c=0$. We also say that $x=\alpha$ is a solution of the quadratic equation or that α satisfies the quadratic equation. And solving a quadratic equation means to find the roots of the quadratic equation. Note that a root of the quadratic equation $ax^2+bx+c=0$ is a zero of the polynomial ax^2+bx+c and vice-versa.

Remark : A quadratic equation can also have roots which are not real. But here, we shall be concerned with real roots only.

A quadratic equation can be solved by two methods namely:

(i) method of factorisation and

(ii) method of completing the perfect square.

5.3 Factorisation method of solving a Quadratic Equation

In class IX, you have learnt how to factorise a quadratic polynomial by splitting the middle term. Here, we shall use the technique to solve a quadratic equation by factorisation method. The method also involves an important result :

$ab=0\ (a,b\in R)\ \Rightarrow a=0$ or $b=0$, which has been discussed and proved in chapter 1.

Example 1. **Solve $x^2–2x–8 = 0$ by the method of factorisation and verify the result.**

Solution : The quadratic polynomial x^2-2x-8 can be factorised as follows :

$$x^2-2x-8 = x-4x^2+2x-8$$
$$= x(x-4)+2(x-4)$$
$$= (x+2)(x-4)$$

$\therefore x^2-2x-8=0$

$\Rightarrow (x+2)(x-4)=0$

$\Rightarrow$ Either $x+2=0$ or $x–4=0$ $[ab=0 \Rightarrow a=0$ or $b=0]$

$\therefore x=-2$ or $x=4$

Thus, –2 and 4 are the roots of the given quadratic equation.

Verification: When $x = –2$

$$x^2-2x-8=(–2)^2-2(–2)-8$$
$$=4+4-8=0$$

and when $x = 4$,

$$x^2-2x-8=4^2-2\times4-8$$
$$=16-8-8=0\cdot$$

Hence, $x=-2$ and $x=4$ satisfy the given equation.

Example 2. **Solve $3x^2–16x+5=0$ by the method of factorisation and verify the result.**

Solution : The quadratic polynomial $3x^2-16x+5$ can be factorised as follows :

$$3x^2-16x+5=3x^2-15x-x+5$$
$$=3x(x-5)-1(x-5)$$
$$=(3x-1)(x-5)$$

$\therefore \quad 3x^2-16x+5=0$

$\Rightarrow (3x-1)(x-5)=0$

$\Rightarrow$ Either $3x-1=0$ or $x–5=0$

$\therefore x=\frac{1}{3}$ or $x = 5$

Thus, $\frac{1}{3}$ and 5 are the roots of the given quadratic equation.

Verification : When $x = \frac{1}{3}$,

$$3x^2 - 16x + 5 = 3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5$$

$$= \frac{1}{3} - \frac{16}{3} + 5 = \frac{1 - 16 + 15}{3} = 0,$$

and when $x = 5$,

$$3x^2 - 16x + 5 = 3 \times 5^2 - 16 \times 5 + 5$$

$$= 75 - 80 + 5 = 0$$

Hence, $x = \frac{1}{3}$ and $x = 5$ satisfy the given equation.

Note : It is sometimes inconvenient to factorise a given quadratic polynomial by splitting the middle term. For example, to factorise the polynomial $x^2 + 5x + 5$, the middle term can not be broken into rational parts. Indeed, $x^2 + 5x + 5 = x^2 + \frac{5+\sqrt{5}}{2}x + \frac{5-\sqrt{5}}{2}x + 5$. Imagine how hard it can be to find the two irrational numbers $\frac{5+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$.

We shall now study the method of completing the perfect square which can be conveniently applied to solve any given quadratic equation.

5.4 Solution of a Quadratic Equation by the method of completing the perfect square.

Consider the quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$

Multiplying both sides by $4a$, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

$$\Rightarrow (2ax)^2 + 2(2ax)b + b^2 - b^2 + 4ac = 0$$

$$\Rightarrow (2ax + b)^2 - (b^2 - 4ac) = 0$$

$$\Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This method is known as Hindu method or Sreedharacharyya's method.

Thus, the roots of the quadratic equation $ax^2+bx+c=0$ are say,

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

This formula for finding the roots of a quadratic equation is known as the quadratic formula. One may find the roots of a quadratic equation directly by using the quadratic formula.

Note: Completion of the perfect square can also be done after dividing throughout by a, the coefficient of x^2.

The process of completing the perfect square and also the use of quadratic formula are illustrated in the following examples.

Example 3. Solve $x^2 + 9x + 18 = 0$ by the method of completing the perfect square and verify the result.

Solution :

$$x^2+9x+18=0$$

$$\Rightarrow 4x^2+36x+72=0 \quad \text{(Multiplying throughout by 4)}$$

$$\Rightarrow (2x)^2+2\,(2x)9+9^2-9^2+72=0$$

$$\Rightarrow (2x+9)^2-9=0$$

$$\Rightarrow (2x+9)^2=9$$

$$\Rightarrow 2x+9=\pm\sqrt{9}=\pm 3$$

$$\Rightarrow x=\frac{-9\pm 3}{2}$$

$$\Rightarrow x=\frac{-9+3}{2} \text{ or } x=\frac{-9-3}{2}$$

$$\Rightarrow x=-3 \text{ or } x=-6$$

Another way of completing the perfect square is as follows :

$$x^2+9x+18=0$$

$$\Rightarrow x^2+2x\times\frac{9}{2}+\left(\frac{9}{2}\right)^2-\left(\frac{9}{2}\right)^2+18=0$$

$$\Rightarrow \left(x+\frac{9}{2}\right)^2-\frac{81}{4}+18=0$$

$$\Rightarrow \left(x+\frac{9}{2}\right)^2-\frac{9}{4}=0$$

$$\Rightarrow \left(x+\frac{9}{2}\right)^2=\frac{9}{4}$$

$$\Rightarrow x+\frac{9}{2}=\pm\sqrt{\frac{9}{4}}=\pm\frac{3}{2}$$

$$\Rightarrow x=-\frac{9}{2}\pm\frac{3}{2}$$

$$\Rightarrow x=\frac{-9}{2}+\frac{3}{2} \text{ or } x=\frac{-9}{2}-\frac{3}{2}$$

$$\Rightarrow x=-3 \quad \text{or} \quad x=-6$$

Verification : When $x=-3$,

$$x^2+9x+18=(-3)^2+9(-3)+18$$
$$=9-27+18=0,$$

and when $x=-6$,

$$x^2+9x+18=(-6)^2+9(-6)+18$$
$$=36-54+18=0$$

Hence, $x=-3$ and $x=-6$ satisfy the given equation.

Example 4. Solve $4x^2 + 2x + 3 = 0$ by the method of completing the perfect square.

Solution : $4x^2+2x+3=0$

$$\Rightarrow (2x)^2+2(2x)\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+3=0$$

$$\Rightarrow \left(2x+\frac{1}{2}\right)^2=\frac{1}{4}-3=-\frac{11}{4}<0$$

But $\left(2x+\frac{1}{2}\right)^2$ cannot be negative for any real value of x. So, there is no real value of x satisfying the given equation. Therefore, the given equation has no real roots.

Note:

Indeed, the roots of this equation are $\dfrac{-\frac{1}{2}\pm\sqrt{-\frac{11}{4}}}{2}$ **which are not real.**

Here, $\sqrt{-\frac{11}{4}}$ **is not real because there is no real number whose square is** $-\frac{11}{4}$**.**
You will be dealing with numbers of this type in higher classes.

Example 5. Solve $2x^2 - 2x - 5 = 0$ by using the quadratic formula.

Solution : Comparing the given equation with $ax^2+bx+c=0$,

we get a = 2, b= –2, c= –5.

Using the quadratic formula, the roots of the given equation are given by

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4\times2\times(-5)}}{2\times2} \qquad \left[x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$=\frac{2\pm\sqrt{44}}{4}$$

$$=\frac{2\pm2\sqrt{11}}{4}$$

$$=\frac{1\pm\sqrt{11}}{2}$$

i.e. $x=\dfrac{1+\sqrt{11}}{2}$ or $x=\dfrac{1-\sqrt{11}}{2}$

Hence, the roots of the given equation are

$\dfrac{1+\sqrt{11}}{2}$ and $\dfrac{1-\sqrt{11}}{2}$.

5.5. Nature of the roots of $ax^2+bx+c=0$

In the previous section, we have seen that the roots of the quadratic equation $ax^2+bx+c=0$ are

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Though $a, b, c \in R$ and hence $b^2 - 4ac \in R$, we cannot say at once whether the roots are real or not. The nature of the roots depends in fact upon the value of $b^2 - 4ac$. The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

According to the value of the discriminant $b^2 - 4ac$, we may consider the following three distinct cases.

Case I : $b^2 - 4ac > 0$

When $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ is real and hence we get two real and distinct roots $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Further if *a, b, c* are rational and $b^2 - 4ac$ is a perfect square, then the roots are rational $\left(\sqrt{b^2 - 4ac} \text{ being rational}\right)$ and if $b^2 - 4ac$ is not a perfect square, then the roots are irrational.

Case II : $b^2 - 4ac = 0$

When $b^2 - 4ac = 0$, $\sqrt{b^2 - 4ac} = 0$ and so the quadratic equation has two real and equal roots, each being $-\dfrac{b}{2a}$.

Case III : $b^2 - 4ac < 0$

When $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ is not real, since there is no real number whose square is the negative number $b^2 - 4ac$.
So, the roots of the quadratic equation are not real.

Thus, a quadratic equation $ax^2 + bx + c = 0$ has

(i) two unequal real roots, if $b^2 - 4ac > 0$,
(ii) two equal real roots, if $b^2 - 4ac = 0$,
(iii) no real roots, if $b^2 - 4ac < 0$.

Note that the roots of the quadratic equation $ax^2 + bx + c = 0$ are real if $b^2 - 4ac \geq 0$.

Example 6. Without solving, examine the nature of the roots of the quadratic equations given below:

(i) $3x^2 - 5x + 2 = 0$ **(ii)** $x^2 - 4x + 1 = 0$

(iii) $x^2 - 6x + 25 = 0$ **(iv)** $9x^2 - 6x + 1 = 0$

Solution : (Compare each of the given quadratic equations with $ax^2 + bx + c = 0$)

(i) $3x^2 - 5x + 2 = 0$

Here, $a = 3$, $b = -5$, $c = 2$.

Now, discriminant = $b^2 - 4ac = (-5)^2 - 4 \times 3 \times 2 = 1 > 0$

Also, *a, b, c* are rational and discriminant is a perfect square.

$\therefore$ the roots are rational and unequal.

(ii) $x^2 - 4x + 1 = 0$

Here, $a = 1$, $b = -4$, $c = 1$

Now, discriminant = $b^2 - 4ac = (-4)^2 - 4 \times 1 \times 1 = 12 > 0$

Also, *a, b, c* are rational but discriminant is not a perfect square.

$\therefore$ the roots are irrational and unequal.

(iii) $x^2 - 6x + 25 = 0$

Here, $a = 1$, $b = -6$, $c = 25$

Now, discriminant = $b^2 - 4ac = (-6)^2 - 4 \times 1 \times 25 = -64 < 0$

Since the discriminant is negative, the roots are not real.

(iv) $9x^2 - 6x + 1 = 0$

Here, $a = 9$, $b = -6$, $c = 1$

Now, discriminant = $b^2 - 4ac = (-6)^2 - 4 \times 9 \times 1 = 0$

$\therefore$ the roots are real and equal.

Example 7. Determine whether the given quadratic equations have real roots and if so, find the roots by using the quadratic formula.

(i) $x^2 - 9x + 20 = 0$ **(ii)** $2x^2 + 3x - 1 = 0$

(iii) $3x^2 + x + 1 = 0$ **(iv)** $x^2 - 4x + 4 = 0$

Solution : **(i)** $x^2 - 9x + 20 = 0$

Here, $a = 1$, $b = -9$, $c = 20$

So, discriminant $= b^2 - 4ac$

$$= (-9)^2 - 4 \times 1 \times 20$$
$$= 1 > 0$$

$\therefore$ the equation has real roots.

By using quadratic formula, we have

$$x = \frac{-(-9) \pm \sqrt{1}}{2 \times 1} = \frac{9 \pm 1}{2} = \frac{10}{2}, \frac{8}{2}$$

i.e. $x = 5$ or $x = 4$.

$\therefore$ the roots are 5 and 4.

(ii) $2x^2 + 3x - 1 = 0$

Here, $a = 2$, $b = 3$, $c = -1$

So, discriminant $= b^2 - 4ac$

$$= 3^2 - 4 \times 2 \times (-1)$$
$$= 17 > 0$$

$\therefore$ the equation has real roots.

By using quadratic formula, we have

$$x = \frac{-3 \pm \sqrt{17}}{2 \times 2}$$

i.e. $x = \frac{-3 + \sqrt{17}}{4}$ or $x = \frac{-3 - \sqrt{17}}{4}$

$\therefore$ the roots are $\frac{-3 + \sqrt{17}}{4}$ and $\frac{-3 - \sqrt{17}}{4}$

(iii) $3x^2 + x + 1 = 0$

Here, $a = 3$, $b = 1$, $c = 1$

So, discriminant $= b^2 - 4ac$

$$= 1^2 - 4 \times 3 \times 1 = -11 < 0$$

$\therefore$ the equation does not have real roots.

(iv) $x^2 - 4x + 4 = 0$

Here, $a = 1$, $b = -4$, $c = 4$

So, discriminant $= b^2 - 4ac$

$$= (-4)^2 - 4 \times 1 \times 4$$
$$= 0$$

$\therefore$ the equation has real roots.

By using quadratic formula, we have

$$x = \frac{-(-4) \pm \sqrt{0}}{2 \times 1} = \frac{4 \pm 0}{2}$$

i.e. $x = 2$ or $x = 2$

$\therefore$ the roots are 2 and 2

5.6 Relation between roots and coefficients

The two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\therefore \alpha + \beta =$ Sum of the roots

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

And, $\alpha\beta =$ Product of the roots

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$= \frac{b^2 - \left(b^2 - 4ac\right)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Important Deductions:

(i) Suppose b = 0

Then, $\alpha + \beta = -\frac{b}{a} = -\frac{0}{a} = 0$

$\Rightarrow \alpha = -\beta$

Thus, if $b = 0$, the roots of the quadratic equation $ax^2 + bx + c = 0$ are equal in magnitude but opposite in sign.

(ii) Suppose $c = 0$

Then, $\alpha\beta = \frac{c}{a} = \frac{0}{a} = 0$

$\Rightarrow \alpha = 0 \; or \; \beta = 0$

Thus, if $c = 0$, then one of the roots is zero

(iii) Suppose $c = a$

Then, $\alpha\beta = \frac{c}{a} = 1 \qquad \Rightarrow \alpha = \frac{1}{\beta}$

Thus, if $c=a$, then one root is the reciprocal of the other.

(iv) Suppose $a + b + c = 0$.

Then, 1 is a root of the quadratic equation since

$a \times 1^2 + b \times 1 + c = 0$

Also, let α be the other root.

Then, $1 \times \alpha = \frac{c}{a} \Rightarrow \alpha = \frac{c}{a}$

Thus, if $a+b+c = 0$, then the roots are 1 and $\frac{c}{a}$.

Note : It is essential that $a \neq 0$, for if $a = 0$, **then the equation** $ax^2 + bx + c = 0$ **fails to be a quadratic equation.**

Example 8. Without solving, find the sum and the product of the roots in each of the following quadratic equations:

(i) $2x^2 + 4x + 1 = 0$ **(ii)** $x^2 + 3x + 3 = 0$ **(iii)** $3x^2 - 2\sqrt{3}x - \sqrt{3} = 0$

Solution : **(i)** $2x^2 + 4x + 1 = 0$

Here, $a = 2$, $b = 4$, $c = 1$

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{4}{2} = -2$

and product of the roots $= \frac{c}{a} = \frac{1}{2}$

(ii) $x^2 + 3x + 3 = 0$

Here, $a = 1$, $b = 3$, $c = 3$

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{3}{1} = -3$

and product of the roots $= \frac{c}{a} = \frac{3}{1} = 3$

(iii) $3x^2 - 2\sqrt{3}x - \sqrt{3} = 0$

Here, a = 3, b $= -2\sqrt{3}$, c $= -\sqrt{3}$

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{-2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$

and product of the roots $= \frac{c}{a} = \frac{-\sqrt{3}}{3}$

Example 9. Without solving, show that the roots of the equation, $6x^2+13x + 6 = 0$ are reciprocal of one another.

Solution : Let α and β be the roots of $6x^2 + 13x + 6 = 0$.

Then, $\alpha\,\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{6}{6} = 1$

$\Rightarrow \alpha = \frac{1}{\beta}$

Hence, the roots are reciprocal of one another.

Example 10. Without solving, show that the roots of the equation $2x^2 - 5 = 0$ are equal in magnitude but opposite in sign.

Solution : The given equation is $2x^2 - 5 = 0$

i.e. $2x^2 + 0x - 5 = 0$

Let α and β be the roots of the equation.

Then, $\alpha + \beta = -\frac{0}{2} = 0$

$\Rightarrow \alpha = -\beta$

$\therefore$ the two roots are equal in magnitude but opposite in sign.

Example 11. If α and β are the roots of the equation $x^2 - px + q = 0$, $q \neq 0$, find the values of the following (in terms of *p* and *q*)

(i) $\alpha^2 + \beta^2$ **(ii)** $\frac{1}{\alpha} + \frac{1}{\beta}$ **(iii)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **(iv)** $(\alpha + 2)(\beta + 2)$

Solution : Since α and β are the roots of the equation $x^2 - px + q = 0$,

we have, $\alpha + \beta = -\frac{-p}{1} = p$

and $\alpha\beta = \frac{q}{1} = q$

Now, **(i)** $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= p^2 - 2q$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{p}{q}$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2q}{q}$

(iv) $(\alpha + 2)(\beta + 2) = \alpha\beta + 2\alpha + 2\beta + 4$

$= \alpha\beta + 2(\alpha + \beta) + 4$

$= q + 2p + 4$

Note : Expressions like $\alpha^2 + \beta^2$, $\frac{1}{\alpha} + \frac{1}{\beta}$, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, $(\alpha+2)(\beta+2)$ etc. which remain unchanged when α and β are interchanged are called symmetric functions of α and β. Such expressions can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$ and hence in terms of the coefficients

5.7 Formation of Quadratic Equation when the roots are given

Suppose, we want to form the quadratic equation whose roots are α and β .

Let $ax^2 + bx + c = 0$ be the required equation.

Then, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now, $ax^2 + bx + c = 0$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Thus, the equation whose roots are α and β is given by $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e. x^2 – (sum of the roots) x + product of the roots = 0

Alternatively, the quadratic equation whose roots are α and β can be written as

$$(x - \alpha)(x - \beta) = 0$$

i.e. $x^2 - \alpha x - \beta x + \alpha\beta = 0$

i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e. x^2 – (sum of the roots) x+ product of the roots = 0

Example 12. Form the quadratic equation whose roots are

(i) –3, – 4 **(ii)** $\sqrt{3}, 2\sqrt{3}$ **(iii)** $\frac{1}{4}, \frac{1}{5}$ **(iv)** $5 + \sqrt{3}$, $5 - \sqrt{3}$

Solution : **(i)** Here, sum of the roots $= -3 + (-4) = -7$

and product of the roots $= -3 \times (-4) = 12$

$\therefore$ the required equation is $x^2 - (-7)x + 12 = 0$

i.e. $x^2 + 7x + 12 = 0$

(ii) Here, sum of the roots $= \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

and product of the roots $= \sqrt{3} \times 2\sqrt{3} = 6$

$\therefore$ the required equation is $x^2 - 3\sqrt{3}x + 6 = 0$

(iii) Here, sum of the roots $= \frac{1}{4} + \frac{1}{5} = \frac{5+4}{20} = \frac{9}{20}$

and product of the roots $= \frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$

$\therefore$ the required equation is $x^2 - \frac{9}{20}x + \frac{1}{20} = 0$

i.e. $20x^2 - 9x + 1 = 0$

(iv) Here, sum of the roots $= (5+\sqrt{3})+(5-\sqrt{3}) = 10$

and product of the roots $= (5+\sqrt{3})(5-\sqrt{3}) = 25-3 = 22$

$\therefore$ the required equation is $x^2 - 10x + 22 = 0$

Example 13. If α and β are the roots of the equation $ax^2 + bx + c = 0$, form the quadratic equation whose roots are

(i) $-\alpha, -\beta$ **(ii)** $\alpha+1,\ \beta+1$ **(iii)** $\frac{1}{\alpha}, \frac{1}{\beta}$

Solution : Since α and β are the roots of $ax^2+bx+c=0$, we have

$\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

(i) Here, we are to form the quadratic equation whose roots are $-\alpha$ and $-\beta$.

Now, sum of the roots $= -\alpha + (-\beta) = -(\alpha+\beta) = -\left(-\frac{b}{a}\right) = \frac{b}{a}$

and product of the roots $= -\alpha \times (-\beta) = \alpha\beta = \frac{c}{a}$

$\therefore$ the required equation is $x^2 - \frac{b}{a}x + \frac{c}{a} = 0$

i.e. $ax^2 + bx + c = 0$

(ii) Here, we are to form the quadratic equation whose roots are $\alpha+1$ and $\beta+1$.

Now, sum of the roots $= (\alpha+1)+(\beta+1) = (\alpha+\beta)+2 = -\frac{b}{a}+2 = \frac{-b+2a}{a}$

and productof the roots $= (\alpha+1)(\beta+1) = \alpha\beta + (\alpha+\beta) + 1$

$$= \frac{c}{a} + \frac{-b}{a} + 1 = \frac{c-b+a}{a}$$

$\therefore$ the required equation is $x^2 - \left(\frac{-b+2a}{a}\right)x + \frac{c-b+a}{a} = 0$

i.e. $ax^2 + (b-2a)x + (a-b+c) = 0$

(iii) Here, we are to form the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Now, sum of the roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$

and product of the roots $= \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$

$\therefore$ the required equation is $x^2 - \left(-\frac{b}{c}\right)x + \frac{a}{c} = 0$

i.e. $cx^2 + bx + a = 0$

Example 14. If the roots of the equation $x^2 - px + q = 0$ differ by 1, then show that $p^2 = 4q + 1$.

Solution : Let α and $\alpha - 1$ be the two roots of $x^2 - px + q = 0$.

Then, $\alpha + (\alpha - 1) = -\frac{-p}{1}$

$\Rightarrow 2\alpha - 1 = p$

$\Rightarrow \alpha = \frac{p+1}{2}$ ...(1)

and, $\alpha(\alpha - 1) = q$

$\Rightarrow \frac{p+1}{2}\left(\frac{p+1}{2} - 1\right) = q$ (using (1))

$\Rightarrow \frac{p+1}{2} \times \frac{p-1}{2} = q$

$\Rightarrow p^2 - 1 = 4q$

$\Rightarrow p^2 = 4q + 1$

Hence shown.

Example 15. If one root of the equation $ax^2 + bx + c = 0$ be n times the other, show that $ac(n + 1)^2 = b^2n$.

Solution : Let α and $n\alpha$ be the roots of the equation $ax^2 + bx + c = 0$,

Then, we have

$$\alpha + n\alpha = -\frac{b}{a}$$

$$\Rightarrow \quad \alpha(n+1) = -\frac{b}{a}$$

$$\Rightarrow \quad \alpha = -\frac{b}{a(n+1)} \qquad \text{----------(1)}$$

And, $\alpha \times n\alpha = \frac{c}{a}$

$$\Rightarrow \quad \alpha^2 n = \frac{c}{a}$$

$$\Rightarrow \quad \left\{-\frac{b}{a(n+1)}\right\}^2 n = \frac{c}{a} \qquad \{\text{using (1)}\}$$

$$\Rightarrow \quad \frac{b^2 n}{a^2(n+1)^2} = \frac{c}{a}$$

$$\Rightarrow \quad a^2 c(n+1)^2 = ab^2 n$$

$$\Rightarrow \quad ac(n+1)^2 = b^2 n$$

Hence shown.

EXERCISE 5.1

1. Solve the following quadratic equations by the method of factorisation:

(i) $x^2 - 7x + 10 = 0$ **(ii)** $2x^2 - 7x - 9 = 0$

(iii) $12x^2 - 7x + 1 = 0$ **(iv)** $4x^2 - 12x + 9 = 0$

(v) $\sqrt{2}y^2 - 3y - 2\sqrt{2} = 0$ **(vi)** $4\sqrt{3}y^2 + 5y - 2\sqrt{3} = 0$

2. Solve the following quadratic equations (real roots only) by the method of completing the perfect square:

(i) $x^2 + 3x - 10 = 0$ **(ii)** $x^2 - 6x + 25 = 0$

(iii) $12x^2 - 17x + 6 = 0$ **(iv)** $4x^2 - 12x + 1 = 0$

(v) $x^2 - 10x + 34 = 0$ **(vi)** $6x^2 + x - 2 = 0$

3. Determine whether the following quadratic equations have real roots and if so, find the roots by using the quadratic formula:

(i) $x^2 - 6x - 16 = 0$ **(ii)** $x^2 - 6x + 7 = 0$

(iii) $4x^2 - x + 1 = 0$ **(iv)** $x^2 - 5\sqrt{3}x + 18 = 0$

(v) $4x^2 + 12x + 9 = 0$ **(vi)** $2x^2 - 3x + 2 = 0$

4. **Without solving, examine the nature of the roots of the following quadratic equations:**

(i) $9x^2 - 6x + 1 = 0$ **(ii)** $x^2 + x + 2 = 0$

(iii) $25x^2 - 25x + 6 = 0$ **(iv)** $x^2 - 3x + 1 = 0$

(v) $3x^2 - 10x + 3 = 0$ **(vi)** $x^2 - x + 1 = 0$

5. **Without solving, find the sum and the product of the roots of the following quadratic equations:**

(i) $x^2 - 3x - 3 = 0$ **(ii)** $5x^2 - 10x + 2 = 0$

(iii) $2x^2 - 3\sqrt{3}x + 3 = 0$ **(iv)** $3x^2 - (2+\sqrt{3})x - 4 = 0$

(v) $\sqrt{5}x^2 - 10x + 5 = 0$ **(vi)** $x^2 + 5x + 5 = 0$

6. **Form the quadratic equations whose roots are**

(i) $-3, -6$ **(ii)** $\frac{1}{2}, \frac{1}{3}$

(iii) $3\sqrt{2}, -4\sqrt{2}$ **(iv)** $3+\sqrt{5}, \quad 3-\sqrt{5}$

(v) $\frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}$ **(vi)** $\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$

7. **If α, β are the roots of the equation $x^2 + px + q = 0$, $q \neq 0$ then find the values of the following (in terms of p and q)**

(i) $\alpha^2\beta + \alpha\beta^2$ **(ii)** $(\alpha + 1)(\beta + 1)$ **(iii)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

(iv) $\alpha^3 + \beta^3$ **(v)** $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ **(vi)** $(\alpha^2 + 2)(\beta^2 + 2)$

8. **If α, β are the roots of the equation $x^2 - px + q = 0$, $q \neq 0$ then form the quadratic equation whose roots are**

(i) $3\alpha, 3\beta$ **(ii)** $\frac{\alpha}{2}, \frac{\beta}{2}$ **(iii)** $\alpha - 1, \beta - 1$

(iv) α^2, β^2 **(v)** $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ **(vi)** α^3, β^3

9. Find the value of p so that the equation $3px^2 - 12x + p = 0$ has equal roots.

10. One root of $x^2 + ax - 21 = 0$ is 3, while $x^2 + ax + b = 0$ has equal roots. Find b.

11. If the sum and the product of the roots of a quadratic equation are respectively 4 and $\frac{15}{4}$, find the equation.

12. If one root of the equation $2x^2 - 5x + k = 0$ be reciprocal of the other, find the value of k.

13. Find the quadratic equation whose roots are each less by 2 than those of $x^2 - 3x + 1 = 0$.

14. If one root of the equation $x^2 - px + q = 0$ be twice the other, show that $2p^2 = 9q$.

15. If one root of the equation $x^2 + px + q = 0$ be the square of the other, then show that $p^3 + q^2 + q = 3pq$.

16. If the sum of the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero, prove that the product of the roots is $-\frac{1}{2}(a^2 + b^2)$.

ANSWER

1. **(i) 2 and 5** **(ii) –1 and** $\frac{9}{2}$ **(iii)** $\frac{1}{3}$ and $\frac{1}{4}$

(iv) $\frac{3}{2}$ *and* $\frac{3}{2}$ **(v)** $2\sqrt{2}$ and $\frac{-\sqrt{2}}{2}$ **(vi)** $\frac{\sqrt{3}}{4}$ and $\frac{-2\sqrt{3}}{3}$

2. **(i) 2 and –5** **(ii) No real roots** **(iii)** $\frac{3}{4}$ and $\frac{2}{3}$

(iv) $\frac{3+2\sqrt{2}}{2}$ and $\frac{3-2\sqrt{2}}{2}$ **(v) No real roots** **(vi)** $\frac{1}{2}$ and $\frac{-2}{3}$

3. **(i) 8 and –2** **(ii)** $3+\sqrt{2}$ and $3-\sqrt{2}$ **(iii) No real roots**

(iv) $2\sqrt{3}$ and $3\sqrt{3}$ **(v)** $\frac{-3}{2}$ and $\frac{-3}{2}$ **(vi) No real roots**

4. **(i) Real and equal** **(ii) Not real**
(iii) Rational and unequal **(iv) Irrational and unequal**
(v) Rational and unequal **(vi) Not real**

5. **(i) Sum=3, Product = –3** **(ii) Sum = 2, Product =** $\frac{2}{5}$

(iii) Sum $= \frac{3\sqrt{3}}{2}$, Product $= \frac{3}{2}$ (iv) Sum $= \frac{2+\sqrt{3}}{3}$, Product $= \frac{-4}{3}$

(v) Sum = $2\sqrt{5}$, Product = $\sqrt{5}$ (vi) Sum = –5, Product = 5

6. (i) $x^2+9x+18=0$ (ii) $6x^2-5x+1=0$ (iii) $x^2+\sqrt{2}x-24=0$

(iv) $x^2-6x+4=0$ (v) $2x^2-6x+3=0$ (vi) $9x^2-9\sqrt{2}x+4=0$

7. (i) $-pq$ (ii) $q-p+1$ (iii) $\frac{p^2-2q}{q^2}$

(iv) $-p^3+3pq$ (v) $\frac{-p^3+3pq}{q}$ (vi) $2p^2+q^2-4q+4$

8. (i) $x^2-3px+9q=0$ (ii) $4x^2-2px+q=0$

(iii) $x^2+(2-p)x+(q-p+1)=0$ (iv) $x^2+(2q-p^2)x+q^2=0$

(v) $qx^2+(2q-p^2)x+q=0$ (vi) $x^2+(3pq-p^3)x+q^3=0$

9. $\pm 2\sqrt{3}$ 10. 4

11. $4x^2-16x+15=0$ 12. $k=2$

5.8 Word Problems based on Quadratic Equation

In the previous classes we have learnt how to solve practical problems with the help of simple linear simultaneous equations. Now, here we shall discuss how problems can be solved by forming quadratic equations from the information given in the question. Some times it may happen that out of the two roots of the quadratic equation, only one root satisfies the condition of the problem. Therefore, in solving a problem with the help of a quadratic equation, the root which does not satisfy the conditions of the problem must be rejected.

Example 16. Divide 15 into two parts such that their product is 56.

Solution : Let x be the first part. Then the second part is $15-x$.

Now, from the given condition, we have

$x(15-x)=56$

$\Rightarrow 15x-x^2=56$

$\Rightarrow x^2-15x+56=0$

$\Rightarrow (x-7)(x-8)=0$

$\therefore$ Either, $x - 7 = 0$, giving $x = 7$.
or, $x - 8 = 0$, giving $x = 8$

[From the step $x^2 - 15x + 56 = 0$, the roots can be written down:

$$x = \frac{15 \pm \sqrt{15^2 - 4 \times 56}}{2} = \frac{15 \pm 1}{2} = 8, 7$$

When $x = 7$, the parts are 7 and (15 –7) i.e. 8 and when $x = 8$, the parts are 8 and (15–8) i.e., 7

Thus, the parts are 7 and 8.]

Example 17. The sum of the squares of three consecutive natural numbers is 77. Find the numbers.

Solution : Let the three consecutive natural numbers be $x - 1$, x and $x + 1$.

Then, from the given condition of the problem, we have

$$(x-1)^2 + x^2 + (x+1)^2 = 77$$

$$\Rightarrow \quad 3x^2 + 2 = 77$$

$$\Rightarrow \quad 3x^2 = 75$$

$$\Rightarrow \quad x^2 = \frac{75}{3} = 25$$

$$\therefore \quad x = \pm\sqrt{25} = \pm 5$$

Natural number cannot be negative. Therefore –5 is obviously inadmissible and so it is rejected.

$\therefore$ we take $x = 5$

Hence, the three consecutive natural numbers are 4, 5 and 6.

Note : If in this question 'natural numbers' be replaced by 'integers', then both values of *x* are admissible and –6, –5 and – 4 would have been another set of solution.

Example 18. The sum of a number and its reciprocal is $\frac{17}{4}$. Find the number.

Solution : Let x be the number. Then its reciprocal is $\frac{1}{x}$.

Therefore, from the given condition, we have

$$x + \frac{1}{x} = \frac{17}{4}$$

$\Rightarrow 4x^2 - 17x + 4 = 0$

$\Rightarrow (x-4)(4x-1) = 0$

$\therefore$ Either $x - 4 = 0$ giving $x = 4$

or, $4x - 1 = 0$ giving $x = \frac{1}{4}$

Both the roots satisfy the condition of the problem.

$\therefore$ the number is either 4 or $\frac{1}{4}$.

Example 19. A number consists of two digits. The product of the digits is 15. When 18 is added to the number, the digits interchange their places. Find the number.

Solution : Let x be the digit in the ten's place and y the digit in the unit's place of the number.

Then,

the number = $10x + y$ ...(1)

and the number with the digits interchanged = $10y + x$.

From the given conditions of the problem, we get the equations

$x.y = 15$ ...(2)

and $10x + y + 18 = 10y + x$

$\Rightarrow 9y - 9x = 18$

$\Rightarrow y - x = 2$ ...(3)

From (3) and (2), we have

$y - \frac{15}{y} = 2$ [From (2), $x = \frac{15}{y}$]

$\Rightarrow y^2 - 2y - 15 = 0$

$\Rightarrow (y-5)(y+3) = 0$

$\therefore y = 5, -3$

But the digit of a number cannot be negative. So, y = –3 is to be rejected.

$\therefore$ y = 5

and $x = \frac{15}{y} = \frac{15}{5} = 3$

Substituting these values of x and y in (1) we have, the required number

$= 10 \times 3 + 5 = 35$.

Example 20. The hypotenuse of a right triangle is 3 m longer than twice the shortest side. The third side is 2 m longer than twice the shortest side. Find the lengths of the sides of the triangle.

Solution : Let the shortest side of the right triangle be x metres long.

From the given conditions of the problem, we have,

length of hypotenuse = $2x + 3$

and that of the third side = $2x + 2$.

Applying Pythagoras theorem on the right triangle, we have

$$(2x+3)^2 = x^2 + (2x+2)^2$$

$\Rightarrow x^2 - 4x - 5 = 0$ [Simplifying and writing in the standard form of quadratic equation]

$\Rightarrow (x-5)(x+1) = 0$

$\therefore x = 5, -1$

Since length cannot be negative, we take $x = 5$

$\therefore$ Length of the shortest side = 5 m

So, length of the hypotenuse = $2 \times 5 + 3 = 13$ m

and that of the third side = $2 \times 5 + 2 = 12$m.

Example 21. A dealer buys a pocket radio set at a certain sum of money. By selling it at ₹ 171 he gains exactly as much percent as the radio set had cost him. What is the price of the set.

Solution : Let ₹ x be the cost price of the radio set.

Then, gain = x% of x = $\frac{x^2}{100}$

Now by question, we have

$$x + \frac{x^2}{100} = 171$$

$\Rightarrow x^2 + 100x - 17100 = 0$

Solving, we get

$$x = \frac{-100 \pm \sqrt{100^2 + 4 \times 1 \times 17100}}{2 \times 1}$$

$$= \frac{-100 \pm \sqrt{78400}}{2} = \frac{-100 \pm 280}{2} = 90, -190$$

Here, $x = -190$ is obviously inadmissible, because the price of the radio set cannot be a negative quantity.

Hence the cost price of the radio set is ₹ 90.

Example 22. The distance between Imphal and Dimapur is 215 km. While going from Imphal to Dimapur, a passenger bus takes $1\frac{19}{24}$ hours less, if its average speed is increased by 10 km/hr. from its usual average speed. What is the usual average speed of the bus?

Solution : Let the usual average speed of the bus be x km/hr. Then, its increased speed is $(x + 10)$ km/hr.

Time taken by the bus while travelling from Imphal to Dimapur with its usual average speed $= \frac{215}{x}$ hr. and corresponding time taken with its increased speed $= \frac{215}{x+10}$ hr.

Now by question,

we have, $\frac{215}{x} - \frac{215}{x+10} = 1\frac{19}{24} = \frac{43}{24}$

$$\Rightarrow \frac{5}{x} - \frac{5}{x+10} = \frac{1}{24}$$

$$\Rightarrow \frac{5(x+10-x)}{x(x+10)} = \frac{1}{24}$$

$$\Rightarrow x^2 + 10x = 50 \times 24$$

$$\Rightarrow x^2 + 10x - 1200 = 0$$

$$\Rightarrow x^2 - 30x + 40x - 1200 = 0$$

$$\Rightarrow x(x-30) + 40(x-30) = 0$$

$$\Rightarrow (x-30)(x+40) = 0$$

$$\therefore x = 30, -40$$

The speed of the bus cannot be a negative quantity. Therefore, $x = -40$ is rejected.

Hence, the usual average speed of the bus is 30 km/hr.

Example 23. A motor boat takes 1hour longer to go 36 km upstream than to return downstream to the same spot. If the speed of the stream is 3 km/hr, find the speed of the motor boat in still water.

Solution : Let the speed of the motor boat in still water be x km/hr

Then, speed upstream $= (x-3)$ km/hr.

and speed downstream $= (x+3)$ km/hr.

$\therefore$ time taken to go 36 km upstream $= \dfrac{36}{x-3}$ hours

and time taken to go 36 km downstream $= \dfrac{36}{x+3}$ hours.

Now by question, we have

$$\frac{36}{x-3} - \frac{36}{x+3} = 1$$

$$\Rightarrow \quad \frac{36[(x+3)-(x-3)]}{(x-3)(x+3)} = 1$$

$$\Rightarrow \quad \frac{36 \times 6}{x^2-9} = 1$$

$$\Rightarrow \quad x^2 - 9 = 216$$

$$\Rightarrow \quad x^2 = 225$$

$$\Rightarrow \quad x = 15 \quad [\because x \neq -15]$$

$\therefore$ The speed of the motor boat in still water is 15 km/hr.

EXERCISE 5.2

1. The sum of the squares of two consecutive natural numbers is 61. Find the numbers.

2. The sum of a number and its reciprocal is $\dfrac{37}{6}$. Find the number.

3. The sum of two numbers is 7 and the difference of their reciprocals is $\dfrac{1}{12}$. What are the two numbers ?

4. The sum of the squares of two consecutive even positive integers is 244. What are the two numbers ?

5. Find two consecutive odd integers if the sum of their squares is 202.

6. A number consists of two digits. The digit in the unit's place is the square of the digit in the ten's place. If the sum of the digits is 12, find the number.

7. A number consists of two digits. The product of the digits is 20. When 9 is added to the number, the digits interchange their places. Find the number.

8. The lengths of the sides (in cm) of a right triangle are x, $x+1$ and $x+2$. Find the lengths of the sides of the triangle. What is its area ?

9. The sides containing the right angle of a right triangle differ in length by 7 cm. If the hypotenuse is 13 cm long, determine the lengths of the two sides of the triangle.

10. The sides containing the right angle of a right triangle are x and $3x + 1$ (measured in cm) long. If the area of the triangle is 100 sq.cm, find the lengths of the sides containing the right angle of the right triangle.

11. The difference between the lengths of the two diagonals of a rhombus is 2 cm. If the area of the rhombus is 24 sq. cm, find the length of the side of the rhombus.

12. The length of a rectangular plot is 4 m more than its breadth. If the area of the plot is 320 sq. m, find the length and breadth of the plot.

13. A rectangular looking glass 18 cm by 12 cm has a wooden frame of uniform width. If the area of the frame is equal to that of the looking glass, find the dimensions of the framed glass.

14. A sum of Rs. 200 lent out at compound interest compounded annually amounts to Rs. 220.50 in two years. What is the rate of interest ?

15. A journey of 224 km. from Imphal to Jiribam takes $2\frac{2}{5}$ hours less by a car than by a passenger bus. If the average speed of the bus is 12 km/hr less than that of the car, find the average speed of the bus and that of the car.

16. A motor boat whose speed is 18 km/hr in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

17. The speed of a boat in still water is 8 km/hr. It can go 15 km upstream and 22 km downstream in 5 hours. Find the speed of the stream.

ANSWER

1. 5 and 6	**2. 6 or $\frac{1}{6}$**	**3. 3 and 4, or 28 and –21**
4. 10 and 12	**5. 9 and 11, or –9 and –11**	**6. 39**
7. 45	**8. 3 cm, 4 cm and 5 cm; 6 sq.cm**	
9. 5cm, 12 cm	**10. 8 cm and 25 cm**	**11. 5 cm**
12. 16 m and 20 m	**13. 24 cm × 18 cm**	**14. 5%**
15. Speed of the bus = 28 km/hr; speed of the car = 40km/hr		
16. 6 km/hr	**17. 3 km/hr**	

SUMMARY

In this chapter, you have studied the following points :

1. An equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ is called a quadratic equation in the variable x .

2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. There can also be roots which are not real.

3. A root of the quadratic equation $ax^2 + bx + c = 0$, is a zero of the quadratic polynomial $ax^2 + bx + c$ and vice-versa.

4. **A quadratic equation can be solved by two methods namely**

(i) method of factorisation and

(ii) method of completing the perfect square.

5. The roots of the quadratic equation $ax^2 + bx + c = 0$ are given by

$\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$,provided

$b^2 - 4ac \geq 0$ (for real roots)

This formula to find the roots of a quadratic equation is known as the quardratic formula.

6. **The quadratic equation $ax^2 + bx + c = 0$ has**

(i) two distinct real roots , if $b^2 - 4ac > 0$

(ii) two equal real roots , if $b^2 - 4ac = 0$, and

(iii) no real roots, if $b^2 - 4ac < 0$

7. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$\alpha + \beta$ = sum of the roots $= \dfrac{-b}{a} = \dfrac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$

and $\alpha\beta$ = product of the roots $= \dfrac{c}{a} = \dfrac{\text{Constant term}}{\text{Coefficient of x}^2}$

8. The quadratic equation whose roots are given is of the form

$x^2 -$ (sum of the roots)x+product of the roots =0

****

CHAPTER

6 ARITHMETIC PROGRESSION (AP)

6.1 Introduction

In our daily life, we come across many things around us which follow a certain pattern, such as the petals of a flower, the holes of a honey comb, the scales of a fish, the grains on a maize cob, the fingerprint of a person , the stripes of a zebra etc. Let us now see the following examples in our day-to-day life which involve certain patterns of numbers.

(i) In a certain savings scheme, the amount doubles itself after every 5 years. The maturity amount (in ₹) of an investment of ₹ 5000 after 5 years, 10 years, 15 years, 20 years, etc. will be respectively 10000, 20000, 40000, 80000 etc.

(ii) Suresh has been selected for a job which offers a starting monthly salary of ₹ 5000, with an annual increment of ₹ 300. His monthly salary (in ₹) for the 1st year, 2nd year, 3rd year, 4th year etc. will be respectively 5000, 5300, 5600, 5900 etc.

(iii) Sunita invests a sum of ₹ 8000 at 5 % per annum simple interest . The interest (in ₹) of her investment at the end of the 1st year, 2nd year, 3rd year, 4th year etc. will be respectively 400, 800, 1200, 1600 etc.

In the examples above, we observe some patterns of numbers like

(A) 10000, 20000, 40000, 80000, ...(from (i))

(B) 5000, 5300, 5600, 5900, ...(from (ii))

and (C) 400, 800, 1200, 1600, ...(from (iii))

In each case, we have a list or succession of numbers following a specific rule. Such lists of numbers are called sequences.

The study of sequences assumes significance in the light of the many questions that we may be confronted with, like (in the examples) “ What is the maturity amount after 25 years?”, “What is the total amount that Suresh has recieved as salary upto the 10^{th} year ?”, “ How much amount does Sunita recieve as interest at the end of the 12^{th} year ?” etc.

A succession of numbers formed according to a specific rule is called a sequence.

The numbers in a sequence are called **terms** of the sequence. The terms are labelled as first term, second term, third term, fourth term etc. which are usually denoted by a_1, a_2, a_3, a_4,...respectively.

For example, in (A), we have

$a_1 = 10000$, $a_2 = 20000$, $a_3 = 40000$, $a_4 = 80000$, etc.

The number of terms of a sequence may be finite or infinite and accordingly the sequence is called finite or infinite. Usually in an infinite sequence, only the first few terms are given followed by dots. But the terms of a sequence follow a specific rule and this enables us to express the nth term (or the general term) a_n in terms of n where $n \in N$. For example, for sequence (A), we have

$$a_1 = 10000 = 10000 \times 1 = 10000 \times 2^{\circ} = 10000 \times 2^{1-1}$$
$$a_2 = 20000 = 10000 \times 2 = 10000 \times 2^1 = 10000 \times 2^{2-1}$$
$$a_3 = 40000 = 10000 \times 4 = 10000 \times 2^2 = 10000 \times 2^{3-1}$$
$$a_4 = 80000 = 10000 \times 8 = 10000 \times 2^3 = 10000 \times 2^{4-1}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Following the pattern, we see that

$$a_n = 10000 \times 2^{n-1}$$

Once we know a_n (in terms of n) of a sequence, we can find any term of the sequence.

Thus, we have

$$a_6 = 10000 \times 2^{6-1} = 320000,$$
$$a_{11} = 10000 \times 2^{11-1} = 10000 \times 2^{10} \text{ etc.}$$

Also, we have the answer to one of the three questions.

Now, the maturity amount after 25 years

$= a_{25} = 10000 \times 2^{25-1} =$ ₹ 10000×2^{24}

[Can you find the n^{th} term for the sequences (B) & (C)?]

Sequences are given special names according to the rules they follow. In some sequences, the succeeding terms are obtained by adding a fixed number as in the case of (B) and (C), in other by multiplying with a fixed number as in the case of (A) and so on.

In this chapter, we shall be dealing with sequences such as the ones in (B) and (C), in which the succeeding terms can be obtained by adding a fixed number to the preceding terms. Such sequences are called **Arithmetic Progression (AP).**

Further in (B), we observe that

$a_2 - a_1 = 5300 - 5000 = 300$

$a_3 - a_2 = 5600 - 5300 = 300$

$a_4 - a_3 = 5900 - 5600 = 300$

... ...

... ...

Thus, for all $n \in N$, $a_{n+1} - a_n = 300$ which is a constant.

Also, in (C), you will observe that

for all $n \in N$, $a_{n+1} - a_n = 400$ which is a constant.

Indeed, this property holds in the case of any AP. We use this to define Arithmetic progression.

6.2 Arithmetic Progression

Definition : A sequence $a_1, a_2, a_3, a_4, ..., a_n, ...$ is called an Arithmetic progression (AP) if $a_{n+1} - a_n$ = constant for all $n \in N$.

The constant difference usually denoted by d is called the common difference of the AP.

In other words, an arithmetic progression is a sequence in which each term other than the 1st, is obtained by adding a fixed number (common difference) to the preceding term.

An AP is completely determined , if we know the first term and the common difference. In fact, if a is the first term and d, the common difference of an AP, then the AP is a, a + d, a + 2d, a + 3d, ...

Note : Throughout this chapter, the usual notations for the first term and the common difference of an AP are respectively a and d unless otherwise stated.

6.3 The n^{th} term of an AP

Let a be the first term and d be the common difference of an AP. Then, the AP is a, a+d, a+2d, a+3d, ...

Here, $a_1 = a = a + (1-1)d$

$a_2 = a + d = a + (2-1)d$

$a_3 = a + 2d = a + (3-1)d$

$a_4 = a + 3d = a + (4-1)d$

... ...

... ...

Looking at the pattern, we can say that

$a_n = a + (n-1)d.$

Thus, for an AP whose first term is a and the common difference is d,

the n^{th} term (or the general term) $a_n = a+(n-1)d$

Example 1. Which of the following sequences are AP. In case of an AP, find the first term a and the common difference d and write the next three terms.

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ ... **(ii)** 4, 2, 0, –2, –4 ...

(iii) $\frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}$ ... **(iv)** 1, 2, 3, 5, 7 ...

Solution : **(i)** We have, $a_1 = 1,\ a_2 = \frac{1}{2}, a_3 = \frac{1}{3},\ a_4 = \frac{1}{4},\ a_5 = \frac{1}{5}$ ...

Now, $a_2 - a_1 = \frac{1}{2} - 1 = -\frac{1}{2}$

$a_3 - a_2 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$

As, $a_3 - a_2 \neq a_2 - a_1$, the given sequence is not an AP.

(ii) We have , $a_1 = 4, a_2 = 2, a_3 = 0,\ a_4 = -2, a_5 = -4$ ...

Now, $a_2 - a_1 = 2 - 4 = -2$

$a_3 - a_2 = 0 - 2 = -2$

$a_4 - a_3 = -2 - 0 = -2$

$a_5 - a_4 = -4 - (-2) = -2$

... ...

... ...

Here, $a_{n+1} - a_n = -2$ for all $n \in N$

$\therefore$ the given sequence is an AP.

Then, a = 4 and d = – 2

And, the next three terms are: $-4 + (-2) = -6, -6 + (-2) = -8$

and $-8 + (-2) = -10$

(iii) We have, $a_1 = \frac{1}{10}, a_2 = \frac{1}{5}, a_3 = \frac{3}{10}, a_4 = \frac{2}{5}, a_5 = \frac{1}{2}, ...$

Now, $a_2 - a_1 = \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$

$a_3 - a_2 = \frac{3}{10} - \frac{1}{5} = \frac{1}{10}$

$a_4 - a_3 = \frac{2}{5} - \frac{3}{10} = \frac{1}{10}$

$a_5 - a_4 = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$

... ...

... ...

Here, $a_{n+1} - a_n = \frac{1}{10}$ for all $n \in N$.

$\therefore$ the given sequence is an AP.

Then, $a = \frac{1}{10}$ and d=$\frac{1}{10}$.

And, the next three terms are : $\frac{1}{2} + \frac{1}{10} = \frac{3}{5}, \frac{3}{5} + \frac{1}{10} = \frac{7}{10}$

and $\frac{7}{10} + \frac{1}{10} = \frac{4}{5}$

(iv) We have , $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 7$................

Now, $a_2 - a_1 = 2 - 1 = 1$

$a_3 - a_2 = 3 - 2 = 1$

$a_4 - a_3 = 5 - 3 = 2$

As, $a_4 - a_3 \neq a_3 - a_2$, the given sequence is not an AP.

Example 2. Write the n[th] term and also the first four terms of the AP whose first term a and the common difference 'd' are given as follows :

(i) a = 5, d = 3 **(ii)** a = 0, d = –2

(iii) a = –10, d = 0.5 **(iv)** a = 6, d = $\frac{1}{5}$

Solution : **(i)** We have, a = 5, d = 3

$\therefore\ a_n = a + (n-1)\,d$

$= 5 + (n-1) \times 3$

$= 3n + 2$

So, $a_1 = 3 \times 1 + 2 = 5$, $a_2 = 3 \times 2 + 2 = 8$, $a_3 = 3 \times 3 + 2 = 11$ and

$a_4 = 3 \times 4 + 2 = 14$.

Thus, the first four terms are 5, 8, 11 and 14.

[The first four terms can also be obtained as

$a_1 = a = 5$

$a_2 = a + d = 5 + 3 = 8$

$a_3 = a + 2d = 5 + 2 \times 3 = 11$

and $a_4 = a + 3d = 5 + 3 \times 3 = 14$]

(ii) We have,

a = 0, d = – 2

$\therefore\ a_n = a + (n-1)\,d$

$= 0 + (n-1) \times (-2)$

$= 2 - 2n$

So, $a_1 = 2 - 2 \times 1 = 0$, $a_2 = 2 - 2 \times 2 = -2$,

$a_3 = 2 - 2 \times 3 = -4$, $a_4 = 2 - 2 \times 4 = -6$.

Thus, the first four terms are 0, – 2, – 4 and – 6.

(iii) We have,

$a = -10,\ d = 0.5 = \frac{1}{2}$

$\therefore\ a_n = a + (n-1)d$

$= -10 + (n-1) \times \frac{1}{2}$

$= \frac{-20 + n - 1}{2} = \frac{n - 21}{2}$

So, $a_1 = \frac{1-21}{2} = -10$,

$a_2 = \frac{2-21}{2} = \frac{-19}{2} = -9.5$, $a_3 = \frac{3-21}{2} = \frac{-18}{2} = -9$,

$a_4 = \frac{4-21}{2} = \frac{-17}{2} = -8.5$

Thus, the first four terms are –10, –9.5, –9 and –8.5.

(iv) We have,

$a = 6,\ d = \frac{1}{5}$

$\therefore\ a_n = a + (n-1)d$

$= 6 + (n-1) \times \frac{1}{5}$

$= \frac{30 + n - 1}{5} = \frac{n + 29}{5}$

So, $a_1 = \frac{1+29}{5} = 6$, $a_2 = \frac{2+29}{5} = \frac{31}{5}$, $a_3 = \frac{3+29}{5} = \frac{32}{5}$,

$a_4 = \frac{4+29}{5} = \frac{33}{5}$

Thus, the first four terms are 6, $\frac{31}{5}, \frac{32}{5}$ and $\frac{33}{5}$.

Example 3. Find the 11th term of the AP : 12, 9, 6, 3, 0, ...

Solution : Here, a = 12 and $d = a_2 - a_1 = 9 - 12 = -3$

$\therefore\ a_n = a + (n-1)d$

$= 12 + (n-1) \times (-3) = 12 - 3n + 3 = 15 - 3n$.

So, $a_{11} = 15 - 3 \times 11 = 15 - 33 = -18$.

Example 4. Is 72 a term of the AP : 7, 10, 13, 16,.........?

Solution : The given AP is 7, 10, 13, 16, ... ?

Here, a = 7 and d = 10– 7 = 3

If possible, let 72 be the n^{th} term of the AP.

Then, 72 = a + (n–1)d

$$\Rightarrow 72 = 7 + (n-1)\times 3$$

$$\Rightarrow 72 = 3n + 4$$

$$\Rightarrow 3n = 72 - 4 = 68$$

$$\Rightarrow n = \frac{68}{3}$$, which is not a natural number.

Therefore, 72 is not a term of the given AP.

Note: For 72 to be a term of the AP, n must be a natural number because there is no term such as $\left(\frac{68}{3}\right)^{th}$ term

Example 5. The n^{th} term of a sequence is given by $a_n = 4n + 5$. Show that the sequence is an AP.

Solution : We have, $a_n = 4n + 5$

$$\therefore \; a_{n+1} = 4(n+1) + 5 = 4n + 9$$

Now, $a_{n+1} - a_n = (4n+9) - (4n+5) = 4$

Thus, $a_{n+1} - a_n$ = constant for all $n \in N$.

$\therefore$ the given sequence is an AP with common difference 4.

Example 6. The 6^{th} and the 17^{th} terms of an AP are respectively 19 and 41. Find the first term and the common difference. Also, find the 25^{th} term.

Solution : Let a be the first term and d, the common difference.

Here, $a_6 = 19$

$$\Rightarrow a + 5d = 19 \qquad ....(1)$$

and $a_{17} = 41$

$$\Rightarrow a + 16d = 41 \qquad ...(2)$$

Solving (1) and (2), we get

a = 9 and d = 2

$\therefore$ first term is 9 and common difference is 2.

Also, $a_{25} = a + 24d$

$= 9 + 24 \times 2$

$= 9 + 48$

$= 57$

Example 7. How many two - digit numbers are divisible by 7 ?

Solution : The sequence of all the two - digit numbers divisible by 7 is :

14, 21, 28, ...,

This sequence is an AP.

Here, a = 14 and d = 21–14= 7. The greatest two-digit number divisible by 7 is 98.

Let 98 be the n^{th} term of the AP.

then, $a_n = 98$

$\Rightarrow \; a + (n-1)d = 98$

$\Rightarrow \; 14 + (n-1) \times 7 = 98$

$\Rightarrow \; (n-1) \times 7 = 84$

$\Rightarrow \; n - 1 = \frac{84}{7} = 12$

$\Rightarrow \; n = 12 + 1 = 13$

Therefore, there are 13 two digit numbers divisible by 7.

Example 8. If the p^{th} term of an AP is q and the q^{th} term is p, prove that its n^{th} term is (p+q–n).

Solution : Let a be the first term and d, the common difference of the AP.

Then, $a_p = q$

$\Rightarrow \; a + (p-1)d = q \qquad ...(1)$

and $a_q = p$

$\Rightarrow \; a + (q-1)d = p \qquad ...(2)$

Subtracting (2) from (1), we have

$(p-q)d = q - p$

$\Rightarrow \; d = \frac{q-p}{p-q} = -1$

And, from (1), we have

$a + (p-1) \times (-1) = q$

$\Rightarrow \; a = p - 1 + q = p + q - 1$

$$\therefore \quad n^{th}\text{ term} = a_n = a + (n-1)d$$
$$= (p+q-1) + (n-1)\times(-1)$$
$$= p+q-1-n+1$$
$$= p+q-n.$$

Example 9. Ramesh got a government job and started work with a monthly salary of ₹ 8000 and enjoying an annual increment of ₹ 400. Find his annual income in the 10^{th} year of service.

Solution : We have, the monthly salary (in ₹) of Ramesh in the first year, second year, third year, fourth year etc. are 8000, 8000 + 400, 8000 + 400 + 400, 8000 + 400 + 400 + 400 etc. i.e., 8000, 8400, 8800, 9200 etc. which form an AP.

Here, a = 8000 and d = 400

Now, the monthly salary (in ₹) in the tenth year

$$a_{10} = a + (10-1)d$$
$$= 8000 + 9\times 400$$
$$= 8000 + 3600$$
$$= 11600$$

$\therefore$ The annual income in the tenth year of service

= ₹ 11600×12

= ₹ 139200.

EXERCISE 6.1

1. **Which of the following sequences are AP ? In case of AP, find the first term a and the common difference d and write the next three terms.**

(i) $1^2, 2^2, 3^2, 4^2, 5^2, ...$ **(ii)** 8, 5, 2, –1, – 4, ...

(iii) $2, \frac{5}{2}, 3, \frac{7}{2}, 4, ...$ **(iv)** 0.6, 1.7, 2.8, 3.9, 5, ...

(v) 3,5, 7, 8, 10, ... **(vi)** –6, –1, 4, 9, 14, ...

2. **Write the n^{th} term and also the first four terms of the AP whose first term a and the common difference d are given as follows :**

(i) a = 2, d = 2 **(ii)** a = 6, d = – 4

(iii) a = –10, d = – 3 **(iv)** $a = \frac{3}{2}, d = -\frac{1}{2}$

(v) $a = 4.5, d = 0.5$ (vi) $a = -\frac{5}{3}, d = \frac{1}{3}$

3. **Find**

(i) the 11th term of the AP : 2, 5, 8, 11, 14, ...

(ii) the 15th term of the AP : 4, 1, –2, –5, –8, ...

(iii) the 20th term of the AP : $1, \frac{3}{2}, 2, \frac{5}{2}, 3, ...$

(iv) the 18th term of the AP : $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}, ...$

4. **(i)** Is 67 a term of the AP : 5, 8, 11, 14, 17, ...?

(ii) Is –204 a term of the AP : 10, 6, 2, – 2,– 6, ...?

5. **(i)** Which term of the AP : 4, 9, 14, 19, 24, ...is 124?

(ii) Which term of the AP : 84, 80, 76, 72, 68, ...is 0?

6. **Find the number of terms in each of the following APs :**

(i) 7, 10, 13, ..., 43 **(ii)** 4, 1, –2, ..., –86

7. The nth term of a sequence is $2 – 5n$. Is the sequence an AP? If so, find the first term and the common difference.

8. Show that the sequence whose nth term is $2n^2 + 3$ is not an AP.

9. Find the 15th term from the last term (towards the first term) of the AP : 3, 7, 11, ...,123.

10. In an AP, the 5th term is 7 and the 11th term is 10. Find the first term and the common difference.

11. The 6th and the 17th terms of an AP are 21 and 54 respectively. Find the 40th term.

12. The 15th term of an AP exceeds the 20th term by 8. Find the common difference.

13. How many two-digit numbers are divisible by 3 ?

14. If the mth term of an AP be $\frac{1}{n}$ and the nth term be $\frac{1}{m}$, then show that the (mn)th term is 1.

15. If a, b, c be the pth, qth and rth terms of an AP, prove that $a(q – r)+b(r–p)+c(p– q) = 0$.

16. A sum of ₹ 5000 is invested at 6% per annum simple interest. Calculate the interest at the end of each year and show that they form an AP. Also, find the interest at the end of the 25th year.

17. In an auditorium, the seats are so arranged that there are 8 seats in the first row, 11 seats in the second, 14 seats in the third etc. thereby increasing the number of seats by 3 every next row. If there are 50 seats in the last row, how many rows of seats are there in the auditorium?

ANSWER

1. **(i)** **The sequence is not an AP.**

(ii) **The sequence is an AP ; a = 8, d = –3; –7, –10, –13**

(iii) **The sequence is an AP ; a = 2,** $d = \frac{1}{2}; \frac{9}{2}, 5, \frac{11}{2}$

(iv) **The sequence is an AP;** $a = 0.6, d = 1.1; 6.1, 7.2, 8.3$

(v) **The sequence is not an AP**

(vi) **The sequence is an AP ;** $a = -6, d = 5; 19, 24, 29$

2. **(i)** $a_n = 2n;\ a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8$

(ii) $a_n = 10 - 4n;\ a_1 = 6,\ a_2 = 2,\ a_3 = -2,\ a_4 = -6$

(iii) $a_n = -7 - 3n;\ a_1 = -10,\ a_2 = -13,\ a_3 = -16,\ a_4 = -19$

(iv) $a_n = \frac{4-n}{2};\ a_1 = \frac{3}{2},\ a_2 = 1,\ a_3 = \frac{1}{2},\ a_4 = 0$

(v) $a_n = 4 + \frac{n}{2};\ a_1 = 4.5,\ a_2 = 5,\ a_3 = 5.5,\ a_4 = 6$

(vi) $a_n = \frac{n-6}{3};\ a_1 = -\frac{5}{3},\ a_2 = \frac{-4}{3},\ a_3 = -1,\ a_4 = \frac{-2}{3}$

3. **(i) 32** **(ii) –38** **(iii)** $\frac{21}{2}$ **(iv) 35**$\sqrt{2}$

4. **(i) No** **(ii) No**

5. **(i) 25th term** **(ii) 22nd term**

6. **(i) 13** **(ii) 31**

7. **The sequence is an AP; first term = –3, common difference = –5**

9. **67**

10. **First term = 5, common difference =** $\frac{1}{2}$

11. **123**

12. $-\frac{8}{5}$

13. **30**

16. **₹ 7500**

17. **15**

6.4 Sum of first n terms of an AP

Let us try to find the sum of the natural numbers from 1 to 100. Carl Friedrich Gauss (1777 - 1855), as a young student, astonished his mathematics instructor by finding the sum in a matter of seconds. Let us see how he did it.

Let $S = 1+2+3+ \ ... + 99 + 100$

Writing the numbers in the reverse order, we get

$S = 100+99+ \ ... + 3+2+1$

Adding these two, we have

$2S = 101+101+101+ \ ... +101+101$ (100 times)

$= 101 \times 100$

$$\Rightarrow S = \frac{101 \times 100}{2} = 5050$$

The technique enables us to find the sum with relative ease. We shall use the same technique to find the sum of the first n terms of an AP.

Let a be the first term and d , the common difference of an AP. Also, let S_n denote the sum of the first n terms of the AP.

Then,

$$S_n = a + (a + d) + ... + [a + (n - 2)d] + [a + (n - 1)d] ...(1)$$

Writing the terms in the reverse order, we have

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + (a + d) + a \ ...(2)$$

Adding (1) and (2), we have

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + ... + [2a + (n - 1)d + [2a + (n\text{-}1)d]$$

(n times)

$$\Rightarrow 2S_n = n[2a + (n - 1)d]$$

$$\Rightarrow \mathbf{S_n = \frac{n}{2}[2a + (n - 1)d]}$$

Also, we have

$$S_n = \frac{n}{2}[a + a + (n - 1)d]$$

$$\Rightarrow S_n = \frac{n}{2}[a + a_n]$$

And, if the AP is finite with only n terms,

then $a_n = l =$ last term

By then, $\mathbf{S_n = \frac{n}{2}[a + l]}$

Example 10. Find the sum of the first 25 terms of the AP : 20, 17, 14, 11, 8, ...

Solution : Here, a = 20 and d = 17 –20 = –3

And, we have $S_n = \frac{n}{2}[2a + (n-1)d]$

Putting n = 25, we have

the required sum $= S_{25} = \frac{25}{2}[2\times 20 + (25-1)\times(-3)]$

$= \frac{25}{2}(40 - 72)$

$= \frac{25}{2}(-32)$

$= 25\times(-16)$

$= -400$

Example 11. Find the sum of all the terms of the finite AP : 2, 5, 8, ..., 179, 182.

Solution : Here, a = 2 and d = 5 –2 = 3

Let there be n terms in the AP.

Then, $a_n = 182$

$\Rightarrow a + (n-1)d = 182$

$\Rightarrow 2 + (n-1)\times 3 = 182$

$\Rightarrow (n-1)\times 3 = 180$

$\Rightarrow (n-1) = \frac{180}{3} = 60$

$\Rightarrow n = 61$

$\therefore$ the required sum $= S_{61} = \frac{61}{2}[2 + 182]$ [using the formula, $S_n = \frac{n}{2}[a + l]$ for a finite AP]

$= \frac{61}{2}\times 184$

$= 61\times 92 = 5612$

Example 12. How many terms of the AP : 30, 28, 26, 24, ...must be taken so that their sum is 240. Explain the double answer.

Solution : Here, a = 30 and d = 28–30 = –2

Let n be the number of terms to be taken so that

$$S_n = 240$$

$$\Rightarrow \quad \frac{n}{2}[2a+(n-1)d] = 240$$

$$\Rightarrow \quad n[2\times 30+(n-1)\times(-2)] = 240\times 2$$

$$\Rightarrow \quad n(62-2n) = 480$$

$$\Rightarrow \quad 62n-2n^2 = 480$$

$$\Rightarrow \quad n^2-31n+240 = 0$$

$$\Rightarrow \quad (n-15)(n-16) = 0$$

$$\Rightarrow \quad n = 15 \text{ or } 16$$

The double answer is explained by the fact that the 16th term here is zero (as you can check out) and thus the sums of the first 15 and the first 16 terms are the same, each being 240.

Example 13. Find the sum of all the multiples of 6 between 100 and 200.

Solution : The sequence of the multiples of 6 between 100 and 200 is : 102, 108, 114, 120, ...,192, 198 which is an AP.

Here, a = 102 and d = $108-102 = 6$

Let 198 be the n^{th} term of the AP

Then, $a_n = 198$

$$\Rightarrow \quad a+(n-1)d = 198$$

$$\Rightarrow \quad 102+(n-1)\times 6 = 198$$

$$\Rightarrow \quad (n-1)\times 6 = 96$$

$$\Rightarrow \quad n-1 = \frac{96}{6} = 16$$

$$\Rightarrow \quad n = 17$$

$\therefore$ the required sum = $S_{17} = \frac{17}{2}[102+198]$

$$= \frac{17}{2}\times 300$$

$$= 17\times 150$$

$$= 2550$$

Example 14. Find the sum of (i) the first n odd natural numbers

(ii) the first n even natural numbers

Solution : (i) The sequence of the odd natural numbers is : 1, 3, 5, 7, 9, 11 ..., which is an AP.

Here, a = 1 and d = 3–1= 2

$$\therefore \text{ the required sum} = S_n = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[2\times1+(n-1)\times2]$$

$$= \frac{n}{2}\times 2n = n^2$$

(ii) The sequence of the even natural numbers is : 2, 4, 6, 8, 10, ..., which is an AP.

Here, a = 2 and d = 2.

$$\therefore \text{ the required sum} = S_n = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[2\times2+(n-1)\times2]$$

$$= \frac{n}{2}[2n+2]$$

$$= \frac{n}{2}\times 2(n+1)$$

$$= n(n+1).$$

Example 15. Show that the sequence whose n^th^ term is given by $a_n = 5 - 3n$, is an AP. And, find the sum of the first 15 terms of the AP.

Solution : Here, $a_n = 5 - 3n$

$$\therefore a_{n+1} = 5 - 3(n+1)$$

$$= 5 - 3n - 3 = 2 - 3n$$

Now, $a_{n+1} - a_n = (2-3n) - (5-3n)$

$= -3$ which is constant for all $n \in N$

$\therefore$ the sequence is an AP.

Then, $a = a_1 = 5 - 3\times1 = 2$

and $d = -3$

$\therefore$ the required sum $= S_{15}$

$$= \frac{15}{2}[2a + (15-1)d]$$

$$= \frac{15}{2}[2\times 2 + 14\times(-3)]$$

$$= \frac{15}{2}\times(-38)$$

$$= 15\times(-19)$$

$$= -285\cdot$$

Example 16. Find the sum of the first 25 terms of an AP whose third and seventh terms are respectively 3 and 5.

Solution : We have , $a_3 = 3$

$\Rightarrow a + 2d = 3$ ...(1)

and $a_7 = 5$

$\Rightarrow a + 6d = 5$ ...(ii)

Solving (i) and (ii), we have

$a = 2$ and $d = \frac{1}{2}$

$\therefore$ the required sum $= S_{25} = \frac{25}{2}\left[2\times 2 + (25-1)\times\frac{1}{2}\right]$

$$= \frac{25}{2}\times 16$$

$$= 25\times 8$$

$$= 200\cdot$$

Example 17. The sum of the first p, q, r terms of an AP are a, b, c respectively. Show that $\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0$

Solution : Let A be the first term and D be the common difference of the AP.

Then, $a = S_p$

$\Rightarrow a = \frac{p}{2}[2A + (p-1)D]$

$$\Rightarrow \frac{a}{p} = \frac{1}{2}[2A+(p-1)D] \qquad ...(i)$$

also, $b = S_q$

$$\Rightarrow b = \frac{q}{2}[2A+(q-1)D]$$

$$\Rightarrow \frac{b}{q} = \frac{1}{2}[2A+(q-1)D] \qquad ...(ii)$$

and $c = S_r$

$$\Rightarrow c = \frac{r}{2}[2A+(r-1)D]$$

$$\Rightarrow \frac{c}{r} = \frac{1}{2}[2A+(r-1)D] \qquad ...(iii)$$

Multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q)$$

$$= \frac{1}{2}[2A+(p-1)D](q-r) + \frac{1}{2}[2A+(q-1)D](r-p) + \frac{1}{2}[2A+(r-1)D](p-q)$$

$$= \frac{1}{2}[2A(q-r+r-p+p-q) + D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}]$$

$$= \frac{1}{2}[2A \times 0 + D \times 0]$$

$$= \frac{1}{2} \times 0$$

$$= 0$$

Hence shown.

Example 18. Anita put ₹ 50 into her daughter's money box on her first birthday and increased the amount by ₹ 10 every next birthday. Find the total amount in the money box by the time the daughter turns 25.

Solution : We have, the amount of money (in ₹) put in the money box on the first birthday, second birthday, third birthday etc. are respectively 50, 60, 70 etc. which form an AP.

Here, $a = 50$ and $d = 10$

Now, the required amount of money (in ₹)

$$= S_{25} = \frac{25}{2}\left[2a + (25-1)d\right]$$
$$= \frac{25}{2}\left[2\times 50 + 24\times 10\right]$$
$$= \frac{25}{2}(100 + 240)$$
$$= \frac{25}{2}\times 340$$
$$= 25\times 170$$
$$= 4250$$

EXERCISE 6.2

1. Find the sum of the following APs :

(i) 40, 36, 32, 28, ..., to 15 terms

(ii) 3, 8, 13, 18, ..., to 12 terms

(iii) $-2, -\frac{3}{2}, -1, -\frac{1}{2}$,..., to 25 terms

(iv) 4.5, 4.3, 4.1, 3.9, ..., to 14 terms

2. Find the sum of all terms of the following finite APs :

(i) 63, 61, 59, ..., 35

(ii) 2, 5, 8, ..., 182

(iii) –10, –2, 6, 14, ..., 78

(iv) $2, \frac{5}{3}, \frac{4}{3}$, 1, ..., –6

3. How many terms of the AP : 1, 6, 11, 16, ... must be taken so that their sum is 148?

4. Find the number of terms of the AP: 32, 28, 24, 20, ... of which the sum is 132. Explain the double answer.

5. Find the sum of

(i) the first n natural numbers

(ii) the first 200 natural numbers

(iii) the first 100 odd natural numbers

(iv) the first 100 even natural numbers.

6. Find the sum of all odd numbers between 100 and 200.

7. Find the sum of the first 25 multiples of 6.

8. Find the sum of all the multiples of 7 between 100 and 300.

9. If the 12^{th} term of an AP is –13 and the sum of the first four terms is 24, find the sum of the first 15 terms.

10. Find the sum of the first 22 terms of an AP whose 8^{th} and 16^{th} terms are respectively 37 and 85.

11. The sum of the first 7 terms of an AP is 10 and that of the next 7 terms is 17. Find the first term and the common difference of the AP.

12. Find the sum of the first 50 terms of an AP whose n^{th} term is 3 –2n.

13. Find the first term and the common difference of an AP if the sum of the first n terms is $\frac{n(5n+7)}{12}$.

14. In an AP, if the sum of the first m terms is equal to n and that of the first n terms is equal to m, then prove that the sum of the first (m+n) terms is –(m+n).

15. The sums of the first n, 2n, 3n terms of an AP are S_1, S_2, S_3 respectively. Show that $S_3 = 3(S_2 - S_1)$.

16. A man repays a debt of ₹ 4860 by paying ₹ 30 in the first month and then increases the amount by ₹ 15 every month. How long will it take him to clear the debt (Assume that no interest is charged) ?

17. A manufacturer of radio sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find the production in the first year and also the total production in 8 years.

ANSWER

1.	**(i) 180**	**(ii) 366**	**(iii) 100**	**(iv) 44.8**
2.	**(i) 735**	**(ii) 5612**	**(iii) 408**	**(iv) –50**
3.	**8**			

4. **6 or 11** ***Explanation:*** **Since d is negative, the terms go on diminishing and the 9^{th} term becomes zero after which the terms become negative. Thus, the positive and the negative terms from the 7^{th} term to the 11^{th} term cancel out each other.**

5.	**(i)** $\frac{n(n+1)}{2}$	**(ii) 20100**	**(iii) 10000**	**(iv) 10100**	
6.	**7500**	**7. 1950**	**8. 5586**	**9. –75**	**10. 1276**
11.	**First term = 1, common difference =** $\frac{1}{7}$				
12.	-2400	**13. First term = 1, common difference =** $\frac{5}{6}$			
16.	**2 years**	**17. Production in first year = 550 units, total production in 8 years = 5100 units.**			

SUMMARY

In this chapter, you have studied the following points.

1. A succession of numbers following a specific rule is called a sequence.

The numbers in the sequence are called the terms of the sequence. They are labelled as first term, second term, third term, fourth term etc. and are respectively denoted by a_1, a_2, a_3, a_4 etc.

2. A sequence is called finite or infinite according as the number of terms in the sequence is finite or infinite.

3. A sequence in which each term except the first term is obtained by adding a fixed number to the preceding term is called an Arithmetic progression (AP).

The fixed number is called the common difference of the AP.

4. A sequence a_1, a_2, a_3,... , a_n, ... is an AP if $a_{n+1} - a_n$ = constant for all $n \in N$.

The constant difference usually denoted by d is the common difference of the AP.

5. If a = first term and d = common difference of an AP, then the AP is a , a+d, a+2d, a +3d, ...

6. In an AP with first term a and common difference d, the n^{th} term (or the general term) is given by $a_n = a + (n-1)d$.

7. The sum of the first n terms of an AP is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a_n].$$

8. If an AP is finite with n terms, then the sum of all the terms is given by

$$S_n = \frac{n}{2}[a + l]$$, where $l = a_n$ = the last term.

****

CHAPTER

7

TRIANGLES

7.1 Introduction.

In earlier classes, you have studied about triangles and many of their properties. In Class IX, you have studied about the congruence of triangles. Recall that two plane figures are congruent if they have the same shape and the same size. In our day-to-day life, we often come across figures of the same shape but of different sizes, for example bangle of different sizes, different size photographs from the same negative etc. To construct a bridge, engineers usually draw a picture (called blue print) of a model, much smaller in size than the actual bridge. The bridge and the model are of the same shape but of different sizes. Two figures having the same shape but not neccessarily the same size are called similar figures. In this chapter, we shall first discuss about the similar figures, particularly, the similar triangles and then apply the knowledge of the similarity of triangles to obtain a simple proof of the Pythagoras Theorem which you have studied in the earlier class.

7.2. Similar Figures

In class IX, you have learnt that two line segments of equal length are congruent and two angles of equal measure are congruent. You have also learnt that two equilateral triangles with the same side lengths are congruent and two circles with the same radii are congruent.

The two circles **(Fig 7.1)** are of the same shape but of different radii. So, they are similar but not congruent.

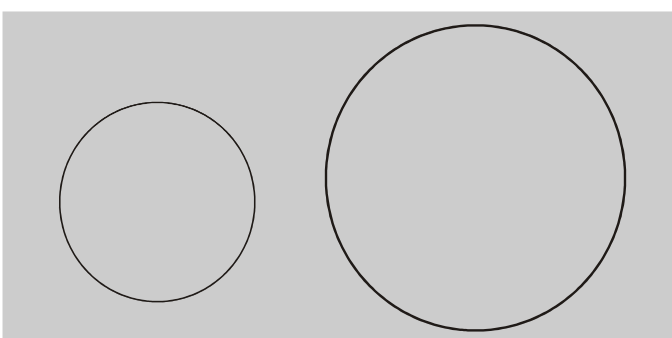

Fig. 7.1

By the same reasoning, the two equilateral triangles in figure 7.2 are similar but not congruent.

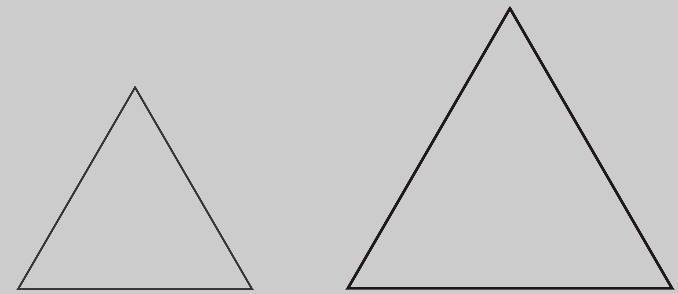

Fig. 7.2

From these examples, it is clear that all congruent figures are always similar but similar figures need not be congruent.

Our experience of solid objects enables us to recognise quite easily when two articles of different sizes have, as we say, 'the same shape". We shall now investigate what is involved in the question of shape. Draw a quadrilateral ABCD (of no special shape) on paper. Suppose we wish to make an enlarged version of the quadrilateral say , exactly two times in size. Let us takes a suitable centre say O (Fig. 7.3) and scale factor (side ratio) 2. The enlargement is done by pushing out radially from the centre O: e.g. A' is on the line OA such that $OA' = 2 \times OA$.

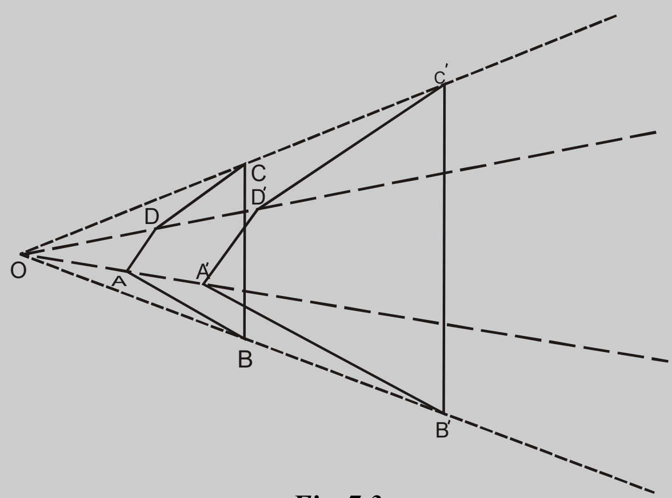

Fig 7.3

In a similar way, take the points B', C', D'. Then, $A'B'C'D'$ is the enlarged copy of ABCD. Compare $A'B'C'D'$ with ABCD. You will observe that $A'B'=2AB, B'C'=2BC, \quad C'D'=2CD, D'A'=2DA$ and $\angle A = \angle A', \angle B = \angle B'$, $\angle C = \angle C', \angle D = \angle D'$ i.e. not only are the sides in the larger figure exactly two times the original sides but also all the corresponding angles are equal in both the figures. This is the essence of the similarity of two quadrilaterals and in general of two polygons. Thus:

Two polygons having the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio.

Now, to understand the similarity of two polygons clearly, let us perform the following activity:

Activity :

Draw a quadrilateral ABCD on a paper and take two photocopies of this paper, one enlarged and other reduced in size. Name the enlarged quadrilateral as $A'B'C'D'$ and the reduced one as $A''B''C''D''$ (Fig. 7.4)

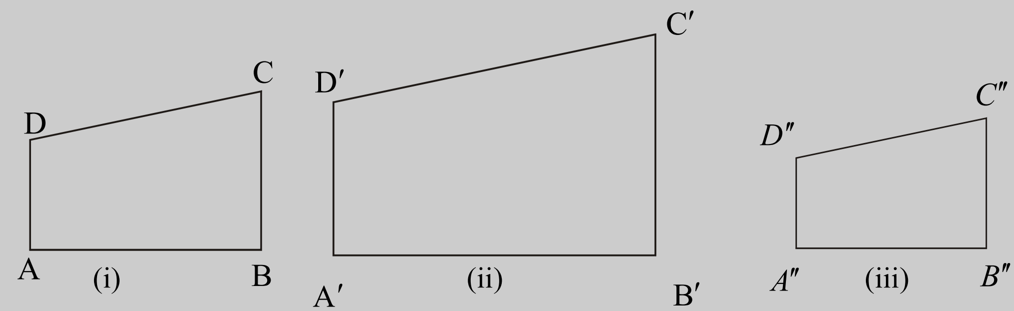

Fig. 7.4

Observe that ABCD, $A'B'C'D'$ and $A''B''C''D''$ have the same shape but are of different sizes i.e., they are similar. Compare ABCD with $A'B'C'D'$. Note that the vertex A' corresponds to vertex A, vertex B' corresponds to vertex B, vertex C' corresponds to vertex C and vertex D' corresponds to vertex D. Measuring the angles and the sides of the two quadrilaterals, you will get

(i) $\angle A = \angle A', \ \angle B = \angle B', \angle C = \angle C', \angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$

Again, comparing ABCD with $A''B''C''D''$. You will get

(i) $\angle A = \angle A'', \ \angle B = \angle B'', \angle C = \angle C'', \angle D = \angle D''$ and

(ii) $\dfrac{AB}{A''B''} = \dfrac{BC}{B''C''} = \dfrac{CD}{C''D''} = \dfrac{DA}{D''A''}$

This result emphasises the requirements for two polygons having the same number of sides to be similar i.e., (i) the corresponding angles should be equal and (ii) their corresponding sides should be in the same ratio. Thus, to examine the similarity of two polygons, we have to examine three things (1) whether the two polygons have the same number of sides or not, (2) whether the corresponding angles are equal or not and (3) whether the corresponding sides are in the same ratio or not. The polygons in Fig 7.5 are not similar because they do not have the same number of sides.

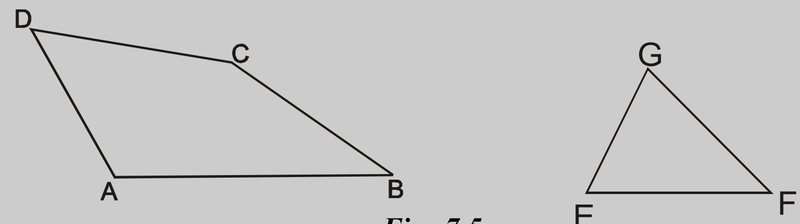

Fig .7.5

The quadrilaterals in Fig 7.6 are not similar as the corresponding angles are not equal.

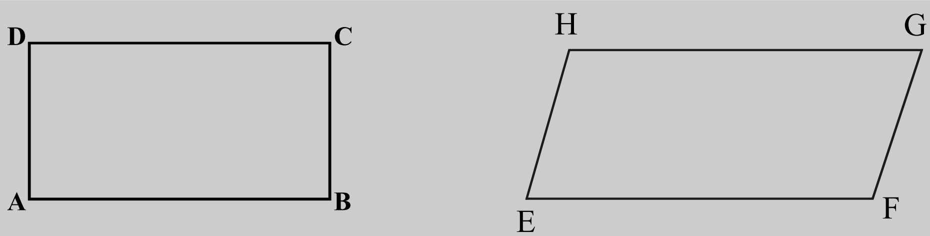

Fig 7.6

The rectangles in Fig 7.7 are not similar as the corresponding sides are not in the same ratio.

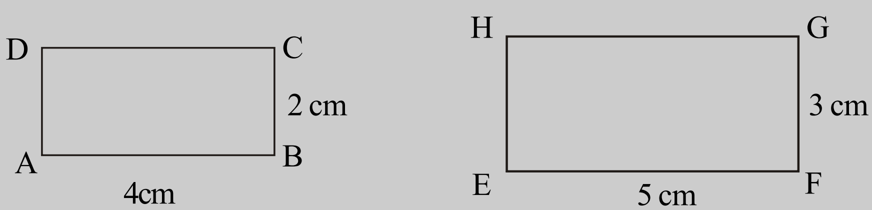

Fig. 7.7

EXERCISE 7.1

1. **Fill in the blanks using the correct word given in brackets :**
   - **(i)** All circles are ... (congruent, similar).
   - **(ii)** All circles of same radii are ... (congruent, not congruent).
   - **(iii)** All squares are ... (similar, congruent).
   - **(iv)** All ... triangles are similar (isosceles, equilateral).
   - **(v)** The reduced and enlarged photographs of an object made from the same negative are ... (similar, congruent).

2. State true or false :

(i) All similar figures are congruent.

(ii) All congruent figures are similar.

(iii) All triangles are similar.

(iv) All equilateral triangles are similar.

(v) All rectangles are similar.

(vi) All squares are not similar.

(vii) Two photographs of the same size of the same person, one at the age of 10years and the other at the age of 60 years are similar.

ANSWER

1.	**(i) similar**	**(ii) congruent**	**(iii) similar**	**(iv) equilateral**
	(v) similar			
2.	**(i) False**	**(ii) True**	**(iii) False**	**(iv) True**
	(v) False	**(vi) False**	**(vii) False.**	

7.3 Similarity of triangles

A triangle is a polygon having three sides. So, according to the definition of similarity of polygons, two triangles will be similar if

(i) Their corresponding sides are proportional and

(ii) Their corresponding angles are equal.

A famous Greek mathematician Thales (600 B.C) stated that 'the ratio of any two corresponding sides in equiangular triangles is always the same irrespective of their sizes'. To get this result, it is believed that he used a theorem called the Basic Proportionality Theorem or Thales Theorem. In view of this result, the two conditions mentioned in the definition of similar triangles are found to be dependent on each other. It means, if one condition holds good, the other holds good automatically. In other words, if any of the two conditions are fulfilled i.e, either the corresponding angles of two triangles are equal or thier corresponding sides are proportional, then the triangles are similar.

Let us perform the following activity.

Activity :

Draw an angle $\angle XAY$ and mark points P_1, P_2, P_3, D, P_4 and B on its arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3D = DP_4 = P_4B = 1$ unit (Fig. 7.8). Through B, draw a line intersecting arm AY at a point say, C. Through D, draw a line parallel to BC to intersect AC at E.

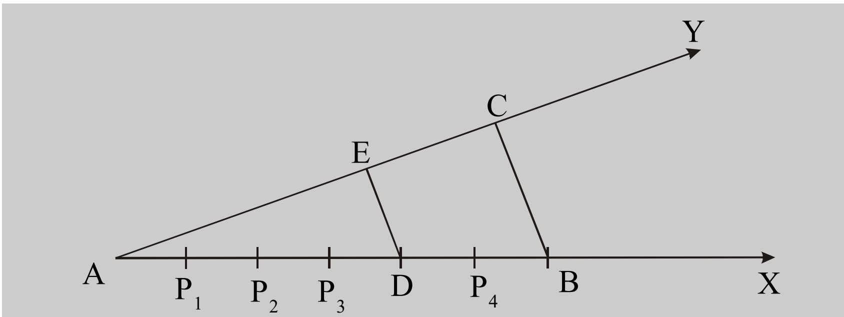

Fig 7.8

Now, $AD = AP_1 + P_1P_2 + P_2P_3 + P_3D = 4$ units

and $DB = DP_4 + P_4B = 2$ units

$\therefore \quad \frac{AD}{DB} = \frac{4}{2} = 2$

Measure AE and EC. You will find that AE= 2EC i.e. $\frac{AE}{EC} = 2$. Here, we have observed that in ΔABC; if $DE \parallel BC$, then we have $\frac{AD}{DB} = \frac{AE}{EC}$

We state this result as a theorem known as **Basic Proportionality Theorem** or **Thale's Theorem** as given below.

Theorem 7.1 If a line is drawn parallel to one side of a triangle intersecting the other two sides , then it divides the other two sides in the same ratio.

Given : A ΔABC and a line $DE \parallel BC$ intersecting AB at D and AC at E (Fig 7.9).

To prove : $\frac{AD}{DB} = \frac{AE}{EC}$

Construction : Join BE,CD and draw $CF \perp AB$, $BG \perp AC$ (Fig 7.9)

Proof : In fig 7.9, area of ΔDBC = area of ΔEBC ...(1) [being Δ^s standing on the same base and between the same parallels]

$\therefore$ area of ΔABC − area of ΔDBC

= area of ΔABC − area of ΔEBC

ie., area of ΔADC = area of ΔAEB ....(2)

From (1) & (2), we have

$$\frac{\text{area of } \Delta ADC}{\text{area of } \Delta DBC} = \frac{\text{area of } \Delta AEB}{\text{area of } \Delta EBC}$$

i.e. $$\frac{\frac{1}{2}AD.CF}{\frac{1}{2}DB.CF} = \frac{\frac{1}{2}AE.BG}{\frac{1}{2}EC.BG}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

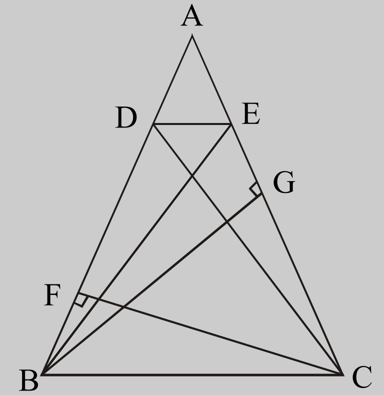

Fig 7.9

Is the converse of this theorem true ? That is, if a line divides any two sides of a triangle in the same ratio, is the line parallel to the third side? To examine this, let us perform the following activity.

Activity: Draw an angle $\angle XAY$ and mark points B_1, B_2, B_3, B on AX such that $AB_1 = B_1B_2 = B_2B_3 = B_3B$ (Fig. 7.10). Also mark points C_1, C_2, C_3 on AY such that $AC_1 = C_1C_2 = C_2C_3 = C_3C$. Join BC and B_3C_3.

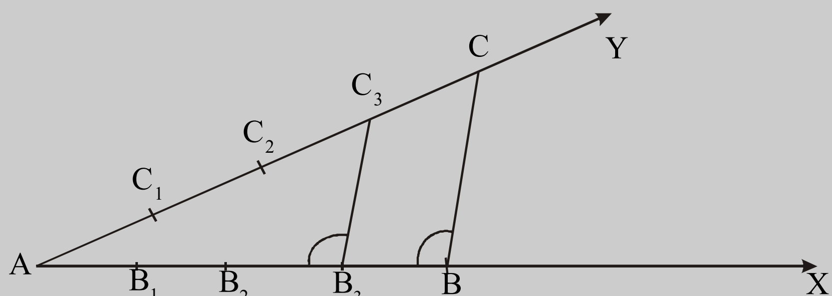

Fig . 7.10

Now, $AB_3 = AB_1 + B_1B_2 + B_2B_3 = 3B_3B$

i.e, $$\frac{AB_3}{B_3B} = \frac{3}{1}$$

Again, $AC_3 = AC_1 + C_1C_2 + C_2C_3 = 3C_3C$

i.e, $$\frac{AC_3}{C_3C} = \frac{3}{1}$$

$$\therefore \frac{AB_3}{B_3B} = \frac{AC_3}{C_3C}$$

Compare the angles AB_3C_3 and ABC. You will find that they are equal thereby showing $B_3C_3 \| BC$. You have observed that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side. This fact is stated and proved as a theorem given below and it is the converse of the basic proportionality theorem.

Theorem 7.2 If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.

Given : A line DE dividing the sides AB, AC of a ΔABC at D, E respectively such that

$$\frac{AD}{DB} = \frac{AE}{EC} \text{ (Fig. 7.11).}$$

To prove : DE || BC

Construction : Through D, draw DF || BC meeting AC at F (Fig 7.11).

Proof : By Basic Proportionality Theorem,

$$\frac{AD}{DB} = \frac{AF}{FC}$$

But, it is given that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore \frac{AF}{FC} = \frac{AE}{EC}$$

$$\Rightarrow \frac{AF}{FC} + 1 = \frac{AE}{EC} + 1$$

$$\Rightarrow \frac{AF+FC}{FC} = \frac{AE+EC}{EC}$$

$$\Rightarrow \frac{AC}{FC} = \frac{AC}{EC}$$

$$\Rightarrow FC = EC$$

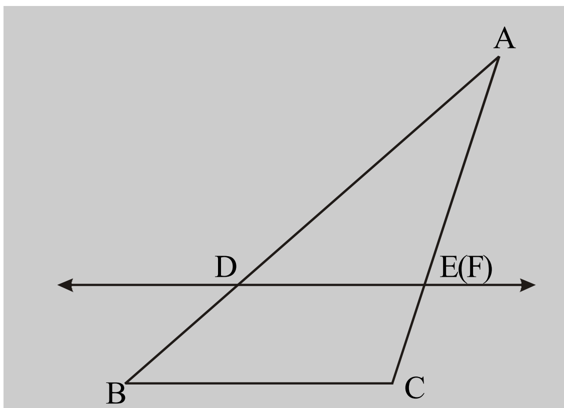

Fig. 7.11

It shows that F concides with E. But DF || BC by construction.

Hence DF || BC.

Example 1. If a line segment DE is drawn parallel to the side BC of a ΔABC meeting AB at D and AC at E, then prove that $\frac{AD}{AB} = \frac{AE}{AC}$ (Fig 7.12)

Solution :

Given : DE||BC in Fig 7.12

To Prove : $\frac{AD}{AB} = \frac{AE}{AC}$

Proof : It is given that DE||BC.

$\therefore$ by Basic Proportionality Theorem,

$\frac{AD}{DB} = \frac{AE}{EC}$

$\Rightarrow \frac{DB}{AD} = \frac{EC}{AE}$

$\Rightarrow 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$

$\Rightarrow \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$

$\Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$

$\therefore \frac{AD}{AB} = \frac{AE}{AC}$

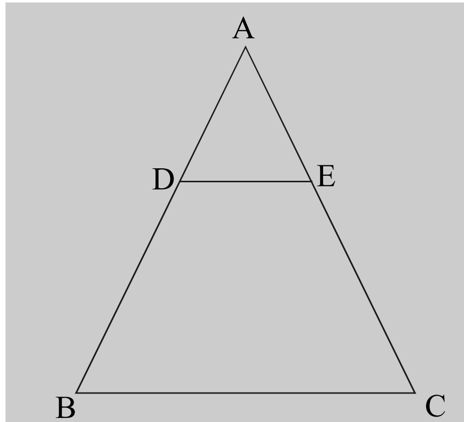

Fig. 7.12

Example 2. In Fig 7.13 , if EF || BC and FG || CD, prove that $\frac{AG}{AD} = \frac{AE}{AB}$

Solution : **Given :** EF || BC and FG || CD

To prove : $\frac{AG}{AD} = \frac{AE}{AB}$

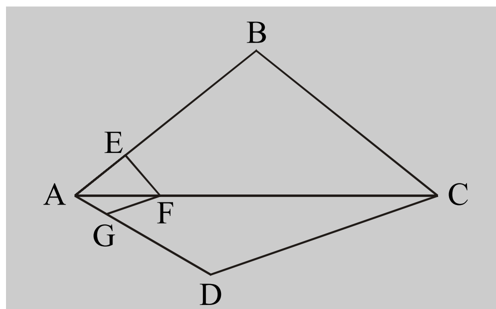

Fig. 7.13

Proof: In ΔABC,

EF || BC (Given)

$\therefore$ by Basic Proportionality Theorem,(vide Example 1.)

$\frac{AE}{AB} = \frac{AF}{AC}$ ...(i)

Again, in ΔACD,

FG || CD

$\therefore \frac{AF}{AC} = \frac{AG}{AD}$ ...(ii)

From (i) and (ii), we have

$$\frac{AE}{AB}=\frac{AG}{AD} \text{ or } \frac{AG}{AD}=\frac{AE}{AB}$$

Example 3. Prove that the line joining the mid - points of two sides of a triangle is parallel to the third side.

Soluition: **Given :** The line DE joining the mid point D of AB and the mid point E of AC of ΔABC (Fig 7.14).

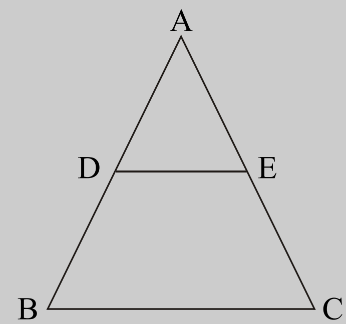

Fig 7.14

To prove : DE || BC

Proof : Since D is the mid point of AB,

$\therefore$ AD = DB

$$\Rightarrow \frac{AD}{DB}=1$$

Similarly, $\frac{AE}{EC}=1$

$$\therefore \frac{AD}{DB}=\frac{AE}{EC}$$

So, by the converse of the Basic Proportionality Theorem, DE || BC.

Example 4. Prove that the line segments, joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

Solution : **Given:** The line segments EF, FG, GH, HE formed by joining the mid points E,F,G, H respectively of AB, BC, CD, DA of a quadrilateral ABCD (Fig 7.15).

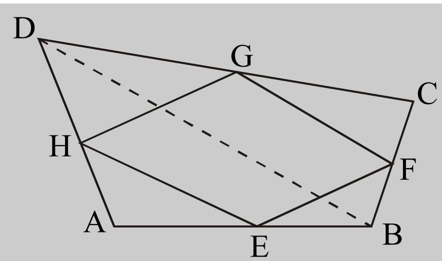

Fig. 7.15

To prove : EFGH is a parallelogram.

Construction : Join BD

Proof : In ΔABD, E and H are the mid - points of AB and AD respectively.

$\therefore$ EH || BD ....(1)

Again, in ΔCDB, F and G are the mid-points of CB and CD respectively.

$\therefore$ FG || BD ....(2)

From (1) and (2), we have

EH || FG

Similarly, EH || FG

$\therefore$ EFGH is a parallelogram.

Example 5 . The diagonals of a quadrilateral ABCD intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that ABCD is a trapezium.

Solution : Draw XO||ABmeeting AD at X. In ΔDAB, XO $\parallel$ AB

$\therefore$ by Basic Proportionality Theorem.

$$\frac{DX}{XA} = \frac{DO}{OB}$$

$$\Rightarrow \frac{XA}{DX} = \frac{OB}{DO} \quad ....(i)$$

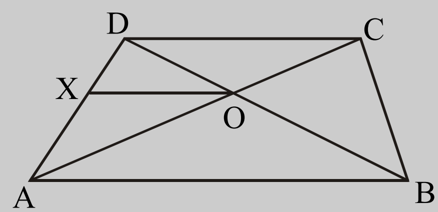

Fig. 7.16

But, it is given that $\frac{AO}{BO} = \frac{CO}{DO}$

or $\frac{AO}{CO} = \frac{BO}{DO}$

or $\frac{AO}{CO} = \frac{OB}{DO}$ ....(ii)

From (i) and (ii), $\frac{XA}{DX} = \frac{AO}{CO}$

$\therefore$ by the converse of the Basic Proportionality Theorem,
XO || DC.

But, by construction XO || AB.

$\therefore$ AB || DC

Hence, ABCD is a Trapezium.

EXERCISE 7.2

1. Prove that any line drawn parallel to the parallel sides of a trapezium divides the oblique sides proportionally.

2. Prove that the line drawn from the mid-point of one side of a triangle parallel to another side bisects the third side.

3. Any point X inside a ΔABC is joined to the vertices. From a point P on AX, PQ is drawn parallel to AB meeting XB at Q and QR is drawn parallel to BC meeting XC at R. Prove that PR || AC.

4. Let ABC be a triangle and D, E be two points on side AB such that AD=BE. If DP||BC and EQ || AC, where P and Q lie on AC and BC respectively, Prove that PQ || AB.

5. ABCD is a quadrilateral. P,Q,R and S are the points on the sides AB, BC, CD and DA respectively such that AP:PB=AS:SD=CQ:QB=CR:RD. Prove that PQRS is a parallelogram.

6. On three line segments OA, OB and OC, points L,M,N respectively are so chosen that LM || AB and MN || BC but neither L, M, N nor A, B, C are collinear. Show that LN || AC.

7. If three or more parallel lines are intersected by two transversals, prove that the intercepts made by them on the transversals are proportional.

8. **ABCD is a parallelogram and P is a point on the side BC.DP when produced meets AB produced at L, prove that**

(i) $\frac{DP}{PL} = \frac{DC}{BL}$ **(ii)** $\frac{DL}{DP} = \frac{AL}{DC}$

9. In a ΔABC, D and E are points on the sides AB and AC respectively, such that $AD \times EC = AE \times DB$. Prove that DE || BC.

10. The side BC of a triangle ABC is bisected at D, O is any point on AD. BO and CO produced meet AC and AB at E and F respectively and AD is produced to X so that D is the mid point of OX . Prove that AO: AX=AF:AB and show that FE || BC.

11. P is the mid-point of the side BC of a triangle ABC and Q is the mid-point of AP. If BQ, when produced, meets AC at L, prove that $LA = \frac{1}{3}CA$.

12. Two triangles ABC and DBC lie on the same side of BC. From a point P on BC, PQ is drawn parallel to BA and meeting AC at Q. PR is also drawn parallel to BD meeting CD at R. Prove that QR || AD.

7.4 Internal and External bisectors of an angle of a triangle.

You are already familiar with the notion of the internal and external bisectors of an angle of a triangle. In this section, we will derive some properties of internal and external bisectors of an angle of a triangle. First, let us learn the notion of internal and external division of a line segment. A point C on the segment AB is said to divide AB internally in the ratio m:n if $\frac{AC}{CB} = \frac{m}{n}$ i.e., if AC:CB = $m:n$ (Fig 7.17).

Fig. 7.17

Whereas, a point C on the line AB, not lying between A and B, is said to divide AB externally in the ratio m : n if $\frac{AC}{CB}=\frac{m}{n}$ (Fig 7.18). In the later case the point C , lies on AB produced or BA produced according as m>n or m<n.

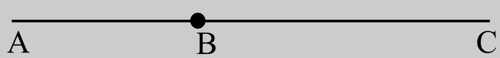

Fig. 7.18

Now, to examine the property of internal bisector of an angle of a triangle , let us perform the following activity.

Activity :

Draw an angle $\angle XAY$ and mark points P_1, P_2, P_3, and B on the arm AX such that $AP_1 = P_1P_2 = P_2P_3 = P_3B = 1$ unit (Fig 7.19). Also, mark points Q_1, Q_2, Q_3, Q_4, Q_5 and C on the arm AY such that $AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4Q_5 = Q_5C = 1$ unit.

Join BC. Here, we have

$$\frac{AB}{AC}=\frac{4}{6}=\frac{2}{3}$$

Draw bisector of $\angle XAY$ to intersect BC at D (Fig. 7.19). Measure lengths BD and DC and compute $\frac{BD}{DC}$.

You will find that $\frac{BD}{DC}=\frac{2}{3}$

$$\therefore \quad \frac{AB}{AC}=\frac{BD}{DC}.$$

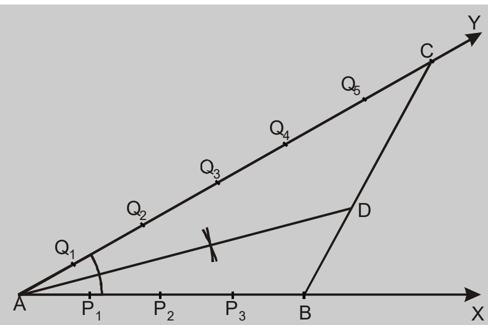

Fig 7.19

It means that the internal bisector of $\angle A$ of ΔABC divides the opposite side BC internally in the ratio AB:AC. This result is stated and proved as a theorem given below.

Theorem 7.3 The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the other two sides.

Given: Internal bisector AD of the $\angle A$ of ΔABC meeting BC at D (Fig 7.20).

To prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw CE parallel to DA meeting BA produced at E (Fig 7.20).

Proof : By construction,

DA || CE

$\therefore$ by Basic Proportionality theorem,

$\frac{BD}{DC} = \frac{BA}{AE}$ ---(1)

DA || CE and BE is a transversal.

$\therefore \angle BAD = \angle AEC$ ---(2)
(corresponding angles)

Again, DA || CE and AC is a transversal.

$\therefore \angle DAC = \angle ACE$ ---(3) (alternative angles)

But, AD bisects $\angle BAC$

$\therefore \angle BAD = \angle DAC$

$\therefore$ from (2) and (3)

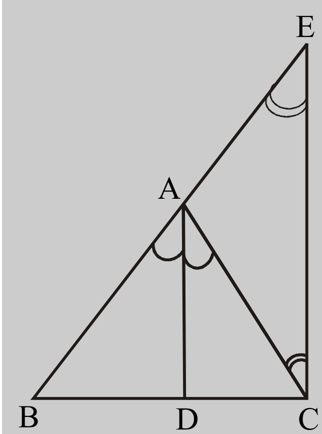

Fig. 7.20

$\angle AEC = \angle ACE$

$\Rightarrow$ AC=AE (sides opposite to equal angles of a triangle)

Putting this value in (1), we have

$\frac{BD}{DC} = \frac{BA}{AC}$

Hence, $\frac{BD}{DC} = \frac{AB}{AC}$.

Example 6. If the diagonal AC of a quadrilateral ABCD bisects both $\angle$A and $\angle$C, show that $\frac{AB}{AD} = \frac{BC}{CD}$.

Solution : **Given :** Diagonal AC of a quadrilateral ABCD bisecting both $\angle$A and $\angle$C (Fig. 7.21)

To prove : $\frac{AB}{AD} = \frac{BC}{CD}$

Construction : Join BD intersecting AC at O.

Proof : In ΔABD, AO is the internal bisector of $\angle$A.

$$\therefore \frac{BO}{OD} = \frac{AB}{AD}$$

$$\Rightarrow \frac{OB}{OD} = \frac{AB}{AD} \quad \text{---(1)}$$

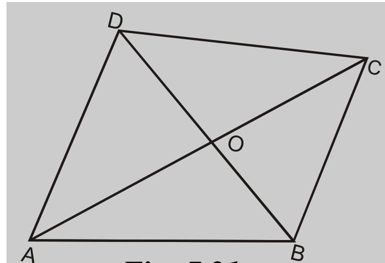

Fig. 7.21

In ΔBCD, CO is the internal bisector of $\angle C$

$$\therefore \frac{BO}{OD} = \frac{BC}{CD}$$

$$\Rightarrow \frac{OB}{OD} = \frac{BC}{CD} \quad \text{---(2)}$$

From (1) and (2), we have

$$\frac{AB}{AD} = \frac{BC}{CD}.$$

Example7. **O is any point inside a triangle ABC. The bisector of ∠AOB, ∠BOC and ∠COA meet the sides AB, BC and CA at points D, E and F respectively. Show that AD×BE×CF = DB×EC×FA.**

Solution : In $\Delta AOB, OD$ is the internal bisector of $\angle AOB$

$$\therefore \frac{AD}{DB} = \frac{OA}{OB} \quad \text{---(1)}$$

In ΔBOC, OE is the internal bisector of $\angle BOC$.

$$\therefore \frac{BE}{EC} = \frac{OB}{OC} \quad \text{---(2)}$$

Again, in ΔAOC, OF is the internal bisector of $\angle AOC$.

$$\therefore \frac{CF}{FA} = \frac{OC}{OA}$$

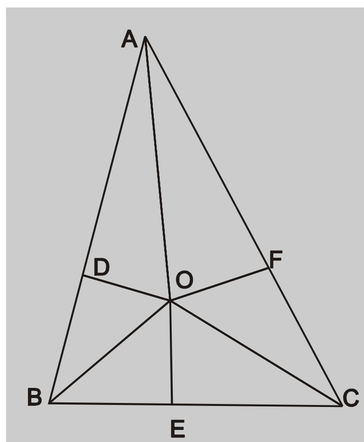

Fig. 7.22

From (1) , (2), (3), we have

$$\Rightarrow \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = \frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA}$$

$$\Rightarrow \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1$$

$$\Rightarrow AD \times BE \times CF = DB \times EC \times FA.$$

EXERCISE 7.3

1. AD is a median of ΔABC. The bisector of $\angle ADB$ and $\angle ADC$ meet AB and AC at E and F respectively. Prove that EF || BC.
2. If the bisector of an angle of a triangle bisects the opposite sides, prove that the triangle is isosceles.
3. In ΔABC, the bisector of $\angle B$ meets AC at D. A line PQ is drawn parallel to **AC** meeting AB, BC and BD at P, Q and R respectively. Show that
(i) $PR \times BQ = QR \times BP$ (ii) $AB \times CQ = BC \times AP$
4. If the medians of a triangle are the bisectors of the corresponding angles of the triangle, prove that the triangle is equilateral.
5. The bisectors of the angles $\angle B$ and $\angle C$ of a triangle ABC meet the opposite sides at D and E respectively. If ED || BC, prove that the triangle is isosceles.

7.5 Criteria for similarity of triangles

In section 7.3, we have already stated that two triangles are similar if

(i) their corresponding angles are equal and

(ii) their corresponding sides are in the same ratio.

In other words, two triangles ABC and DEF are similar, if

(i) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and **(ii)** $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

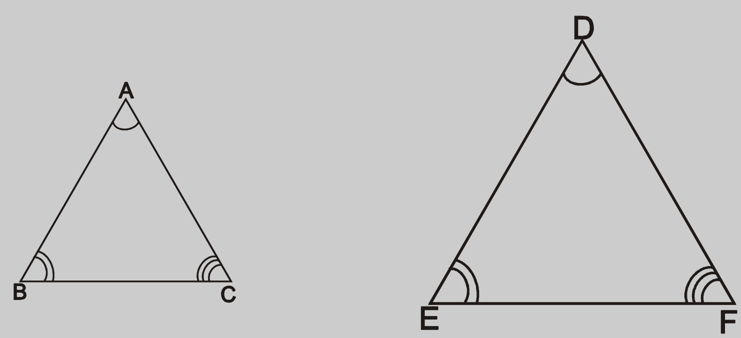

Fig. 7.23

Here, we see that A corresponds to D, B corresponds to E and C corresponds to F. At this situation we write $\Delta ABC \parallel \Delta DEF$ [read as "triangle ABC is similar to triangle DEF"] . As done in the case of congruency, for the similarity of the triangles also, correct correspondence of their vertices should be maintained. For example, in figure 7.25, we write $\Delta ABC \parallel \Delta DEF$ [but neither as $\Delta ABC \parallel \Delta EDF$ nor $\Delta ABC \parallel \Delta EFD$ nor $\Delta ABC \parallel \Delta FED$ nor $\Delta ABC \parallel \Delta FDE$ nor $\Delta ABC \parallel \Delta DFE$].

In this section, using the theorems discussed in earlier sections, we shall derive some criteria for similarity of triangles which in turn will imply that either of the above two conditions is sufficient to define the similarity of two triangles. Now, let us perform the following activity.

Activity:

Draw two line segments BC and EF of lengths say, 4cm and 6cm respectively. At B and C, construct angles XBC and YCB of some mesures say, 60^o and 45^o respectively.

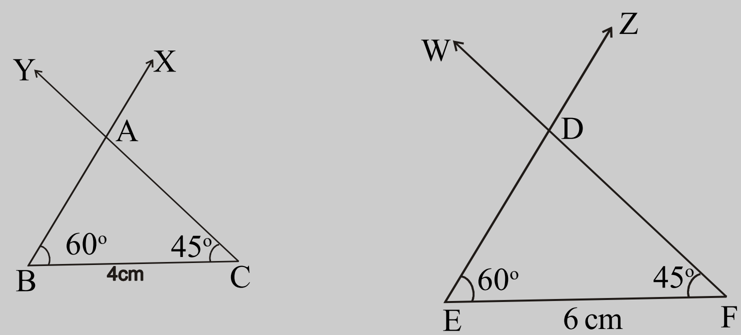

Fig. 7.24

Also, construct angles ZEF and WFE of 60^o and 45^o respectively. Suppose rays BX and CY intersect each other at A and rays EZ and FW intersect each other at D (Fig 7.24)

Now, in ΔABC, and ΔDEF

$\angle B = \angle E$, $\angle C = \angle F$, and so $\angle A = \angle D$

i.e. corresponding angles are equal

Observe that $\frac{BC}{EF} = \frac{4}{6} = \frac{2}{3}$

Now, measure AB, DE, CA and FD. You will find that

$$\frac{AB}{DE} = \frac{2}{3} = \frac{CA}{FD}$$

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

i.e. the corresponding sides are in the same ratio. It follows that $\Delta ABC \sim \Delta DEF$.

Repeating this activity by constructing several pairs of triangles having their corresponding angles equal, you will get the same result. Thus, we have the following criterion for the similarity of two triangles.

Theorem 7.4 (AAA Similarity): If the corresponding angles of two triangles are equal, then the triangles are similar.

Given : Two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To prove : $\Delta ABC \sim \Delta DEF$.

Construction : Take points M and N on $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ respectively such that AM=DE and AN=DF. Join MN.

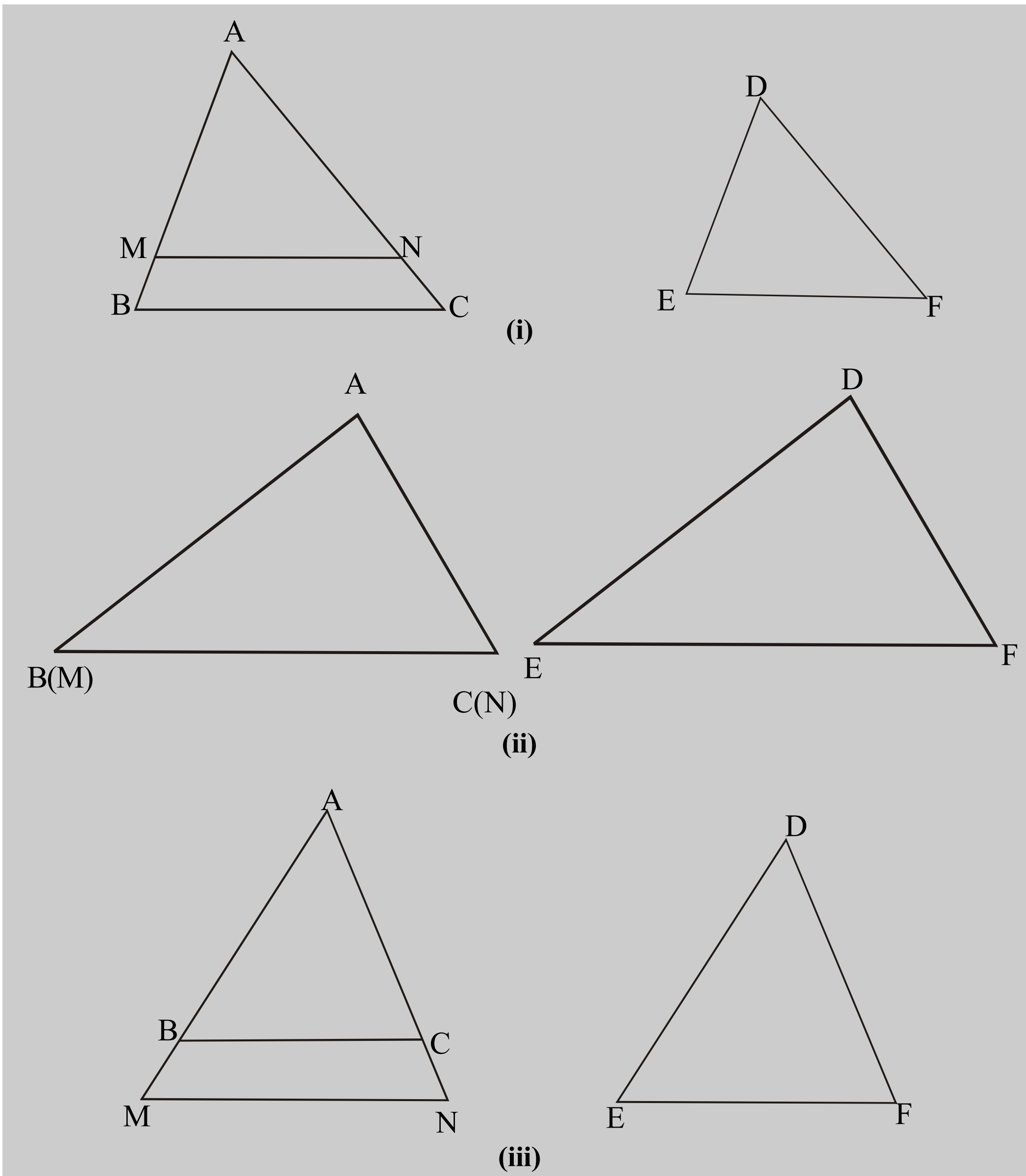

Fig. 7.25

Proof : Three cases arise :

Case I AB>DE.

Here, M lies in AB {Fig. 7.25(i)}.

In Δ^s AMN and DEF

AM= DE, AN=DF and $\angle A = \angle D$.

$\therefore$ by SAS axiom

$\Delta AMN \cong \Delta DEF$

It gives $\angle AMN = \angle DEF$

But $\angle DEF = \angle ABC$ (given)

$\therefore \angle AMN = \angle ABC$

$\Rightarrow$ MN || BC

$\Rightarrow \dfrac{AB}{AM} = \dfrac{AC}{AN}$ [by a deduction from the Basic Proportionality Theorem]

$\Rightarrow \dfrac{AB}{DE} = \dfrac{AC}{DF}$

Similarly, it can be shown that

$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Case II AB = DE

Here, M coincides with B [fig. 7.25(ii)]

Then, by ASA congruence theorem,

$\Delta ABC \cong \Delta DEF$

$\therefore$ AB = DE, BC = EF, AC = DF

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Case III AB<DE

Here, M lies on AB produced [fig 7.25 (iii)]

As in case I, it can be shown that BC || MN and ultimately.

we obtain

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Thus, in all possible cases we get

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

i.e. the corresponding sides of ΔABC and ΔDEF are proportional.As the corresponding angles are given to be equal, we can conclude that $\Delta ABC \sim \Delta DEF$.

Cor. (AA similarity)

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

The proof follows immediately after applying the angle sum property of a triangle.

In the proof of the AAA similarity, we have seen that if the corresponding angles of two triangles are equal, then the corresponding sides are in the same ratio.

Is the converse of this result true ? That is, if the sides of a triangle are respectively, proportional to the sides of another triangle, are the corresponding angles equal? To examine it, let us perform the following activity.

Activity :

Draw two triangles ABC and DEF such that
AB = 4cm, BC = 5cm, CA = 6cm and
DE = 6cm, EF = 7.5cm, FD = 9cm (Fig. 7.26)

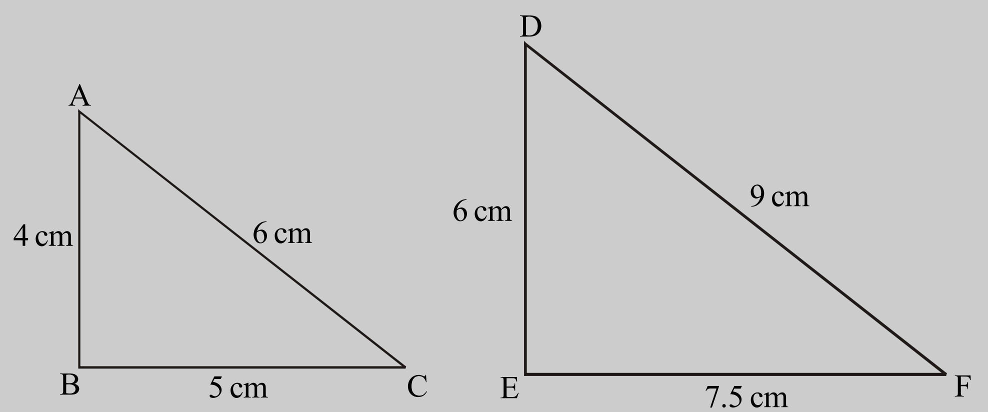

Fig. 7.26

Then , we have $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{2}{3}$ Measure $\angle A, \angle D, \angle B, \angle E, \angle C, \angle F$.

Observe that $\angle A=\angle D, \angle B=\angle E$ and $\angle C=\angle F$ i.e. the corresponding angles are equal. Repeating this activity by taking corresponding sides of two triangles in the same ratio, we will find that the corresponding angles are equal and hence they are similar. Let us now prove this result as a criterion of similarity of two triangles.

Theorem 7.5 (SSS similarity)

If the corresponding sides of two triangles are in the same ratio, then the triangles are similar.

Given : Two triangles ABC and DEF such that

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF} \quad \text{(Fig 7.25)}$$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Take points M and N on $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ respectively such that AM=DE and AN=DF. Join MN(Fig 7.25).

Proof : Three cases arise.

Case I AB>DE

Here, M lies in AB (Fig 7.25(i)).

Now, $\frac{AB}{DE}=\frac{AC}{DF}$

$\Rightarrow \frac{AB}{AM}=\frac{AC}{AN}$

$\Rightarrow \frac{AB}{AM}-1=\frac{AC}{AN}-1$

$\Rightarrow \frac{AB-AM}{AM}=\frac{AC-AN}{AN}$

$\Rightarrow \frac{MB}{AM}=\frac{NC}{AN}$

$\Rightarrow \frac{AM}{MB}=\frac{AN}{NC}$

$\therefore$ by the converse of the Basic Proportionality Theorem, we have,
MN || BC

$\therefore \angle AMN = \angle ABC$ and $\angle ANM = \angle ACB$ (corresponding angles)

So, by AA similarity

$\Delta ABC \sim \Delta AMN$

$\therefore \frac{AB}{AM}=\frac{BC}{MN}$

$\Rightarrow \frac{AB}{DE}=\frac{BC}{MN}$ ---(1)

But, $\frac{AB}{DE}=\frac{BC}{EF}$ ---(2)

From (1) and (2), $\frac{BC}{MN} = \frac{BC}{EF}$

$\therefore$ MN = EF

So, by SSS congruence Theorem,

$\Delta AMN \cong \Delta DEF$

Therefore, ΔAMN and ΔDEF are equiangular ie, the corresponding angles are same.

But, $\Delta ABC \sim \Delta AMN$ and so, they are equiangular. Therefore , ΔABC and ΔDEF are equiangular. It is also given that the corresponding sides of ΔABC and ΔDEF are in the same ratio. Hence $\Delta ABC \sim \Delta DEF$.

Case II AB=DE.

Here M coincides with B (Fig 7.25(ii))

Now, $1 = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

$AB = DE$, $BC = EF$, $AC = DF$.

So, by SSS congruence theorem,

$\Delta ABC \cong \Delta DEF$

Hence, $\Delta ABC \sim \Delta DEF$.

Case III AB<DE.

Here, M lies on AB produced (Fig 7.25(iii)). As in the case of (i), we can show that MN || BC and ultimately we can obtain $\Delta ABC \sim \Delta DEF$.

Thus, in all possible cases, $\Delta ABC \sim \Delta DEF$.

In view of the above two theorems, the following definitions of the similarity of two triangles can be given.

Definition 1. Two triangles are similar if the corresponding angles are equal.

Definition 2. Two triangles are similar if the corresponding sides are in the same ratio.

In class IX, we have learnt about various criteria for congruency of two triangles. Observe that, corresponding to the SSS congruency criterion there is SSS similarity criterion. Again, there is SAS congruency cretirion for congruency of two triangles. It is natural to look for a similarity criterion corresponding to SAS congruency criterion of triangles. To check the existence of such criterion, let us perform the following activity.

Activity :

Draw two triangles ABC and DEF such that

$AB = 8\ cm,\ \angle B = 30^\circ,\ BC = 4\ cm$ and $DE = 6\ cm,\ \angle E = 30^\circ,\ EF = 3\ cm$.(Fig 7.27).

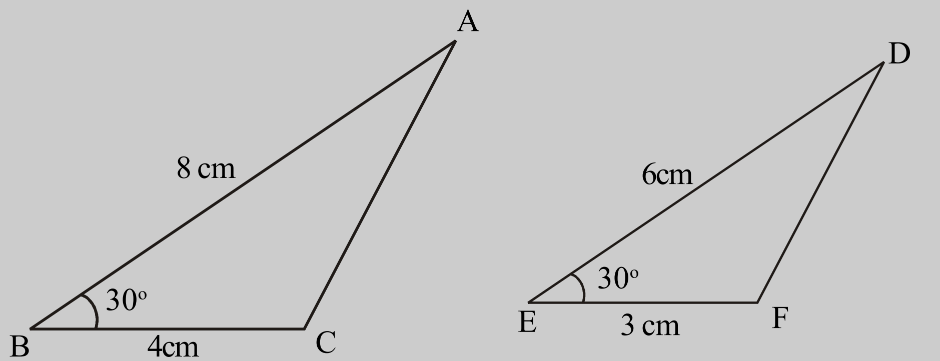

Fig. 7.27

Then, $\frac{AB}{DE} = \frac{4}{3} = \frac{BC}{EF}$ i.e. $\frac{AB}{EF} = \frac{BC}{EF}$ and $\angle B = \angle E$. That is, one angle of ΔABC equals one angles of ΔDEF and the sides including these angles are in the same ratio. Measure $\angle A$, $\angle C$, $\angle D$ and $\angle F$. Observe that $\angle A = \angle D$ and $\angle C = \angle F$. So, by AA similarity, $\Delta ABC \sim \Delta DEF$.

This observation is stated and proved as a theorem given below.

Theorem 7.6 **(SAS similarity)**

If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio, the triangles are similar.

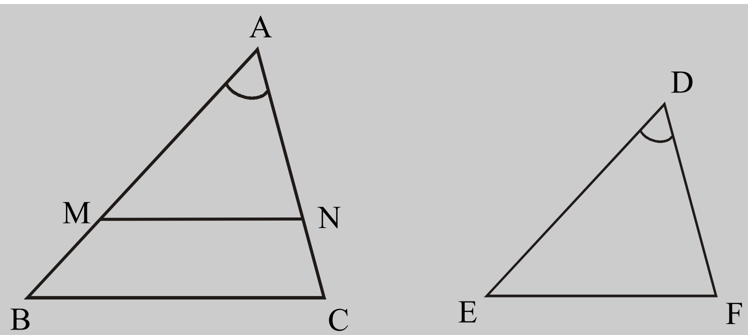

Fig. 7.28

Given : Δ^sABC and DEF in which $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$ (Fig 7.28).

To prove : $\Delta ABC \sim \Delta DEF$.

Proof : If AB=DE, then $\frac{AC}{DF} = \frac{AB}{DE} = 1$

$\Rightarrow AC = DF$

$\therefore$ by SAS axiom, $\Delta ABC \cong \Delta DEF$

So, $\Delta ABC \sim \Delta DEF$

If $AB \neq DE$, then one is greater than the other. Without loss of generality, we can take AB>DE. Take points M and N on AB and AC respectively such that AM=DE and AN=DF. Join MN(Fig 7.28)

In Δ^s AMN and DEF

$AM = DE, AN = DF$ and $\angle A = \angle D$.

$\therefore$ by SAS congruence axiom,

$\Delta AMN \cong \Delta DEF$

$\therefore \angle AMN = \angle DEF, \angle ANM = \angle DFE$ ----(1)

Now, $\frac{AB}{DE} = \frac{AC}{DF}$ (given)

$\therefore \frac{AB}{AM} = \frac{AC}{AN}$ (by construction AM=DE, AN=DF)

$\Rightarrow \frac{AM}{AB} = \frac{AN}{AC}$

So, MN || BC. (by converse of the Basic proportionality theorem)

$\therefore \angle AMN = \angle ABC$ and $\angle ANM = \angle ACB$

ie. $\angle DEF = \angle ABC$ and $\angle DFE = \angle ACB$ (using (1)

So, by AA similarity,

$\Delta DEF \sim \Delta ABC$

Hence, $\Delta ABC \sim \Delta DEF$.

Using the above criteria of similarity of two triangles, we can have the following theorem.

Theorem 7.7

If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

Given : Right ΔABC right angled at A and $AD \perp BC$ (Fig 7.29)

To prove : (i) $\Delta DBA \sim \Delta ABC$
(ii) $\Delta DAC \sim \Delta ABC$
(iii) $\Delta DBA \sim \Delta DAC$

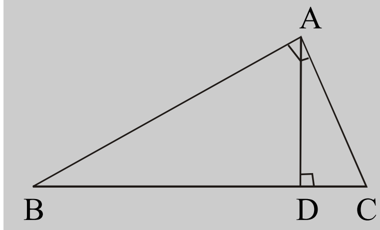

Fig. 7.29

Proof : (i) In Δ^s DBA and ABC ,
$\angle DBA = \angle ABC$ (same angle)
$\angle ADB = \angle CAB = 90^\circ$
$\therefore$ by AA similarity $\Delta DBA \sim \Delta ABC$

(ii) In ΔDAC and ΔABC
$\angle DCA = \angle ACB$ (same angle)
$\angle ADC = \angle BAC = 90^\circ$
$\therefore$ by AA similarity $\Delta DAC : \Delta ABC$.

(iii) In ΔADB ,
$\angle DBA + \angle DAB = 90^\circ$ $\left[\because \angle ADB = 90^\circ\right]$
But $\angle DBA + \angle DCA = 90^\circ$ $\left[\text{in } \Delta ABC,\ \angle BAC = 90^\circ\right]$
$\therefore$ $\angle DAB = \angle DCA$
Also, $\angle ADB = \angle CDA = 90^\circ$
$\therefore$ by AA similarity
$\Delta DBA \sim \Delta DAC$.

Example 8. D is a point on the side BC of a ΔABC such that $\angle ADC = \angle BAC$. Prove that $CA^2 = CB \times CD$.

Solution : Given : The point D on BC of ΔABC such that $\angle ADC = \angle BAC$.
To prove : $CA^2 = CB \times CD$.
Proof : Δ^s ABC and DAC ,
$\angle ACB = \angle DCA$ (same angle)
$\angle BAC = \angle ADC$ (given)
$\therefore$ by AA similarity
$\Delta ABC \sim \Delta DAC$

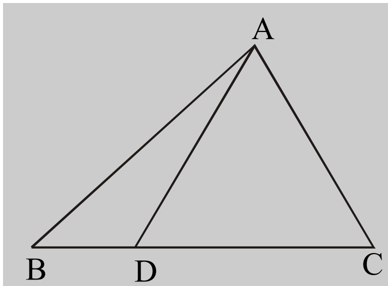

Fig. 7.30

$$\therefore \frac{AB}{DA} = \frac{BC}{AC} = \frac{CA}{CD}$$
$$\Rightarrow \frac{BC}{AC} = \frac{CA}{CD}$$
$$\Rightarrow \frac{BC}{CA} = \frac{CA}{CD}$$
$$\Rightarrow BC \times CD = CA^2$$
Hence, $CA^2 = CB \times CD$.

Example 9. Two vertical poles of heights a metres and b metres are p metres apart. Prove that the height of the point of intersection of the lines joining the top of each pole to the foot of the other is $\frac{ab}{a+b}$ metres.

Solution : Let AB and CD be the two vertical poles of heights a metres and b metres where A,C are the feet of the poles so that, AC= p metres.

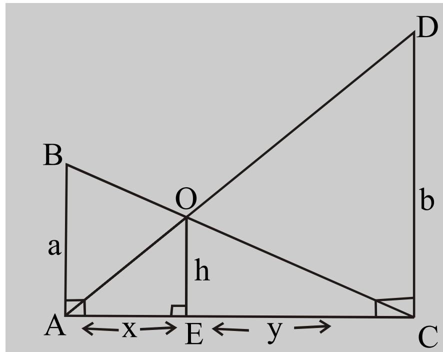

Fig. 7.31

Let AD and CB intersect at O (Fig 7.31). Draw $OE \perp AC$.

Let AE = x metres, CE = y metres. Also , let OE = h metres.

Then, p = AC = AE+EC = x+y

$\Rightarrow \quad p = x + y$ ---(1)

In Δ^s ABC and EOC

$\angle BAC = \angle OEC = 90^\circ$

$\angle ACB = \angle ECO$ (same angle)

$\therefore$ by AA similarity,

$\Delta ABC \sim \Delta EOC$

$\therefore \quad \frac{AB}{EO} = \frac{AC}{EC}$

$\Rightarrow \quad \frac{a}{h} = \frac{p}{y}$

$\Rightarrow \quad y = \frac{ph}{a}$ ---(2)

In Δ^s ACD and AEO ,

$\angle ACD = \angle AEO = 90^\circ$

$\angle DAC = \angle OAE$ (same angle)

$\therefore$ by AA similarity,

$\Delta ACD \sim \Delta AEO$

$\therefore \quad \frac{CD}{EO} = \frac{AC}{AE}$

$\Rightarrow \quad \frac{b}{h} = \frac{p}{x}$

$\Rightarrow \quad x = \dfrac{ph}{b}$ ---(3)

From (1), (2), (3) we have

$$p = \frac{ph}{b} + \frac{ph}{a}$$

$$\Rightarrow 1 = h\left(\frac{1}{b} + \frac{1}{a}\right)$$

$$\Rightarrow 1 = h\left(\frac{a+b}{ab}\right)$$

$$\Rightarrow h = \frac{ab}{a+b} \text{ metres.}$$

Hence, the required height is $\dfrac{ab}{a+b}$.

Example 10. BP and EQ are medians of the triangles ABC and DEF respectively. If $\dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{BP}{EQ}$, prove that $\Delta ABC \sim \Delta DEF$.

Solution : **Given :** medians BP and EQ of $\Delta^s ABC$ and DEF respectively.

Also, $\dfrac{BC}{EF} = \dfrac{AC}{DF} = \dfrac{BP}{EQ}$.

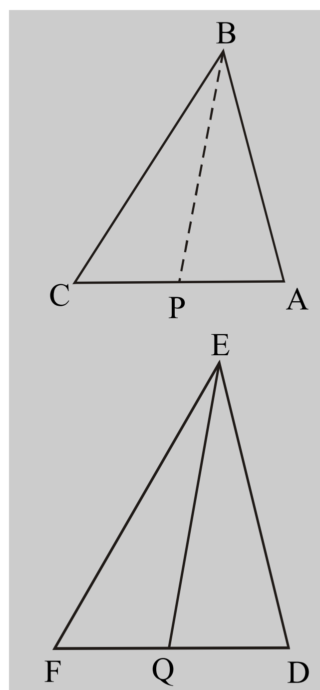

Fig. 7.32

To prove : $\Delta ABC \sim \Delta DEF$.

Proof : Since BP is the median of ΔABC,

2 AP = 2PC = AC.

Similarly, 2DQ = 2QF = DF.

It is given that

$$\frac{BC}{EF} = \frac{AC}{DF} = \frac{BP}{EQ}$$

$$\Rightarrow \frac{BC}{EF} = \frac{2\times PC}{2\times QF} = \frac{BP}{EQ}$$

$$\Rightarrow \frac{BC}{EF} = \frac{PC}{QF} = \frac{BP}{EQ}$$

$\therefore$ by SSS similarity,

$\Delta BPC \sim \Delta EQF$

$\therefore \angle BCP = \angle EFQ$

$\Rightarrow \angle BCA = \angle EFD$

But, $\frac{BC}{EF} = \frac{AC}{DF}$

$\therefore$ by SAS similarity

$\Delta ABC \cong \Delta DEF$.

Example 11. ABCD is a trapezium in which AB || DC. If the diagonals AC and BD intersect at P and ΔAPD : ΔBPC, prove that AD = BC.

Solution : **Given :** Trapezium ABCD in which AB || DC and ΔAPD : ΔBPC where P is the point of intersection of the diagonals AC and BD.

To prove : AD = BC

Proof : Since $\Delta APD \sim \Delta BPC$,

$\frac{AP}{BP} = \frac{DP}{CP} = \frac{AD}{BC}$ ------(1)

Again, it is given that AB || DC

$\therefore \angle CDP = \angle ABP$ (alternate angles)

$\angle DCP = \angle BAP$ (alternate angles)

$\therefore$ by AA similarity,

$\Delta ABP \sim \Delta CDP$

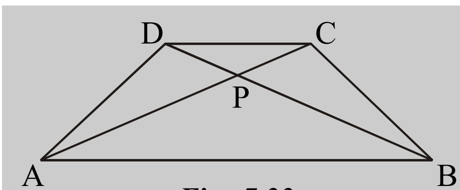

Fig. 7.33

$\therefore \frac{AP}{CP} = \frac{BP}{DP}$

$\Rightarrow \frac{AP}{BP} = \frac{CP}{DP}$

$\Rightarrow \frac{DP}{CP} = \frac{CP}{DP}$ [using (1)]

$\Rightarrow DP^2 = CP^2$

$\Rightarrow DP = CP$

$\therefore$ From (1), $\frac{AP}{BP} = \frac{AD}{BC} = \frac{DP}{CP} = 1$

$\therefore$ AD = BC.

Example 12. ABC is right triangle right angled at A and AD is drawn perpendicular to BC meeting it at D. Prove that AB² = BD × BC.

Solution : **Given :** A right ΔABC right angled at A and $AD \perp BC$.

To prove : $AB^2 = BD \times BC$

Proof : Since AD is the altitude of the rt ΔABC,
right angled A,

$\Delta ABC \sim \Delta DBA$

$\therefore \frac{AB}{DB} = \frac{BC}{BA} = \frac{AC}{DA}$

$\Rightarrow \frac{AB}{DB} = \frac{BC}{AB}$

$\Rightarrow AB^2 = DB \times BC$

$\therefore AB^2 = BD \times BC$

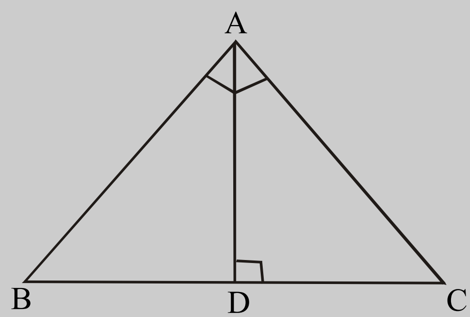

Fig. 7.34

EXERCISE 7.4

1. **If two triangles are similar, prove that the corresponding**
(i) medians are proportional
(ii) altitudes are proportional

2. ABCD is a parallelogram and E is the mid-point of AB. If F is the point of intersection of $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$, prove that BC = BF.

3. ABC is an isosceles triangle in which AB = AC and D is a point on AC such that $BC^2 = AC.CD$. Prove that BC=BD.

4. If in a triangle ABC, $AD \perp BC$ and $AD^2 = BD.DC$, prove that $\angle BAC = 90^\circ$.

5. Find the height of a vertical tower which casts a shadow of length 36m at the time when the shadow of a vertical post of length 5m is 3m. (Ans. 60m)

6. Prove that the line segments joining the mid-points of the sides of a triangle form four triangles, each of which is similar to the original triangle.

7. In ΔABC, DE is parallel to base BC with D on AB and E on AC. If $\frac{AD}{DB} = \frac{2}{3}$, find $\frac{BC}{DE}$. (Ans. $5/2$)

8. In a ΔABC, P and Q are points on AB and AC respectively such that PQ || BC. Prove that median AD bisects PQ.

9. ABC is an isosceles triangle in which AB = AC. AD is drawn perpendicular to BC. From a point E on CB produced, EF is drawn perpendicular to AC. Prove that $\Delta ADC \sim \Delta EFC$.

10. **ABC is a right triangle right angled at B and D is the foot of the perpendicular drawn from B on AC. If DM $\perp$ BC and DN $\perp$ AB where M, N lie on BC, AB respectively, prove that**

(i) $DM^2 = DN \times MC$ **(ii)** $DN^2 = DM \times AN$.

11. Through the vertex D of parallelogram ABCD, a line is drawn to intersect the sides BA produced and BC produced at E and F respectively. Prove that $\frac{AD}{AE} = \frac{BF}{BE} = \frac{CF}{CD}$

12. The perimeter of two similar trianglesABC and PQR are respectively, 72 cm and 48 cm. If PQ = 20 cm, find AB. (Ans: 30cm)

13. Two right triangles ABC and DBC are drawn on the same hypotenuse BC and on the same side of BC. If AC and BD intersect at P, prove that $AP \times PC = BP \times PD$.

14. If one angle of a triangle is equal to one angle of another triangle and the bisectors of these equal angles divide the opposite sides in the same ratio, prove that the triangles are similar.

15. If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding altitudes.

16. If two triangles are equiangular, prove that the ratio of the corresponding sides is same as the ratio of the corresponding angle bisector segments.

7.6 Areas of Similar Triangles

In this section, we will discuss a theorem concerning the ratio of areas of similar triangles.

Theorem 7.8 The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

Given : $\Delta ABC \sim \Delta DEF$

To prove : $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$

Construction : Draw $AM \perp BC$ and $DN \perp EF$ (Fig. 7.34).

Proof : $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{\frac{1}{2} BC \times AM}{\frac{1}{2} EF \times DN}$

$$= \frac{BC}{EF} \times \frac{AM}{DN} \qquad \text{-----(1)}$$

In Δ^s ABM and DEN,

$\angle ABM = \angle DEN$ $[\because \Delta ABC \sim \Delta DEF],$

$\therefore \angle B = \angle E\ ]$

$\angle AMB = \angle DNE = 90^\circ$

$\therefore$ by AA similarity

$\Delta ABM \sim \Delta DEN$

$$\therefore \frac{AB}{DE} = \frac{AM}{DN}$$

But $\frac{AB}{DE} = \frac{BC}{EF}$ $[\because \Delta ABC \sim \Delta DEF]$

$$\therefore \frac{AM}{DN} = \frac{BC}{EF}$$

Then, from (1)

$$\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{BC}{EF} \times \frac{BC}{EF} = \frac{BC^2}{EF^2}$$

Again, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

$$\Rightarrow \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

Hence, $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$.

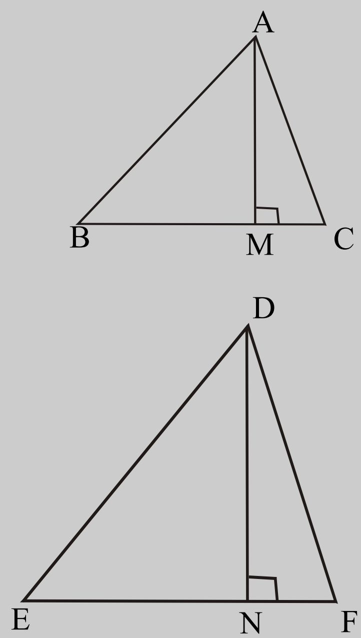

Fig. 7.34

Example 13. If two triangles are similar and their areas are in the ratio 4:9 ; prove that their perimeters are in ratio 2:3.

Solution : Let ABC and DEF be two triangles such that

$\Delta ABC \sim \Delta DEF$ and $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{4}{9}$

But $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$

$$\Rightarrow \frac{4}{9} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

$$\Rightarrow \frac{2}{3} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AB+BC+AC}{DE+EF+DF}$$ [taking square roots]

$$\Rightarrow \frac{AB+BC+CA}{DE+EF+FD} = \frac{2}{3}$$

Hence, $\dfrac{\text{perimeter of } \Delta ABC}{\text{perimeter of } \Delta DEF} = \dfrac{2}{3}$.

EXERCISE 7.5

1. If the areas of two similar triangles are equal, prove that they are congruent.
2. If D, E, F are respectively the mid-points of the sides BC, CA, AB of a ΔABC, prove that $\dfrac{ar(\Delta DEF)}{ar(\Delta ABC)} = \dfrac{1}{4}$
3. Prove that the areas of two similar triangles are in the ratio of the square of the corresponding altitudes.
4. Prove that the areas of two similar triangles are in the ratio of the square of the corresponding medians.
5. Prove that the area of the equilateral triangle described on the side of a squre is half the area of the equilateral triangle described on its diagonal.

7.7 Pythagoras Theorem

In this section, we shall prove an important Theorem known as Pythagoras Theorem (also known as Baudhayan Theorem) using the concept of similarity of triangles.

Theorem 7.9 (Pythagoras Theorem)

In a right triangle the square of the hypotenuse is equal to the sum of squares of the other two sides.

Given : A right ΔABC right angled at A (Fig 7.35).

To prove : $BC^2 = AB^2 + AC^2$

Construction : Draw $AD \perp BC$ (Fig 7.35)

Proof : In Δ^s ABC and DBA,

$\angle BAC = \angle BDA = 90^\circ$

$\angle ABC = \angle DBA$ (same angle)

$\therefore$ by AA similarity,

$\Delta ABC \sim \Delta DBA$

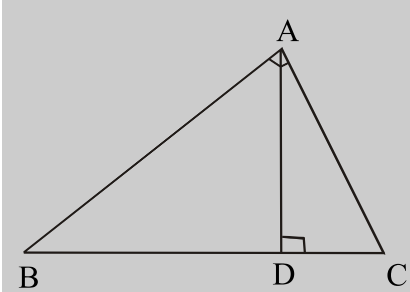

Fig . 7.35

$\therefore \frac{AB}{DB} = \frac{BC}{BA}$

$\Rightarrow AB^2 = BC.BD$ ....(1)

Similarily, from Δ^s ABC and DAC, it can be shown that

$AC^2 = BC.DC$ ....(2)

From (1) amd (2) , we have

$$AB^2 + AC^2 = BC(BD + DC)$$
$$= BC \times BC$$
$$= BC^2$$

Hence, $BC^2 = AB^2 + AC^2$.

The converse of the above theorem is also true and is proved below.

Theorem 7.10 (Converse of Pythagoras Theorem)

In a triangle, if the square of one side is equal to the sum of the squares of the remaining two, the angle opposite to the first side is a right angle.

Given : A ΔABC in which $BC^2 = AB^2 + AC^2$ (Fig.7.36).

To prove : $\angle A = 90^o$

Construction : Draw a right angle QPR such that PQ=AB and PR = AC. Join QR.

Proof : In ΔPQR , by Pythagoras theorem,

$QR^2 = PQ^2 + PR^2$

$= AB^2 + AC^2$ (by construction)

But, $AB^2 + AC^2 = BC^2$ (given)

$\therefore QR^2 = BC^2$

$\Rightarrow QR = BC$

Now, in Δ^s ABC and PQR

AB=PQ, AC=PR and BC = QR

$\therefore$ by SSS congruence,

$\Delta ABC \cong \Delta PQR$

$\therefore \angle BAC = \angle QPR$

But $\angle QPR = 90^\circ$ (by construction)

$\therefore \angle BAC = 90^\circ$

i.e. $\angle A = 90^\circ$.

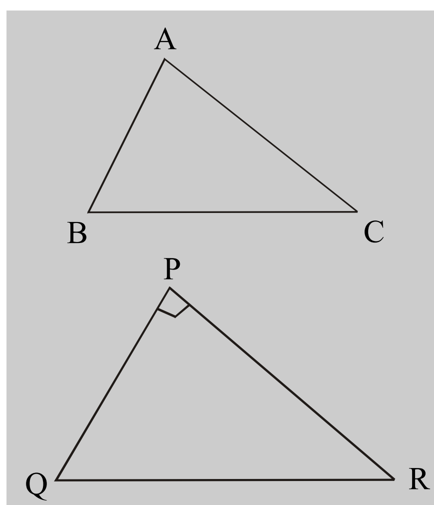

Fig. 7.36

Example 14. Prove that the sum of squares on the four sides of a rhombus is equal to the sum of squres on the diagonals.

Solution : **Given :** A rhombus ABCD

To prove : $AB^2 + BC^2 + CD^2 + DA^2$

$= AC^2 + BD^2$

Proof : We know that the diagonals of rhombus bisect each other at 90°. Let O be the point of intersection of AC and BD (Fig 7.37).

Then, $BO = DO = \frac{1}{2} BD$

and $CO = AO = \frac{1}{2} AC$.

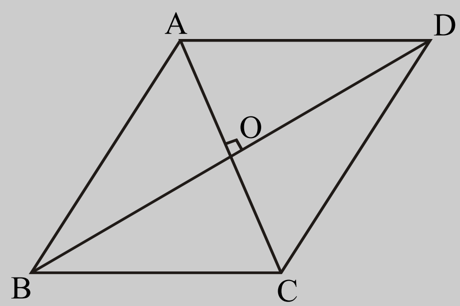

Fig. 17.37

From the rt. ΔAOB, by Pythagoras theorem,

$AB^2 = BO^2 + AO^2$

Similarly, $BC^2 = BO^2 + CO^2$

$CD^2 = CO^2 + DO^2$

and $DA^2 = DO^2 + AO^2$

$$\therefore \; AB^2 + BC^2 + CD^2 + DA^2 = 2\left[AO^2 + BO^2 + CO^2 + DO^2\right]$$

$$= 2\left[\left(AO^2 + CO^2\right) + \left(BO^2 + DO^2\right)\right]$$

$$= 2\left[2.\frac{1}{4}AC^2 + 2.\frac{1}{4}BD^2\right]$$

$$= AC^2 + BD^2$$

Example 15. D and E are points on the sides CA and CB respectively of a triangle ABC right angled at C. Prove that AE^2 + $\mathbf{BD^2 = AB^2 + DE^2}$**.**

Solution : **Given :** Points D and E on CA and CB respectively of a ΔABC in which $\angle C = 90^\circ$.

To prove : $AE^2 + BD^2 = AB^2 + DE^2$

Construction : Join AE, BD , DE (Fig. 7.38)

Proof : From the rt. ΔACE, by Pythagoras theorem

$AE^2 = AC^2 + CE^2$ ...(1)

From the rt. ΔBCD,

$BD^2 = BC^2 + CD^2$ ---(2)

From the rt. ΔACB,

$AB^2 = AC^2 + BC^2$ ---(3)

From the rt. ΔDCE,

$DE^2 = CD^2 + CE^2$ ---(4)

From (1) and (2),

$$AE^2 + BD^2 = (AC^2 + CE^2) + (BC^2 + CD^2)$$

$$= (AC^2 + BC^2) + (CD^2 + CE^2)$$

$$= AB^2 + DE^2 \text{ (using (3) \& (4))}$$

$$\therefore AE^2 + BD^2 = AB^2 + DE^2.$$

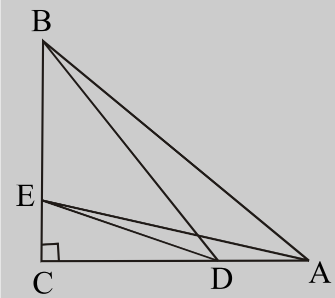

Fig. 7.38

Example 16. Two poles of height 9 m and 14 cm stand on a horizontal ground. If the distance between their feet is 12 m, find the distance between their tops.

Solution : Let AB and CD be the poles of respective heights 9m and 14 cm where the ends B and D are on the ground (Fig 7.39).

Draw $AE \perp CD$.

Then DE=AB=9m

$\therefore \quad CE = CD - DE$

$= 14 - 9$

$= 5$ m

From the rt ΔAEC, by Pythagoras theorem,

$AC^2 = AE^2 + CE^2$

$= BD^2 + CE^2$

$= 12^2 + 5^2$

$= 169$

$\Rightarrow \quad AC = 13$ m

$\therefore$ the distance between their tops is 13 m.

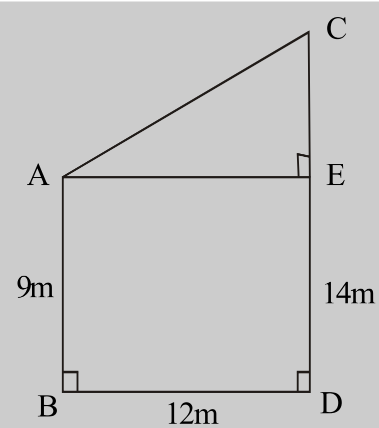

Fig. 7.39

EXERCISE 7.6

1. **Sides of triangles are given below. Determine which of them are right triangles.**
   (i) 3 cm, 4cm, 5cm **(ii)** 5cm, 12cm, 13cm
   (iii) 4cm, 5cm, 7cm **(iv)** 8cm, 11cm, 15cm
   (v) 9cm, 40cm, 41cm
2. A man goes 15m due west and then 8m due north, Find his distance from the starting point.
3. The length of a side of a rhombus is 5 cm and the length of one of its diagonal is 6cm. Find the length of the other diagonal.
4. A ladder 13 m long reaches a window which is 12 m above the ground on one side of a street keeping its foot at the same point, the ladder is turned to other side of the street and it just reaches a window 5m high. Find the width of the street.
5. A ladder reaches 1m below the top of a vetical wall when its foot is at a distance of 6m from the wall. When the foot is shifted 2m nearer the wall, the ladder just reaches the top of the wall; find the height of the wall.
6. Find the length of the altitude and area of an equilateral triangle having 'a' as the length of a side.
7. If D, E, F are the mid-points of the sides BC, CA, AB of a right ΔABC (rt. $\angle$ ed at A) respectively, prove that $3(AB^2+BC^2+CA^2) = 4(AD^2+BE^2+CF^2)$
8. In a right tringle ABC right angled at C, if p is the length of the perpendicular segment drawn from C upon AB, then prove that

   (i) ab = pc **(ii)** $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$, where a = BC, b = CA and c = AB.
9. In an equilateral tringle ABC the side BC is trisected at D. Prove that $9AD^2 = 7AB^2$.
10. If A be the area of a right triangle and b one of the sides containing the right angle, prove that the length of the altitude on the hypotenuse is $\frac{2Ab}{\sqrt{b^4+4A^2}}$.
11. Prove that the equilateral triangles described on the two sides of a right angled triangle are together equal to the equilateral triangle on the hyplotenuse in terms of their area.
12. If O is any point in the interior of a rectangle ABCD, prove that $OA^2 + OC^2 = OB^2 + OD^2$.

ANSWER

1.	**(i) Yes**	**(ii) Yes**	**(iii) No**	**(iv) No**	**(v) Yes**	**2. 17m**
3.	**8cm**	**4. 17m**	**5. 10.5m**	**6.** $\frac{\sqrt{3}}{2}a, \frac{\sqrt{3}}{4}a^2$.		

SUMMARY

In this chapter, you have studied the following points:

1. Two figures having the same shape and not necessarily the same size are called similar figures.
2. All congruent figures are always similar but the similar figures need not be congruent.
3. Two polygons having the same number of sides are similar if (i) the corresponding angles are equal and (ii), the corresponding sides are proplortional
4. If a line is drawn parallel to one side of a triangle to intersect the other two in distinct points, the other two sides are divided in the same ratio.
5. If a line divides two sides of a triangle in the same ratio, the line is parallel to the third side.
6. The internal bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides.
7. If in two triangles, the corresponding angles are equal, their corresponding sides are equal and the two triangles are similar.
8. If the corresponding sides of two triangles are proportional their corresponding angles are equal and the two triangles are similar.
9. If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are proportional, the two triangles are similar.
10. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.
11. The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
12. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
13. In a triangle, if the square of one side is equal to the sum of squares of the remaining two, the angle opposite to the first side is a right angle.

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CHAPTER

8

CIRCLES

8.1 Introduction

In the previous classes, you have studied that a circle is a closed figure consisting of all those points which are at a constant distance (radius) from a fixed point (centre) in the plane. You have also studied various related terms like chord, diameter, circumference, arc, segment, sector etc.

Let us now examine the different possible situations when a circle and a line say l are given in a plane. Indeed, there are three possibilities (in terms of the number of common points between the circle and the line) as given in Fig. 8.1 below:

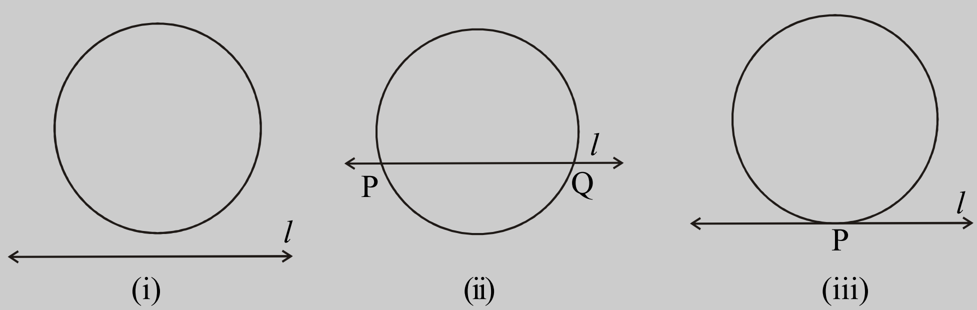


Fig. 8.1

In Fig. 8.1 (i), the circle and the line have no common point i.e. the line does not intersect the circle.

In Fig. 8.1 (ii), the circle and the line have two common points P and Q i.e. the line intersects the circle at two distinct points P and Q. The line l in this case is called a **secant** of the circle.

In Fig. 8.1 (iii), there is only one point P common to the circle and the line i.e. the line intersects the circle at only one point P. The line l in this case, is called a **tangent** to the circle.

In this chapter, we shall study about tangents to a circle, their existence and also study some of their properties.

8.2 Tangent to a Circle

You have just seen in the previous section that a tangent to a circle is a line which intersects the circle at only one point.

Let us perform the following activity to have a better understanding about a tangent to a circle and its existence.

Activity : Firmly fix a rectangular drawing paper on a table and draw a circle on the paper. Then attach a thin straight wire at a point say P on the circle so that it can rotate about P in the plane of the circle.

Start from a position of the wire when it intersects the circle at P and another point Q. Now gently rotate the wire about P in the direction of the minor segment.As the wire rotates, the point Q of intersection of the circle and the wire moves (along the circle) closer and closer to P. And when Q ultimately coincides with P, you will observe that the wire PQ comes to the position AB (see Fig. 8.2) when it intersects the circle at the point P only . This shows that there exists a tangent to the circle at P. On further rotating, you will observe that at all other positions except at AB, the wire intersects the circle at P and another point. This means that there is only one tangent to the circle at P.

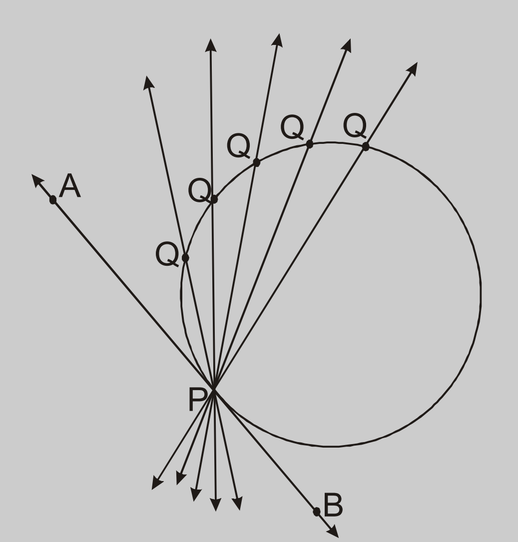


Fig. 8.2

And, you will have the same observation when P is taken anywhere on the circle. Thus, there is one and only one tangent to a circle at any point on it.

Also, from the above activity, you have observed that the tangent to a circle is a special case of secant, when the two endpoints of its corresponding chord coincide.

The common point of the tangent and the circle is called the point of contact (the point P in Fig. 8.1 (iii)) and the tangent is said to touch the

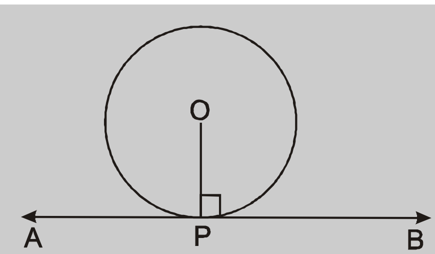


Fig. 8.3

circle at the point of contact. Also, note that all points of the tangent except the point of contact are exterior points of the circle.

Let us now explore some interesting properties of a tangent. To start with, draw a circle with centre O and draw a tangent AB to the circle at a point P (Fig. 8.3). Join the radius OP. Now, measure the angle between the radius OP and the tangent AB. You will find that OP is perpendicular to AB. This property of tangent is proved in the following theorem.

Theorem 8.1 The tangent at any point of a circle is perpendicular to the radius through the point of contact.

Given : A tangent AB to a circle whose centre is O, the point of contact being P. (Fig. 8.4)

To Prove : $OP \perp AB$

Construction : Let us take a point Q on AB other than P and join OQ.

Proof: Since all points on the tangent other than the point of contact are outside the circle, Q lies in the exterior of the circle.

$\therefore$ OQ>OP

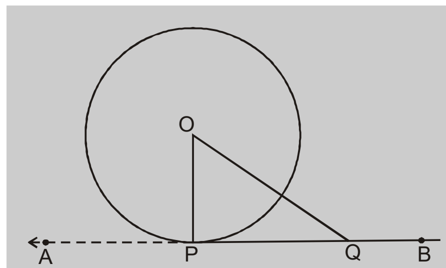


Fig. 8.4

Since this is true for every point on the tangent except the point P, OP is the shortest of all the segments that can be drawn from the centre O to the points of the tangent AB.

And, we know that the shortest segment that can be drawn from a point to a line is the perpendicular segment from the point to the line.

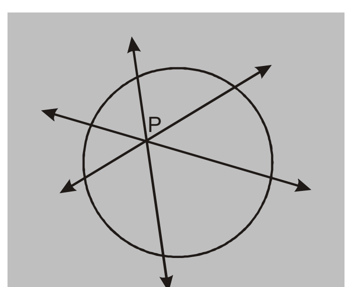
Hence, $OP \perp AB$

8.3 Tangents to a Circle through a point

Let us take a circle and a point P. The following three cases will arise.

Case I (Fig. 8.5 (i)) : If P lies inside the circle, then any line through P intersects the circle at two points. So, it is not possible to draw any tangent to the circle through P.

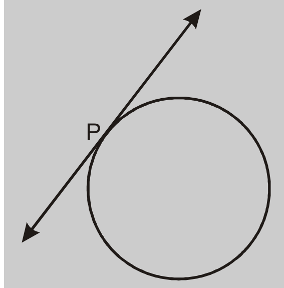
Thus, there is no tangent to a circle passing through a point inside the circle.



(i)

Case II(Fig. 8.5 (ii)) : If P lies on the circle, then as already observed, we can draw one and only one tangent to the circle through P.

Thus, there is one and only one tangent to a circle passing through a point lying on the circle.

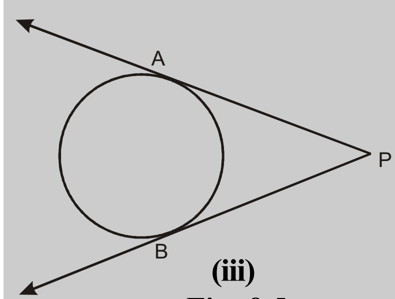


(ii)

Case III (Fig. 8.5(iii)) : If P lies outside the circle, then you will find that exactly two tangents can be drawn to the circle through P. (Refer sec. 9.4 of the next chapter).

Thus, there are exactly two tangents to a circle through (from) a point lying outside the circle.

In Fig. 8.5 (iii) , if A and B are the points of contact of the tangents from P, then PA and PB are called the tangent segments drawn from P to the circle.



(iii)

Fig. 8.5

Now try measuring the lengths PA and PB in the figure.

You will find that they are equal. The equal length PA or PB is known as the length of tangent from P to the circle. This fact of equality of length of the two tangents is proved in the following theorem.

Theorem 8.2 The lengths of tangents drawn from an external point to a circle are equal.

Given : Tangent segments PA and PB drawn to a circle with centre O from an external point P (Fig.8.6).

To prove : PA=PB

Construction : Join OP, OA and OB.

Proof : In the Δ^s OPA and OPB, we have,

$\angle OAP = \angle OBP = 90^o$ (Theorem 8.1)

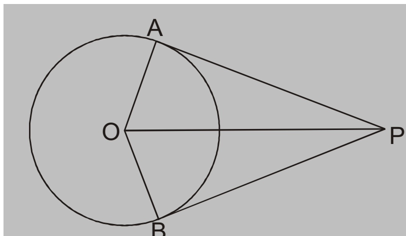


Fig. 8.6

OA = OB (radii of the same circle)

and OP = OP (common)

$\therefore$ Δ OPA $\cong$ Δ OPB (RHS congruence)

Hence, PA=PB (corresponding sides)

Remark : The above theorem can also be proved by using the Pythagoras theorem and is left to the students as an exercise.

Example 1. The length of a tangent to a circle from a point P which is at a distance of 10 cm from the centre of the circle is 8 cm. Find the radius of the circle.

Solution : Let O be the centre of the circle and PA be a tangent segment drawn from P to the circle (Fig. 8.7).

It is given that OP = 10 cm and PA= 8 cm.

We are to find the radius of the circle. Join OA.

Then, $\angle OAP = 90^\circ$ (Theorem 8.1)

Now, in the right ΔOAP,

we have by Pythagoras Theorem,

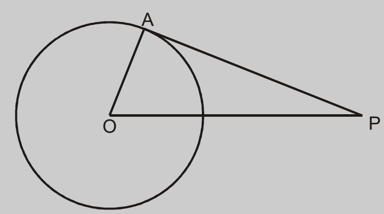


Fig. 8.7

$$OP^2 = PA^2 + OA^2$$

$$\Rightarrow OA^2 = OP^2 - PA^2$$

$$\Rightarrow OA = \sqrt{OP^2 - PA^2}$$

$$= \sqrt{10^2 - 8^2} = 6$$

Therefore, the radius of the circle is 6 cm.

Example 2. PA and PB are tangent segments drawn from an external point P to a circle with centre O. If $\angle$AOB=120°, find $\angle$APB.

Solution : PA and PB are tangent segments drawn from an external point P to a circle with centre O and it is given that $\angle$AOB=120° (Fig. 8.8).

We are to find $\angle$APB.

Here, we have

$\angle$OAP=$\angle$OBP=90° (Theorem 8.1)

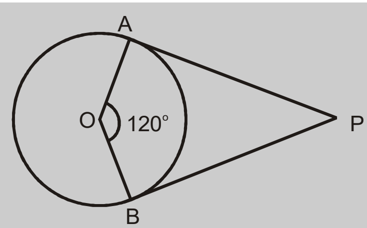


Fig. 8.8

Now, in the quadrilateral AOBP,

$$\angle AOB + \angle OBP + \angle APB + \angle PAO = 360^o$$

$$\Rightarrow 120^\circ + 90^\circ + \angle APB + 90^\circ = 360^\circ$$

$$\Rightarrow \angle APB + 300^\circ = 360^\circ$$

$$\Rightarrow \angle APB = 360^\circ - 300^\circ = 60^\circ$$

Example 3. OP is a radius of a circle with centre O. Prove that the line drawn through P, perpendicular to OP is the tangent to the circle at P.

Given : The radius OP of a circle with centre O and the line APB perpendicular to OP (Fig. 8.9).

To prove : AB is the tangent to the circle at P.

Proof : If possible, let us suppose that AB is not the tangent to the circle at P.

Then, AB will intersect the circle at P and another point say, Q. Join OQ.

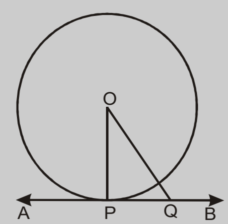


Fig. 8.9

Now, in the ΔOPQ,

$$OP = OQ \quad \text{(radii of the same circle)}$$

$$\Rightarrow \angle OPQ = \angle OQP \quad \text{(angles opposite to equal sides)}$$

But $\angle OPQ = 90^o$

$\therefore \angle OQP = 90^o$

This means that the sum of the three angles of the ΔOPQ is greater than 180^o, which is not possible.

$\therefore$ our supposition is wrong

Hence, AB is the tangent to the circle at P.

Note: The above result is indeed the converse of Theorem 8.1.

Example 4. In two concentric circles, prove that a chord of the larger circle, which is a tangent to the smaller circle, is bisected at the point of contact.

Given : Two concentric circles with common centre O, a chord AB of the larger circle touching the smaller circle at P (Fig. 8.10).

To prove : AP= PB

Construction : OA and OB are joined.

Proof : AB is a tangent to the smaller circle at P and OP is its radius.

$\therefore$ $OP \perp AB$ (Theorem 8.1)

$\Rightarrow$ $\angle OPA = \angle OPB = 90^\circ$

Now in the two right Δ^s AOP and BOP,

OA = OB (radii)

OP = OP (common)

$\therefore$ By RHS congruence theorem,

$\Delta AOP \cong \Delta BOP$

$\therefore$ AP = BP (corresponding sides)

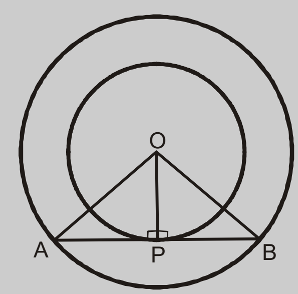


Fig. 8.10

Example 5. A circle touches the side BC of a ΔABC at P and the sides AB and AC produced, at Q and R respectively. Prove that AQ is half the perimeter of ΔABC.

Given : A circle touching the side BC of a ΔABC at P and AB and AC produced, at Q and R respectively (Fig. 8.11).

To prove : AQ = $\frac{1}{2}(AB+BC+AC)$

Proof : BP and BQ are tangent segments drawn from the external point B to the circle.

$\therefore$ BQ = BP (Theorem 8.2)--- (i)

Similarly,

CP = CR ---(ii)

Also, AQ =AR ---(iii)

Now, 2AQ = AQ+ AQ

= AQ+AR [Using (iii)]

= (AB + BQ)+(AC+CR)

= (AB+BP)+(AC+CP) [Using (i) & (ii)]

= AB+(BP+CP)+AC

=AB+BC+AC

$\Rightarrow AQ = \frac{1}{2}(AB+BC+AC)$

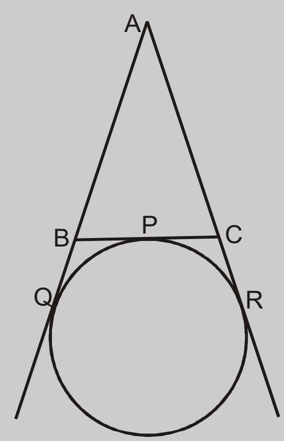


Fig. 8.11

EXERCISE 8.1

1. A point P is at a distance of 13 cm from the centre O of a circle. If the radius of the circle is 5 cm, find the length of tangent from P to the circle.

2. PA and PB are tangent segments drawn from an external point P to a circle with centre O. If $\angle AOP = 70^{\circ}$, find at what angle the two tangents are inclined to each other.

3. Prove that tangents at the ends of a diameter of a circle are parallel.

4. If PA and PB are tangent segments drawn from an external point P to a circle whose centre is O, prove that OP bisects AB and hence $OP \perp AB$.

5. Two concentric circles are of radii 6 cm and 10 cm. Find the length of the chord of the larger circle which touches the smaller circle.

6. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

7. Prove that the opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

8. If a circle touches all the four sides of a quadrilateral ABCD, prove that AB+CD=BC+DA.

9. ΔABC is isosceles with AB=AC. The incircle of the ΔABC touches BC at P. Prove that BP=CP.

10. Prove that the parallelogram circumscribing a circle is a rhombus.

11. The incircle of a ΔABC touches the sides BC, CA and AB at D, E and F respectively. Show that AF+BD+CE=AE+BF+CD=$\frac{1}{2}$ (perimeter of ΔABC).

12. If PA and PB are tangent segments drawn from an external point P to a circle with centre O, prove that $\angle OAB = \frac{1}{2}\angle APB$.

13. Prove that the intercept of a tangent between two parallel tangents to a circle subtends a right angle at the centre.

ANSWER

1. 12 cm   2. 40°   5. 16 cm

SUMMARY

In this chapter, you have studied the following points :

1. A line may intersect a circle at no point, two distinct points or only one point.

2. A line which intersects a circle at two points is called a secant of the circle.

3. A tangent to a circle is a line that intersects the circle at only one point. The point is called the point of contact and the tangent is said to touch the circle at the point of contact.

4. The tangent to a circle is a special case of the secant, when the two ends of the corresponding chord coincide.

5. There is one and only one tangent to a circle at any point on it.

6. The tangent at any point of a circle is perpendicular to the radius through the point of contact.

7. The number of tangents that can be drawn to a circle through a given point is

(i) zero if the point lies inside the circle

(ii) one if the point lies on the circle

(iii) two if the point lies outside the circle.

8. The lengths of tangents drawn from an external point to a circle are equal.

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CHAPTER

9

CONSTRUCTION

9.1 Introduction

In the previous class, you have learnt about the need and importance of geometrical construction. You have also learnt construction of triangle under certain given data and construction of circumcircle and incircle of a given triangle. Here in this chapter we shall deal with some more construction problems. You are reminded once again that as far as practicable, only two geometrical instruments, namely a ruler and a compass will be used in geometrical construction. The analysis part of a construction is given only to reveal the clues leading to the construction process. It is not a part of the construction and will be omitted when the steps of construction are self evident.

9.2 Division of a line segment in a given ratio

There are two methods for dividing a given line segment in a given ratio, one based on the Basic Proportionality theorem and the other on the property of similar triangles. We deal with both the methods by taking an example. Suppose, a given line segment AB is to be divided in a given ratio, say 2:3.

First method :

Given : A line segment AB

Required : To divide AB in the ratio 2:3

Analysis: [Let C be the point of division. If a triangle ABP_5 is formed with AB as a side and CP_2 is drawn parallel to BP_5 meeting AP_2 at P_2 then $AP_2:P_2P_5=AC:BC=2:3$. So, P_2P_5 will be of 3 units when AP_2 is of 2 units and as such C can be located]

Steps of Construction :

(i) Draw a ray AX inclined to AB at a certain angle preferably an acute angle.

(ii) Taking a suitable length as unit, mark five (2+3) points P_1, P_2, P_3, P_4, P_5, on $\overrightarrow{AX}$ such that $AP_1=P_1P_2=P_2P_3=P_3P_4=P_4P_5=$ 1 unit

(iii) Join P_5B and through P_2 draw P_2C parallel to P_5B meeting AB at C.

Then C is the point on AB such that AC:CB=2:3

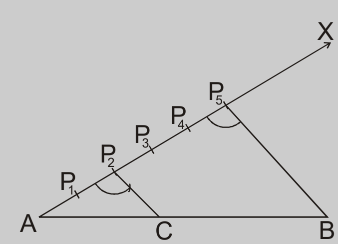

Fig. 9.1

Proof: In the triangle ABP_5, since CP_2 is parallel to BP_5 therefore

$AC:CB = AP_2:P_2P_5$ (Basic Proportional Theorem)

But $AP_2:P_2P_5 = 2:3$ (by construction)

Hence $AC:CB = 2:3$

Second Method:

Given : A line segment AB

Required : To divide AB in the ratio 2:3

Analysis: [If similar triangles ACP_2 and BCQ_3 are drawn so that the point of division C lies on the segment P_2Q_3, then AP_2 is parallel to BQ_3 and $AP_2:BQ_3 = AC:BC = 2:3$. Hence BQ_3 is of 3 unit when AP_2 is of 2 units and as such C can be located]

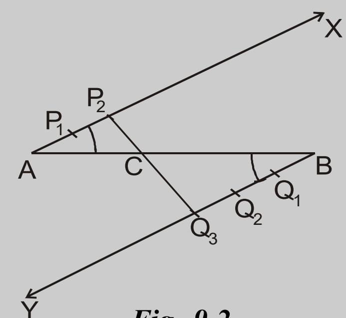

Fig. 9.2

Steps of construction :

(i) Draw a ray AX inclined to AB at a certain angle.

(ii) Draw a ray BY parallel to AX so that $\angle ABY = \angle BAX$.

(iii) Mark Points P_1, P_2 on AX and Q_1, Q_2, Q_3 on BY such that $AP_1 = P_1P_2 = BQ_1 = Q_1Q_2 = Q_2Q_3$ (=1 unit)

(iv) Join P_2Q_3 interesting AB at C.

Then AC:CB = 2:3

Proof : In the triangle ACP_2 and BCQ_3,

$\angle P_2AC = \angle Q_3BC$ ($\because AP_2 \parallel BQ_3$)

and $\angle ACP_2 = \angle BCQ_3$ (vertically opposite angles)

By AA similarity theorem

$\Delta ACP_2 \sim \Delta BCQ_3$

$\therefore AC:CB = AC:BC$

$= AP_2:BQ_3$

$= 2:3$

9.3 Construction of a triangle similar to a given triangle as per given scale factor

The construction can be illustrated by means of examples. Here, 'scale factor' means the ratio of the sides of the triangle to be constructed to the corresponding sides of the given triangle. Two cases arise according as the scale factor is less than or greater than unity. We deal with both the cases separately in the following examples.

Example 1. Construct a triangle similar to given triangle ABC, with its sides equal to $\frac{3}{5}$ (scale factor <1) of the corresponding sides of the triangle ABC.

Solution:

Given : A triangle ABC

Required : To construct a triangle similar to the triangle ABC with its sides equal to $\frac{3}{5}$ of the corresponding sides of the triangle ABC.

Steps of construction :

(i) Draw any ray BX inclined to BC at a certain angle on the opposite sides of A.

(ii) Mark 5 points P_1, P_2, P_3, P_4, P_5, on BX such that $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5$

(iii) Join P_5C and draw P_3C' parallel to P_5C, meeting BC at C'.

(iv) Draw $C'A'$ parallel to CA, meeting AB at A'.

Then $\Delta A'BC'$ is the required triangle.

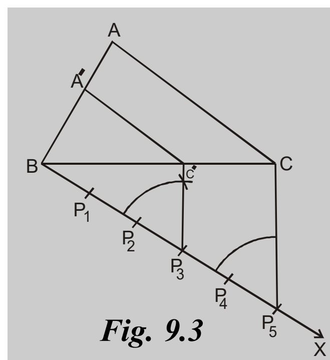

Fig. 9.3

Proof : By construction

$$\frac{BC'}{C'C} = \frac{BP_3}{P_3P_5} = \frac{3}{2}$$

$$\therefore \frac{BC}{BC'} = \frac{BC' + C'C}{B'C} = 1 + \frac{C'C}{B'C} = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\therefore \frac{BC'}{BC} = \frac{3}{5}$$

Also, as $C'A' \parallel CA$ $\quad \Delta A'BC' \sim \Delta ABC$

$$\therefore \frac{A'B}{AB} = \frac{A'C}{AC} = \frac{BC'}{BC} = \frac{3}{5}.$$

Example 2. Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{5}{3}$ (scale factor>1) of the corresponding sides of the triangle ABC.

Solution :

Given: A triangle ABC

Required : To construct a triangle similar to the triangle ABC with its sides equal to $\frac{5}{3}$ of the corresponding sides of the triangle ABC.

Steps of Construction :

(i) Draw any ray BX inclined to BC at a certain angle, on the opposite side of A.

(ii) Mark 5 points P_1, P_2, P_3, P_4, P_5, on BX such that

$BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5$

(iii) Join P_3C and draw P_5C' parallel to P_3C and meeting BC produced at C'

(iv) Draw $C'A'$ parallel to CA, meeting BA produced at A'

Then $\Delta A'BC'$ is the required triangle.

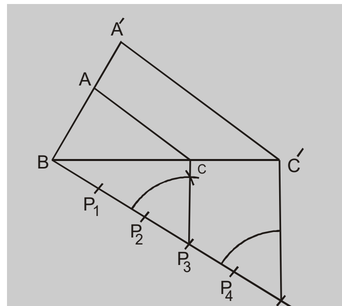

Fig. 9.4

Proof : Since $AC \parallel A'C'$

$\therefore \quad \angle BAC = \angle BA'C'$

and $\quad \angle ACB = \angle A'C'B$

By AA similarity $\Delta ABC \sim \Delta A'BC'$

$$\therefore \quad \frac{AB}{A'B} = \frac{AC}{A'C'} = \frac{BC}{BC'}$$

$$\text{But} \quad \frac{BC}{BC'} = \frac{BP_3}{BP_5} = \frac{3}{5}$$

$$\text{or} \quad \frac{BC'}{BC} = \frac{5}{3}$$

$$\text{Hence,} \quad \frac{A'B}{AB} = \frac{A'C'}{AC} = \frac{BC'}{BC} = \frac{5}{3}$$

Note : In the constructions of examples 1 and 2, you could take any ray through any vertex of the given triangle inclined at a certain angle to any of the sides through the vertex and proceed similarly.

EXERCISE 9.1

1. **Draw any line segment and divide it in the ratio**

   (i) 3:7 **(ii)** 4:7 **(iii)** 3:8 **(iv)** 4:9

2. Draw a line segment of 8.7 cm and divide it in the ratio 5:9. Measure the two parts correct to a millimetre. (Answer : 3.1cm and 5.6 cm)

3. Construct a right triangle with sides 3cm, 4cm and 5cm. And then construct a triangle similar to the right triangle, the scale factor being $\frac{7}{4}$.

4. Draw a triangle with sides 5cm, 7cm and 8cm. Construct a triangle with sides $\frac{4}{5}$ of the corresponding sides of first triangle.

5. Draw a triangle ABC in which AB=6cm, BC=7cm and $\angle B = 45^\circ$. Then construct a triangle whose sides are $\frac{5}{8}$ of the corresponding sides of the triangle ABC.

6. Construct a triangle ABC in which BC=5cm, $\angle B = 60^\circ$ and AB+AC=8cm. Then construct a triangle whose sides are $\frac{5}{4}$ of the corresponding sides of the triangle ABC.

7. Construct an isosceles triangle of given base and altitude. Then construct another isosceles triangle whose sides are $\frac{3}{2}$ times the corresponding sides of the first triangle.

9.4 Construction of a tangent (a pair of tangents) to a circle from an external point

Given: A circle with centre O and an external point P.

Required: To construct a tangent (a pair of tangents) from P to the circle.

Analysis : [If PA is a tangent touching the circle at A, then $OA \perp PA$ and hence the ΔPOA is right angled at A. So, the circle described on the segment PO as a diameter, passes through A. As such, A can be located]

Steps of construction:

(i) Join PO.

(ii) Bisect PO at M

(iii) Draw a circle with centre M and radius MP (or MO) cutting the given circle at A and A′

(iv) Draw rays PA and PA'.

They are the pair of tangents from P to the circle.

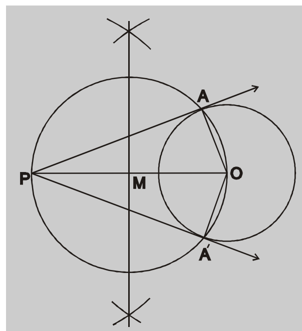

Fig . 9.5

Proof : The angles PAO and $PA'O$ are angles in semicircle. Therefore each of them is a right angle. Thus, $PA \perp OA$ and also $PA' \perp OA'$. Hence PA and PA' are tangents to the circle from the external point P.

Note: The circle having centre at M and radius equal to MP intersects the given circle at two points. Therefore, only two tangents can be drawn from the external point P to the given circle.

EXERCISE 9.2

1. Construct a pair of tangents from an external point which is at a distance of $5{\cdot}5\,\text{cm}$ from the centre of a circle whose radius is 2.5 cm.

2. Draw a circle of radius 3 cm. Construct a pair of tangents to the circle from a point 5 cm away from the centre. Measure the lengths of the tangents. Verify your measurement by geometrical reasoning.

3. Draw a circle of radius 3.5 cm. Costruct a pair of tangents to the circle, inclined to each other at 60^{o}. Find the distance of the point of intersection of the two tangents from the centre of the circle (Answer : 7cm)

4. Draw a circle with a given line segment AB as a diameter. Then construct a tangent to the circle from a given external point P.

SUMMARY

In this chapter you have studied the following constructions :

1. Division of a line segment in a given ratio.

2. Construction of a triangle similar to a given triangle as per given scale factor.

3. Construction of a pair of tangents from an external point to a circle.

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CHAPTER

10

TRIGONOMETRY

10.1 Introduction

You have already known what the term "Trigonometry" means. It is the study of the relationships between the sides and angles of a triangle. Astronomers used trigonometry to calculate distance from the Earth to the planets and stars. It is also used in geography and in navigation and in many technologically advanced methods in Engineering.

In this chapter, we shall study some ratios of the sides of a right triangle called trigonometric ratios of an acute angle of the right triangle. We will restrict our discussion to trigonometric ratio of acute angles only, though it can be extended to those of other angles also. Here we shall establish some relationships between the trigonometric ratios and also we shall calculate numerical values of trigonometric ratio of some standard angles.

We shall use " an angle and its measure' and " a side and its measure" synonymously.

10.2 Trigonometric Ratios of an Acute Angle

Let us take a right triangle ABC, right angled at B, as shown in Fig. 10.1. AC is the **hypotenuse** of the right triangle. We shall try to give names to the other two sides in view of their positions with respect to angle A. Note the position of the side BC with respect to $\angle A$. It faces $\angle A$. We call it the side opposite to $\angle A$ or simply the "opposite side" with reference to $\angle A$. Similarly, the side AB is called the side adjacent to $\angle A$ or simply "adjacent side".

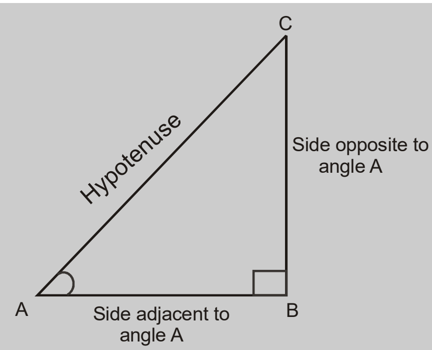

Fig. 10.1

Further, let us note that relative to the $\angle C$, AB becomes the opposite side, BC the adjacent side while the hypotenuse AC remains the same (Fig. 10.2).

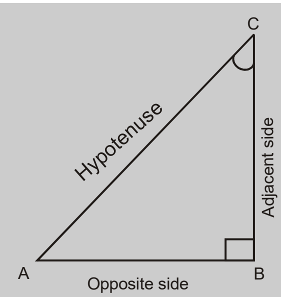

Fig. 10.2

We have studied the concept of 'ratio' in earlier classes. We now define certain ratios involving sides of a right triangle and call them 'trigonometric ratios'.

The trigonometric ratios of $\angle A$ in the right triangle ABC (Fig 10.3) are defined as follows :

sine of $\angle A = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{BC}{AC}$

cosine of $\angle A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{AB}{AC}$

tangent of $\angle A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{BC}{AB}$

cotangent of $\angle A = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{AB}{BC}$

secant of $\angle A = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{AC}{AB}$

cosecant of $\angle A = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{AC}{BC}$

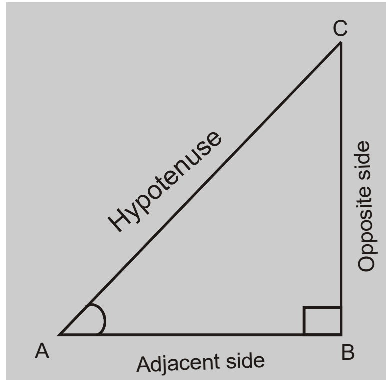

Fig. 10.3

The ratio defined above are abbreviated as sin A, cos A, tan A, cot A, sec A and cosec A respectively. You must have seen from the above definitions that the trigonometric ratios of an acute angle in a right triangle express the relationships between the angle and the ratios of its sides.

A point of caution is that sin A is used as an abbreviation of 'sine of $\angle A$ '. sin A is not the product of 'sin' and 'A'. 'sin' left alone from A has no meaning. Similarly tan A is not the product of 'tan' and 'A'. Similar interpretations follow for other trigonometric ratios.

Now, try to define trigonometric ratios of acute angle C in the right triangle ABC (Fig 10.3).

Note: Since the hypotenuse is the longest side of a right triangle, the value of sin A or cos A is always less than 1 for an acute angle A.

10.3 Uniqueness of Trigonometric Ratios

In this unit let us investigate the uniqueness of trigonometric ratios.

Consider a right ΔABC, right angled at B.

Take a point P on the hypotenuse AC (P may be within AC or AC produced as shown in Fig 10.4), and draw PM perpendicular to AB. Will the trigonometric ratios of $\angle A$ in the right triangle PAM differ from those of $\angle A$ in the right triangle ABC?

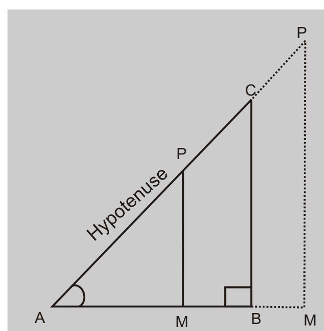

Fig. 10.4

To answer this, first let us note the ralationship between these two triangles. Are these triangles similar ? Yes, using the appropiate similarity criterion (vide chapter 5), you will see that they are similar. Therefore, by the property of similar triangles, the corresponding sides of the triangles are proportional.

Thus, we have

$$\frac{AM}{AB}=\frac{AP}{AC}=\frac{MP}{BC}$$

$$\Rightarrow \frac{MP}{AP}=\frac{BC}{AC}=\sin A$$

Similarly, $\Rightarrow \frac{AM}{AP}=\frac{AB}{AC}=\cos A$

$$\Rightarrow \frac{MP}{AM}=\frac{BC}{AB}=\tan A \text{ and so on.}$$

We thus see that the trigonometric ratios of $\angle A$ in Δ PAM do not differ from those of $\angle A$ in Δ CAB.

From this observation, we can conclude that the values of trigonometric ratios of an angle do not vary with the size of the right triangle considered. In short, the trigonometric ratios of an angle are uniquely defined.

10.4 Motivation of Ratios of 0^o and 90^o

Let us see what happens to the trigonometric ratios of angle A if it is made smaller and smaller in the right triangle ABC (Fig. 10.5) , till it becomes zero.

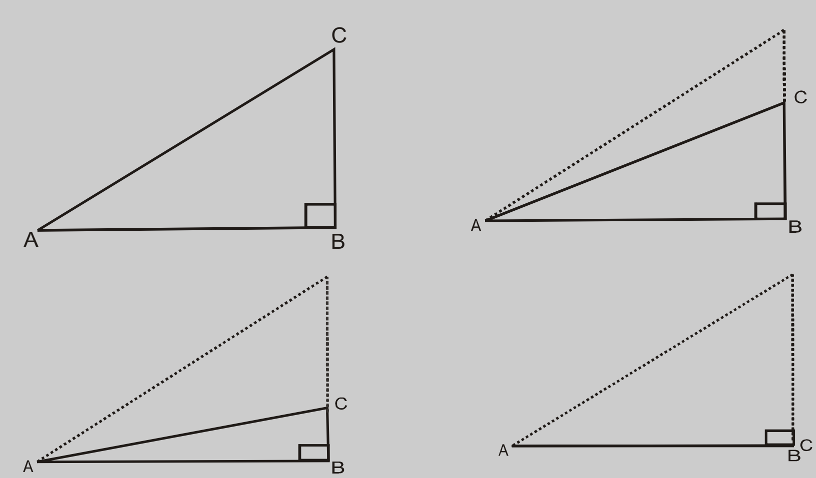

Fig. 10.5

Keeping AB fixed, as $\angle A$ gets smaller and smaller, BC becmes shorter and shorter and finally when $\angle A$ measures 0°, C coincides with B i.e. BC = 0 and AB = AC.

This helps to find the values of sin A and cos A when A = 0°. In fact, we have

$$\sin 0^o = \frac{BC}{AC} = \frac{O}{AC} = 0$$

$$\cos 0^o = \frac{AB}{AC} = \frac{AC}{AC} = 1$$

$$\tan 0^o = \frac{BC}{AB} = \frac{O}{AB} = 0$$

$$\cot 0^o = \frac{AB}{BC} = \frac{AB}{O}$$, which is not defined. (why?).

$$\sec 0^o = \frac{AC}{AB} = \frac{AC}{AC} = 1$$

and $$\text{cosec } 0^o = \frac{AC}{BC} = \frac{AC}{O},$$ which is again not defined.

Next let us see what happens to the trigonometric ratios of $\angle A$ when it is made larger and larger till it measures 90° (Fig 10.6). Keeping BC fixed as $\angle A$ gets larger and larger, AB gets shorter and shorter and finally when $\angle A$ measures 90°, A coincides with B.

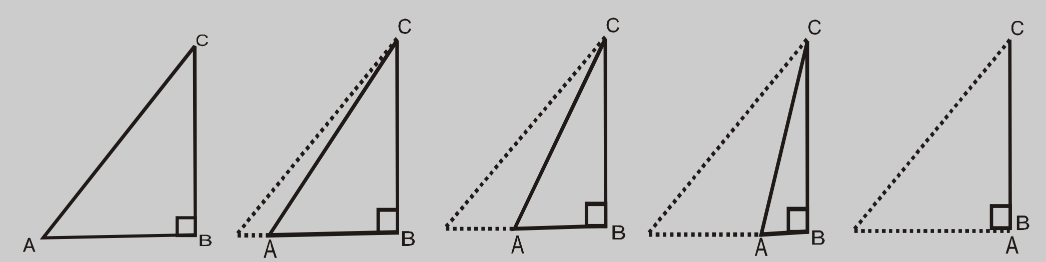

Fig. 10.6

Thus AB = 0, AC = BC. As such,

$$\sin 90^o = \frac{BC}{AC} = \frac{AC}{AC} = 1, \quad \cos 90^o = \frac{AB}{AC} = \frac{0}{AC} = 0.$$

Could you find the values of the other trigonometric ratios of 90^o?

10.5 Trigonometric Ratios of Some Standard Angles

In this section, we will find the values of trigonometric ratios of angles 30^o, 45^o and 60°.

In the previous section you have already known the values of trigonometric ratios of 0° and 90° whenever defined.

(a) Trigonometric Ratios of 45°

In the right ΔABC, right angled at B, if one angle is 45°, then the other angle is also 45°, ie. if $\angle A = 45^\circ$, then

$\angle C = 45^\circ$ (fig 10.7)

So, AB = BC (why)

Suppose, AB = BC= k

Now $AC^2 = AB^2 + BC^2$ (Pythogoras relation)

$= k^2 + k^2$

$= 2k^2$

$\therefore \quad AC = \sqrt{2}k$

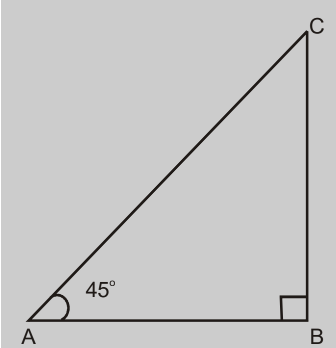

Fig. 10.7

Then, using definitions of trigonometric ratios, we have,

$$\sin 45^\circ = \frac{BC}{AC} = \frac{k}{\sqrt{2}k} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{k}{\sqrt{2}k} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{k}{k} = 1$$

and
$$\cot 45^\circ = \frac{AB}{BC} = \frac{k}{k} = 1$$

$$\sec 45^\circ = \frac{AC}{AB} = \sqrt{2},$$

$$\text{cosec } 45^\circ = \frac{AC}{BC} = \sqrt{2}.$$

(b) Trigonometric Ratios of 30° and 60°

To calculate the values of trigonometric ratios of 30° and 60°, take an equilateral triangle ABC. (Fig 10.8). In an equilateral triangle each angle is 60°. Therefore,

$\angle A = \angle B = \angle C = 60^\circ$.

From A draw AD perpendicualar to BC meeting it at D.

Now, $\Delta ABD \cong \Delta ACD$ (why)

$\therefore \quad BD = DC$

and $\angle BAD = \angle CAD$

Consequently, $BD = DC = \frac{1}{2}BC$

and $\angle BAD = \angle CAD = \frac{1}{2}\angle A = \frac{1}{2}\times 60^\circ = 30^\circ$

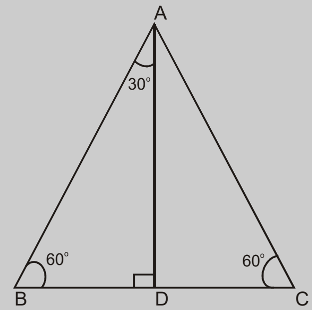

Fig. 10.8

Observe that ΔABD is a right triangle, right angled at D with $\angle BAD = 30^\circ$ and $\angle ABD = 60^\circ$ (Fig. 10.8)

Now, let us find the length of sides of ΔABD,

Suppose AB = 2k,

then $BD = \frac{1}{2}BC = \frac{1}{2}AB = k$

and $AD^2 = AB^2 - BD^2 = (2k)^2 - k^2 = 3k^2$

$\therefore \quad AD = \sqrt{3}k$

Then, we have,

$$\sin 30^\circ = \frac{BD}{AB} = \frac{k}{2k} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{BD}{AD} = \frac{k}{\sqrt{3}k} = \frac{1}{\sqrt{3}}$$

Also, $\operatorname{cosec} 30^\circ = \frac{AB}{BD} = \frac{2k}{k} = 2$

$$\sec 30^\circ = \frac{AB}{AD} = \frac{2k}{\sqrt{3}k} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{AD}{BD} = \frac{\sqrt{3}k}{k} = \sqrt{3}$$

Similarly, from definitions,

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{k}{2k} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}k}{k} = \sqrt{3}$$

$$\cot 60^\circ = \frac{BD}{AD} = \frac{k}{\sqrt{3}k} = \frac{1}{\sqrt{3}}$$

$$\sec 60^\circ = \frac{AB}{BD} = \frac{2k}{k} = 2$$

$$\operatorname{cosec} 60^\circ = \frac{AB}{AD} = \frac{2k}{\sqrt{3}k} = \frac{2}{\sqrt{3}}$$

We shall now display the values of all trigonometric ratios of $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ in a table for ready reference as follows :

$\angle A$	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

The use of the values of trigonometric ratios of these angles will be illustrated in the following examples.

Example 1. **In a right triangle ABC, right angled at C, AB = 10 cm and $\angle BAC = 30^\circ$ (Fig. 10.9). Find the length of sides BC and AC.**

Solution : To find the length of the side BC , we will choose the trigonometric ratio involving BC and the given side AB. Since BC is opposite to $\angle A$ and AB is hypotenuse.

we have,

$$\frac{BC}{AB} = \sin A$$

i.e. $$\frac{BC}{10} = \sin 30^\circ = \frac{1}{2}$$

$$\therefore \; BC = \frac{10}{2} = 5cm$$

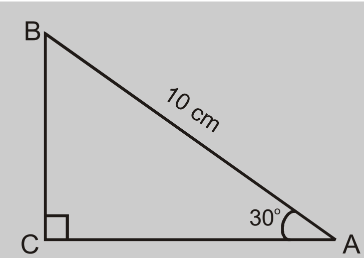

Fig. 10.9

To find the length of AC, we have similarly,

$$\frac{AC}{AB} = \cos A$$

i.e. $\frac{AC}{10} = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\therefore$ $AC = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ *cm*.

Note : To find AC, alternatively we could have used the Pythagoras theorem. Thus,

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{10^2 - 5^2} = \sqrt{75}$$

$$= \sqrt{25 \times 3} = 5\sqrt{3}.$$

Example 2. If $\sin(A+B) = \frac{\sqrt{3}}{2}$, $\cos(A-B) = \frac{\sqrt{3}}{2}$, $0^o < A+B \leq 90^o$, $A > B$, find A and B.

Solution : Given, $\sin(A+B) = \frac{\sqrt{3}}{2} = \sin 60^\circ$

$\therefore$ $A+B = 60^\circ$ ...(1)

And $\cos(A-B) = \frac{\sqrt{3}}{2} = \cos 30^\circ$

$\therefore$ $A-B = 30^\circ$ ...(2)

Solving (1) and (2), we get,

$A = 45^\circ$ and $B = 15^\circ$.

EXERCISE 10.1

1. **Find value of the following.**

(i) $\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$ (ii) $\frac{2\tan 30^\circ}{1 - \tan^2 30^\circ}$

(iii) $\frac{\sin 30^\circ + \tan 45^\circ - \operatorname{cosec} 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ}$ (iv) $\frac{5\cos^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ}$

2. In the right ΔPQR, right angled at Q, PQ = 4cm and $\angle PRQ = 30^o$. Find the length of sides PR and QR.

3. If $\tan(A+B) = \sqrt{3}$ and $\tan(A-B) = \frac{1}{\sqrt{3}}$; $0^o < A+B < 90^o$, $A > B$, find A and B.

ANSWER

1. (i) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (ii) $\sqrt{3}$ (iii) $\frac{3\sqrt{3}-4}{4+3\sqrt{3}}$ (iv) $\frac{67}{12}$

2. PR = 8cm, QR = $4\sqrt{3}$ cm. **3.** A = 45^o, B = 15^o.

10.6 Relationships between the Trigonometric Ratios

Let us establish certain relationships that exist between the trigonometric ratios of an angle. Remember the definitions of sin A and cosec A. In fig 10.10, we have

$$\sin A = \frac{\text{opposite side of } \angle A}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosec } A = \frac{\text{hypotenuse}}{\text{opposite sides of } \angle A} = \frac{AC}{BC}$$

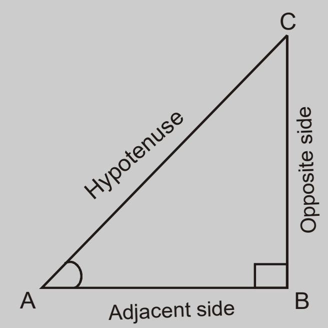

Fig.10.10

So, what do you see ?

$$\text{cosec} A = \frac{AC}{BC} = \frac{1}{\frac{BC}{AC}} = \frac{1}{\sin A} \quad (1)$$

Similarly,

$$\sec A = \frac{AC}{AB} = \frac{1}{\frac{AB}{AC}} = \frac{1}{\cos A} \quad (2)$$

$$\cot A = \frac{AB}{BC} = \frac{1}{\frac{BC}{AB}} = \frac{1}{\tan A} \quad (3)$$

The above three relations (1), (2) and (3) are known as **reciprocal relations** of the trigonometric ratios.

Further, observe that $\tan A = \frac{BC}{AB}$

$$= \frac{\frac{BC}{AC}}{\frac{AB}{AC}}$$

$$= \frac{\sin A}{\cos A} \text{ ....(4)}$$

and, $$\cot A = \frac{\cos A}{\sin A} \text{ ....(5)}$$

These two relations (4) and (5) are known as **quotient relations.**

Now, refer to Fig 10.10. In the ΔABC, right angled at B,

we have $AB^2 + BC^2 = AC^2$

Hence, $$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\Rightarrow \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

$$\Rightarrow (\cos A)^2 + (\sin A)^2 = 1$$

$$\Rightarrow \cos^2 A + \sin^2 A = 1 \text{ ....(6)}$$

This relation holds for every A such that $0 \le A \le 90^\circ$.

Again, $AB^2 + BC^2 = AC^2$

$$\Rightarrow \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\Rightarrow \left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$$1 + \tan^2 A = \sec^2 A \text{ ....(7)}$$

Is this equation true for A = 0 ? Yes, it is . What about A = 90° ? We have seen that tan A and sec A are not defined when A = 90°. So (7) is true for all A such that $0^o \le A < 90^\circ$.

Again,

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\Rightarrow \left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

$$\Rightarrow \cot^2 A + 1 = \text{cosec}^2 A \text{ ....(8)}$$

Remember that cot A and cosec A are not defined for A = 0°. Therefore (8) is true for all A such that $0^o < A \le 90^\circ$.

The above three relations (6), (7), and (8) are known as **Pythagorean relations of trigonometric ratios.**

From (6), (7) and (8) we see that

$$\cos^2 A = 1 - \sin^2 A, \qquad \sin^2 A = 1 - \cos^2 A,$$

$$\sec^2 A - \tan^2 A = 1, \qquad \tan^2 A = \sec^2 A - 1.$$

$$\operatorname{cosec}^2 A - \cot^2 A = 1, \qquad \cot^2 A = \operatorname{cosec}^2 A - 1$$

Note : For the sake of convenience , we write $\cos^2 A$, $\sin^2 A$ etc. in place of $(\cos A)^2$, $(\sin A)^2$ etc. But $\cot A = (\tan A)^{-1} \neq \tan^{-1} A$ (it is called tangent inverse A).

The symbol $\tan^{-1} A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for other trigonometic ratios as well.

10.7 Trigonometric Identities

Let us recall that an equation is called an identity if it is true for all values of the variables involved. Thus, an equation involving trigonometric ratios of an angle (this being a variable) is called a trigonometric identity if it is true for all admissible values of the angle involved.

All the trigonometric relations established in the previous section are standard trigonometric identities.

Further consider the following relation:

$$\cos^2 A - \cos A = \cos A(\cos A - 1).$$

This is an identity because it is true for all values of A;

whereas $\cos^2 A - \cos A = 0$ is not an identity, since it is satisfied only for some particular values of A (in fact for A = 0° and A = 90° only).

Using the standard identities we can express each trigonometric ratio in terms of a given ratio. Of course, from definition of trigonometric ratios we can do the same. The process is illustrated in the following examples.

Example 3. Given $\sin A = \frac{3}{4}$, find the other trigonometric ratios of A.

Solution : First method:

we know that $\cos^2 A + \sin^2 A = 1$

$$\therefore \quad \cos^2 A + \left(\frac{3}{4}\right)^2 = 1$$

$$\Rightarrow \cos^2 A = 1 - \left(\frac{3}{4}\right)^2$$

$$= 1 - \frac{9}{16}$$

$$= \frac{16-9}{16}$$

$$= \frac{7}{16}$$

$$\therefore \cos A = \frac{\sqrt{7}}{4} \text{ (neglecting negative value)}$$

Then, $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\frac{3}{4}}{\frac{\sqrt{7}}{4}} = \dfrac{3}{\sqrt{7}}$

$$\cot A = \frac{1}{\tan A} = \frac{\sqrt{7}}{3}$$

$$\sec A = \frac{1}{\cos A} = \frac{4}{\sqrt{7}}$$

and $\operatorname{cosec} A = \dfrac{1}{\sin A} = \dfrac{4}{3}$.

Second method: Draw a right ΔABC right angled at B (Fig. 10.11)

we know that $\sin A = \dfrac{BC}{AC} = \dfrac{3}{4}$.

Therefore, if BC = 3k , then AC = 4k, where k is a positive number.

We also know that $AC^2 = AB^2 + BC^2$

$$\therefore (4k)^2 = AB^2 + (3k)^2$$

$$\Rightarrow 16k^2 = AB^2 + 9k^2$$

$$\Rightarrow AB^2 = 16k^2 - 9k^2$$

$$= 7k^2$$

$$\therefore AB = \sqrt{7}k .$$

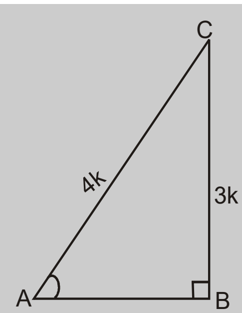

Fig. 10.11

Now, by definitions,

$$\cos A = \frac{AB}{AC} = \frac{\sqrt{7}k}{4k} = \frac{\sqrt{7}}{4}$$

$$\tan A = \frac{BC}{AB} = \frac{3k}{\sqrt{7}k} = \frac{3}{\sqrt{7}},$$

$$\cot A = \frac{AB}{BC} = \frac{\sqrt{7}k}{3k} = \frac{\sqrt{7}}{3},$$

$$\sec A = \frac{AC}{AB} = \frac{4k}{\sqrt{7}k} = \frac{4}{\sqrt{7}}$$

$$\operatorname{cosec} A = \frac{AC}{BC} = \frac{4k}{3k} = \frac{4}{3}.$$

Example 4. In a right triangle ABC , right angled at C, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of $\cos^2 A - \sin^2 A$.

Solution : In ΔABC, $\tan A = \frac{1}{\sqrt{3}}$ (Fig. 10.12)

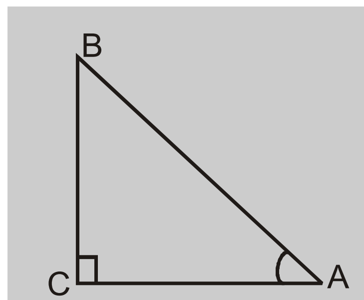

Fig. 10.12

$$\Rightarrow \frac{BC}{AC} = \frac{1}{\sqrt{3}}.$$

$\sqrt{3}BC = AC$ ....(1)

Let BC = k, then AC = $\sqrt{3}k$

Now, $AB^2 = AC^2 + BC^2$

$$= \left(\sqrt{3}k\right)^2 + k^2$$

$$= 3k^2 + k^2$$

$$= 4k^2$$

$\therefore$ $AB = \sqrt{4k^2} = 2k$

Then $\cos A = \frac{AC}{AB} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$

and $\sin A = \frac{BC}{AB} = \frac{k}{2k} = \frac{1}{2}$

$$\therefore \cos^2 A - \sin^2 A = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = 1/2.$$

Example 5. In a right triangle DEF, right angled at E, EF = 24 cm. DE = 7 cm. Find sinF and cos F.

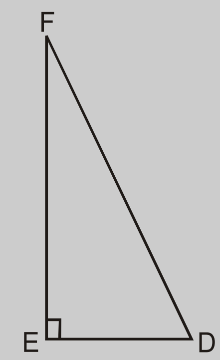

Fig. 10.13

Solution : In ΔDEF, we have,

$$DF^2 = DE^2 + EF^2$$
$$= 7^2 + 24^2$$
$$= 49 + 576$$
$$= 625$$
$$\therefore \quad DF = \sqrt{625} = 25cm$$

Then, $\sin F = \dfrac{DE}{DF} = \dfrac{7}{25}$

$\cos F = \dfrac{EF}{DF} = \dfrac{24}{25}$

Example 6. Express tan A, sec A and sin A in terms of cotA.

Solution : We know that $\cot A = \dfrac{1}{\tan A}$

$$\therefore \ \tan A = \frac{1}{\cot A}$$

Next,

$$\sec^2 A = 1 + \tan^2 A = 1 + \left(\frac{1}{\cot A}\right)^2$$

$$= 1 + \frac{1}{\cot^2 A} = \frac{\cot^2 A + 1}{\cot^2 A}$$

$$\therefore \ \sec A = \pm\sqrt{\frac{\cot^2 A + 1}{\cot^2 A}}$$

This gives, $\sec A = \dfrac{\sqrt{\cot^2 A + 1}}{\cot A}$ (why)

Further, $\text{cosec}^2 A = 1 + \cot^2 A$

$$\Rightarrow \ \frac{1}{\sin^2 A} = 1 + \cot^2 A$$

$$\Rightarrow \ \sin^2 A = \frac{1}{1 + \cot^2 A}$$

$$\therefore \ \sin A = \frac{1}{\sqrt{1 + \cot^2 A}}$$

Example 7. Determine whether the following is an identity :
$\tan^2 q + \cot^2 q = 2$

Solution : $\tan^2\theta + \cot^2\theta = 2$

$$\Rightarrow \tan^2\theta + \frac{1}{\tan^2\theta} = 2$$

$$\Rightarrow \tan^4\theta + 1 = 2\tan^2\theta$$

$$\Rightarrow \tan^4\theta + 1 - 2\tan^2\theta = 0$$

$$\Rightarrow (\tan^2\theta - 1)^2 = 0$$

$$\Rightarrow \tan^2\theta - 1 = 0$$

$$\Rightarrow \tan^2\theta = 1$$

This relation cannot hold for all admissible values of θ. For instance the relation does not hold when $\theta = 60^o$. Hence it is not an identity

Second method : If it can be shown that the equality does not hold for some particular value of θ, then the relation is not an identity.

Taking $\theta = 30^o$, we have,

L.H.S $= \tan^2 30^o + \cot^2 30^o$

$$= \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\sqrt{3}\right)^2$$

$$= \frac{1}{3} + 3 = \frac{1+9}{3}$$

$$= \frac{10}{3} \neq 2 \text{ (R.H. S)}$$

$\therefore$ the relation is not an identity

Example 8. Prove the following identities

(i) $\dfrac{1}{\text{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\text{cosec}\theta + \cot\theta}$

(ii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

(iii) $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = \sec\theta\ \text{cosec}\theta + 1$

Solution : **(i)** $\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}-\dfrac{1}{\sin\theta}=\dfrac{1}{\sin\theta}-\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}$

i.e $\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}+\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}=\dfrac{1}{\sin\theta}+\dfrac{1}{\sin\theta}$

i.e. $\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}+\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}=\dfrac{2}{\sin\theta}$ ....(1)

Now, L.H.S. of (1) $=\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}+\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}$

$$=\frac{\operatorname{cosec}\theta+\cot\theta+\operatorname{cosec}\theta-\cot\theta}{\operatorname{cosec}^2\theta-\cot^2\theta}$$

$$=\frac{2\operatorname{cosec}\theta}{1} \qquad (\because\ \operatorname{cosec}^2\theta=1+\cot^2\theta)$$

$$=\frac{2}{\sin\theta}=\text{R.H.S of (1)}$$

This establishes the given identity.

Alternative method :

L.H.S. $=\dfrac{\operatorname{cosec}\theta+\cot\theta}{(\operatorname{cosec}\theta-\cot\theta)(\operatorname{cosec}\theta+\cot\theta)}-\operatorname{cosec}\theta$

$$=\frac{\operatorname{cosec}\theta+\cot\theta}{\operatorname{cosec}^2\theta-\cot^2\theta}-\operatorname{cosec}\theta$$

$$=\operatorname{cosec}\theta+\cot\theta-\operatorname{cosec}\theta$$

$$=\operatorname{cosec}\theta+\frac{(\cot\theta-\operatorname{cosec}\theta)(\cot\theta+\operatorname{cosec}\theta)}{(\cot\theta+\operatorname{cosec}\theta)}$$

$$=\operatorname{cosec}\theta+\frac{\cot^2\theta-\operatorname{cosec}^2\theta}{\cot\theta+\operatorname{cosec}\theta}$$

$$=\operatorname{cosec}\theta-\frac{\operatorname{cosec}^2\theta-\cot^2\theta}{\operatorname{cosec}\theta-\cot\theta}$$

$$=\operatorname{cosec}\theta-\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

$$=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta+\cot\theta}=\text{R.H.S}$$

(ii) $\sin^6\theta + \cos^6\theta = (\sin^2\theta)^3 + (\cos^2\theta)^3$

$$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$$

$$= 1^3 - 3\sin^2\theta\cos^2\theta$$

$$= 1 - 3\sin^2\theta\cos^2\theta \cdot$$

(iii) L.H.S. $= \dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta}$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}$$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} - \frac{\cos^2\theta}{\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{\sin^3\theta-\cos^3\theta}{\cos\theta\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{(\sin\theta-\cos\theta)(\sin^2\theta+\cos\theta\sin\theta+\cos^2\theta)}{\cos\theta\sin\theta\,(\sin\theta-\cos\theta)}$$

$$= \frac{(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)}{\cos\theta\sin\theta}$$

$$= \frac{1+\sin\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\cos\theta\sin\theta} + 1$$

$$= \sec\theta\,\mathrm{cosec}\,\theta + 1 \qquad \text{............. (ii)}$$

= R.H.S.

EXERCISE 10.2

1. **In a right ΔABC, right angled at C, AC = 3cm, AB = 5cm.**
Find the value of (i) $\sin A, \cos A$ **(ii)** $\sin B, \cos B$

2. **In a right ΔDEF, right angled at E, DF = 13 cm, DE = 12 cm, find**
(i) $\tan D - \cot F$
(ii) $\tan D - \cot D$

3. If $\cos A = \frac{3}{5}$, calculate sin A and tan A.

4. If $\tan A = \frac{3}{4}$, find cosA and cosec A.

5. If $\sec A = \frac{13}{12}$, find tan A and sin A.

6. If $\cot A = \frac{4}{3}$, find the value of $\cos^2 A - \sin^2 A$

7. If $\tan\theta = \frac{a}{b}$, show that $\frac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta} = \frac{a^2 - b^2}{a^2 + b^2}$.

8. In a right ΔPQR, right angled at Q, PR - PQ = 1cm and QR = 5cm. Find the values of sin R, cos R and tan R.

9. Express $\cos\theta, \tan\theta$ and $\sec\theta$ in terms of $\sin\theta$.

10. **Determine whether the following equations are identities :**

(i) $\cot^2\theta + \cos\theta = \sin^2\theta$ **(ii)** $\frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta} = \frac{\sec\theta + 1}{\sec\theta - 1}$

(iii) $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$
(iv) $\tan\phi \operatorname{cosec}\phi = \sec\phi$

11. **Prove the following identities :**

(i) $\frac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2$

(ii) $\frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$

(iii) $\frac{\cos A}{1+\sin A}+\frac{1+\sin A}{\cos A}=2\sec A$

(iv) $\sec^2\theta+\operatorname{cosec}^2\theta=\sec^2\theta\operatorname{cosec}^2\theta$

(v) $\frac{1+\sin\theta}{1-\sin\theta}=(\sec\theta+\tan\theta)^2$

(vi) $\frac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)}=\cot\theta$

(vii) $\frac{\cos\theta}{\operatorname{cosec}\theta+1}+\frac{\cos\theta}{\operatorname{cosec}\theta-1}=2\tan\theta$

(viii) $\cos^2\theta-\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

(ix) $\sin^4\theta-\cos^4\theta=\sin^2\theta-\cos^2\theta$

(x) $\left(1+\frac{1}{\tan^2 A}\right)\left(1+\frac{1}{\cot^2 A}\right)=\frac{1}{\sin^2 A-\sin^4 A}$

(xi) $\tan^2 A+\cot^2 A+2=\sec^2 A\operatorname{cosec}^2 A$

(xii) $\frac{\sin^3 A+\cos^3 A}{\sin A+\cos A}=1-\sin A\cos A$

(xiii) $\frac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\frac{1+\sin\theta}{\cos\theta}$

(xiv) $\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}=\sin A+\cos A$

(xv) $\sin A(1+\tan A)+\cos A(1+\cot A)=\sec A+\operatorname{cosec} A$

(xvi) $\sin A(\sec A+\operatorname{cosec} A)-\cos A(\sec A-\operatorname{cosec} A)=\sec A\operatorname{cosec} A$.

ANSWER

1. **(i)** $\sin A=\frac{4}{5},\ \cos A=\frac{3}{5}$, **(ii)** $\sin B=\frac{3}{5},\ \cos B=\frac{4}{5}$.

2. **(i) 0** **(ii)** $\frac{-119}{60}$.

3. $\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$

4. $\cos A = \frac{4}{5}$, $\text{cosec} A = \frac{5}{3}$

5. $\tan A = \frac{5}{12}$, $\sin A = \frac{5}{13}$.

6. $\frac{7}{25}$

8. $\sin R = \frac{12}{13}$, $\cos R = \frac{5}{13}$, $\tan R = \frac{12}{5}$.

9. $\cos\theta = \sqrt{1 - \sin^2\theta}$, $\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$, $\sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$.

10.8 Trigonometric Ratios of Complementary Angles

We know that in any right triangle, there are two acute angles. What is the relation between them? They are complementary i.e. their sum is 90^o.

Fig. 10.14 shows a right triangle ABC, whose acute angles are $\angle A$ and $\angle C$.

Hence, $\angle A + \angle C = 90^o$ i.e. they are complementary.

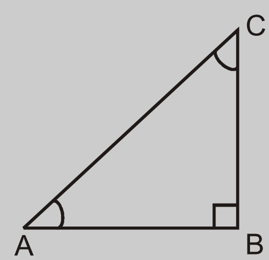

Fig. 10.14

We have,

$$\left.\begin{array}{l} \sin A = \frac{BC}{AC},\ \cos A = \frac{AB}{AC},\ \tan A = \frac{BC}{AB} \\ \text{cosec} A = \frac{AC}{BC},\ \sec A = \frac{AC}{AB},\ \cot A = \frac{AB}{BC} \end{array}\right\} \quad \text{........(i)}$$

Let us find the trigonometric ratios of $C = 90^o - A$.

First let us check what should be the side opposite to and adjacent to angle $90^o - A$. You will find that AB is the opposite side and BC, the adjacent side for the angle C i.e. $90^o - A$.

$$\therefore \quad \left.\begin{array}{l} \sin(90^\circ - A) = \dfrac{AB}{AC},\ \cos(90^\circ - A) = \dfrac{BC}{AC},\ \tan(90^\circ - A) = \dfrac{AB}{BC}, \\ \operatorname{cosec}(90^\circ - A) = \dfrac{AC}{AB},\ \sec(90^\circ - A) = \dfrac{AC}{BC},\ \cot(90^\circ - A) = \dfrac{BC}{AB} \end{array}\right\} \ldots\ldots(ii)$$

Let us now compare the ratios in (i) and (ii) . We will find that

$$\sin(90^\circ - A) = \frac{AB}{AC} = \cos A \text{ and } \cos(90^\circ - A) = \frac{BC}{AC} = \sin A,$$

$$\tan(90^\circ - A) = \frac{AB}{BC} = \cot A \text{ and } \cot(90^\circ - A) = \frac{BC}{AB} = \tan A$$

$$\sec(90^\circ - A) = \frac{AC}{BC} = \operatorname{cosec} A, \text{ and } \operatorname{cosec}(90^\circ - A) = \frac{AC}{AB} = \sec A.$$

Thus, $\sin(90^\circ - A) = \cos A$, $\quad \cos(90^\circ - A) = \sin A$

$\tan(90^\circ - A) = \cot A$, $\quad \cot(90^\circ - A) = \tan A$

$\sec(90^\circ - A) = \operatorname{cosec} A$, $\operatorname{cosec}(90^\circ - A) = \sec A$,

for all values of A lying between 0° and 90°.

It would be a good exercise to check whether these hold for A = 0° and A = 90°.

Example 9. Find the value of $\dfrac{\sin 18^\circ}{\cos 72^\circ}$

Solution : We know that cos A = $\sin(90^\circ - A)$

$$\therefore \quad \cos 72^\circ = \sin(90^\circ - 72^\circ) = \sin 18^\circ$$

Hence, $\dfrac{\sin 18^\circ}{\cos 72^\circ} = \dfrac{\sin 18^\circ}{\sin 18^\circ} = 1$.

Example 10. If $\tan 2A = \cot(A - 18^\circ)$, where 2A is an acute angle, find the value of A.

Solution : We know that

$\tan 2A = \cot(90^\circ - 2A)$

Hence, the given relation can be written as

$\cot(90^\circ - 2A) = \cot(A - 18^\circ)$

$\therefore \quad 90^\circ - 2A = A - 18^\circ$. [Since $90^\circ - 2A$ and $A - 18^\circ$ are acute angles.]

Then, $3A = 90^{o} + 18^{o}$

$= 108^{o}$

$\therefore \quad A = \dfrac{108^{o}}{3} = 36^{o}.$

Example 11. Express $\sin 67^{o} + \cos 65^{o}$ in terms of trigonometric ratios of angles between 0^{o} and 45^{o}.

Solution : $\sin 67^{o} + \cos 65^{o}$

$= \cos(90^{o} - 67^{o}) + \sin(90^{o} - 65^{o})$

$= \cos 23^{o} + \sin 25^{o}\cdot$

Example 12. In the triangle ABC and DEF, $\angle A=50^{o}$, $\angle B=90^{o}$ and AC=4cm, $\angle D=40^{o}$, $\angle E=90^{o}$ and DE = 4cm.

Prove that $\sqrt{\dfrac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF}} = \sin 50^{o}$.

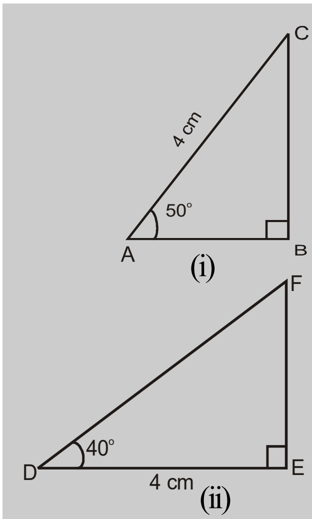

Fig. 10.15

Solution : In the Fig. 10.15. (i) , $\dfrac{AB}{AC} = \cos 50^{o}$

$\Rightarrow$ AB=AC $\cos 50^{o}$

$= 4\cos 50^{o}$ cm $\cdot$

and, $\dfrac{BC}{AC} = \sin 50^{o}$

$\Rightarrow$ BC=AC $\sin 50^{o}$

$= 4 \sin 50^{o}$ cm

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2} AB.BC$

$= \dfrac{1}{2} \times 4 \cos 50^{o} \times 4 \sin 50^{o} \text{ cm}^2$

$= 8 \cos 50^{o} \sin 50^{o} \text{ cm}^2$

In Fig 10.15 (ii), $\dfrac{EF}{DE} = \tan 40^{o}$

$\Rightarrow$ EF = DE $\tan 40^{o}$ cm.

$= 4 \tan 40^{o}$ cm.

$\therefore$ Area of $\Delta DEF = \frac{1}{2}$ DE.EF

$= \frac{1}{2}\ 4.4 \tan 40^o \text{cm}^2$

$= 8 \tan 40^o \text{cm}^2$

Now, $\frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF} = \frac{8\cos 50^o \sin 50^o}{8 \tan 40^o}$

$= \frac{\cos 50^o \sin 50^o}{\frac{\sin 40^o}{\cos 40^o}}$

$= \frac{\cos 40^o \cos 50^o \sin 50^o}{\sin 40^o}$

$= \frac{\sin(90^o - 40^o) \sin(90^o - 50^o) \sin 50^o}{\sin 40^o}$

$= \frac{\sin 50^o \sin 40^o \sin 50^o}{\sin 40^o}$

$= \sin^2 50^o$

$\therefore \sqrt{\frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF}} = \sin 50^o$

EXERCISE 10.3

1. Find the value of

(i) $\frac{\tan 49^o}{\cot 41^o}$ **(ii)** $\frac{\cos 70^o}{\sin 20^o}$ **(iii)** $\frac{\tan 26^o}{\cot 64^o}$

(iv) $\frac{\text{cosec } 35^o}{\sec 55^o}$ **(v)** $\cos 37^o - \sin 53^o$

(vi) $\text{cosec } 47^o - \sec 43^o$

2. Show that

(i) $\sec 53^o \sin 53^o \tan 37^o = 1$

(ii) $\tan 48^o \tan 23^o \tan 42^o \tan 67^o = 1$

(iii) $\sin 40^o \sec 40^o \sec 50^o = \text{cosec} 50^o$

(iv) $\cos 38^o \cos 52^o - \sin 38^o \sin 52^o = 0$

3. If $\tan A = \cot B$, show that $A + B = 90^o$.

4. If $\sec 4A = \text{cosec}(A - 20^o)$, where 4A is an acute angle, find the value of A.

5. **Find the value of θ, where all the angles are acute or equal to 90^o :**

(i) $\text{cosec } 4\theta = \sec 6\theta$

(ii) $\tan (2\theta + 35^o) = \cot 3\theta$

(iii) $\sin (\theta + 55^o) = \cos (7\theta - 5^o)$

6. Express $\cot 85^o + \cos 70^o$ in terms of trigonometric ratios of angles between 0^o and 45^o.

7. If A, B, C are angles of a triangle ABC, show that $\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$.

8. Compare the areas of the right triangles ABC and DEF in which $\angle A = 40^o$, $\angle B = 90^o$ and AC = 5cm; $\angle D = 50^o$, $\angle E = 90^o$ and DE = 5cm.

ANSWER

1. **(i) 1 (ii) 1. (iii) 1 (iv) 1 (v) 0 (vi) 0**

4. **$A = 22^o$.**

5. **(i) $\theta = 9^o$ (ii) $\theta = 11^o$ (iii) $\theta = 5^o$.**

6. $\tan 5^o + \sin 20^o$

8. $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \sin^2 40^o (\text{or } \cos^2 50^o)$

10.9 HEIGHTS AND DISTANCES

In this section, we shall see how trigonometry is applied in finding the heights and distances of various objects without actually measuring them. While dealing with such problems we will come across two terms known as the **angle of elevation** and the **angle of depression.**

Angle of Elevation :

Suppose you are looking up at an object B, say a Kite, (Fig 10.6) A being the position of your eye. Then AB, the line drawn from eye to the object viewed, is the line of sight. It makes an angle BAC with the horizontal line AC through your eye. This $\angle BAC$ is called the angle of elevation of B from A. Thus the angle of elevation of a point observed is the angle formed by the line of sight with the horizontal, when the point being observed is above the horizontal through the eye.

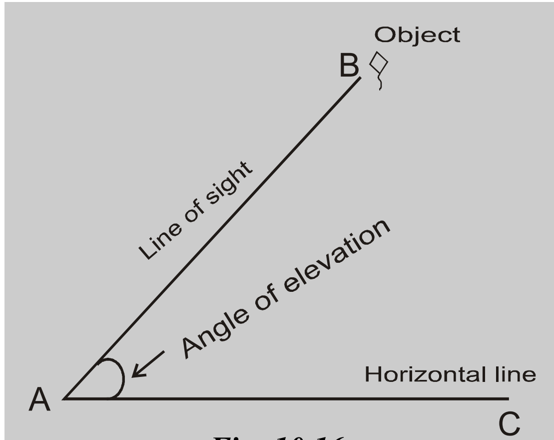

Fig. 10.16

Angle of Depression :

Suppose you are sitting on a balcony of a house and are looking down at a football B on the ground. (Fig 10.17), A being the position of your eye. AB is the line of sight and it makes an angle BAC with AC, the horizontal through A.

This $\angle BAC$ is called the angle of depression of B from A. Thus, the angle of depression of a point observed is the angle formed by the line of sight with the horizontal when the point observed is below the horizontal through the eye.

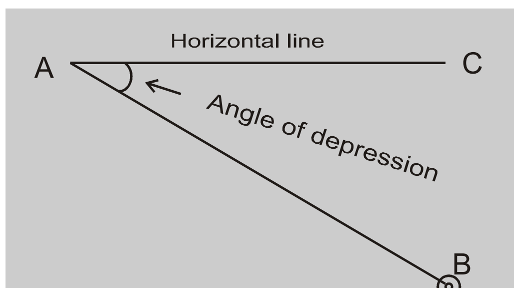

Fig. 10.17

After knowing these terms, we illustrate the process of using trigonometry in finding heights and distances with the help of some examples.

Example 13. A pole stands vertically on the ground. From a point on the ground, 15m away from the foot of the pole, the angle of elevation of the top of the pole is 60°. Find the height of the pole.

Solution : First let us draw a simple representative diagram of the problem.

Let BC represent the pole and let A be the point of observation.

Then, AC is the line of sight.

Hence $\angle BAC = 60^o$,and $\angle ABC = 90^o$ (why?) and AB = 15m. Think of a trigonometric ratio involving BC and the known side AB.

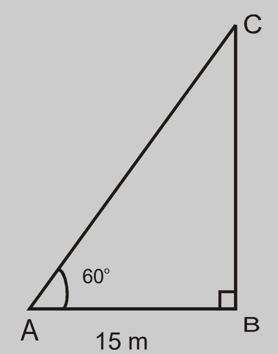

Fig. 10.18

Clearly, $\frac{BC}{AB} = \tan A = \tan 60^o = \sqrt{3}$

$\therefore$ BC = $AB\sqrt{3} = 15\sqrt{3}$.

i.e. height of the pole = $15\sqrt{3}$ m.

Example 14. Find the angle of elevation of the top of a tower of height $100\sqrt{3}$ m from a point on the ground , 100m away from its foot.

Solution : Let BC be the tower and A be the point of observation.

Then, $BC = 100\sqrt{3}$,

$\angle BAC = \theta$, the angle of elevation

and AB = 100 m.

In right triangle ABC,

$$\frac{BC}{AB} = \tan\theta$$

$$\Rightarrow \frac{100\sqrt{3}}{100} = \tan\theta$$

i.e. $\tan\theta = \sqrt{3}$

$$\Rightarrow \theta = 60^o$$

$\therefore$ the required angle of elevation is 60^0.

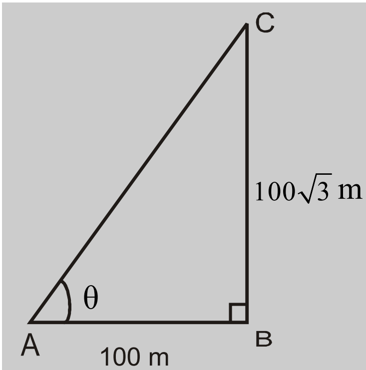

Fig. 10.19

Example 15. Find the height of a tree, if its shadow is 10m long when the altitude of the sun is 30^o.

Solution : Let BC represent the tree and AB its shadow. In the right ΔABC (right angled at B) we have $\angle A = 30^o$ (Fig. 10.20) and AB = 10m.

Now, $\frac{BC}{AB} = \tan 30^o$

$\Rightarrow BC = AB \frac{1}{\sqrt{3}}$

$\Rightarrow BC = \frac{10\sqrt{3}}{3} m$

$\therefore$ the height of the tree is $\frac{10\sqrt{3}}{3}$ m.

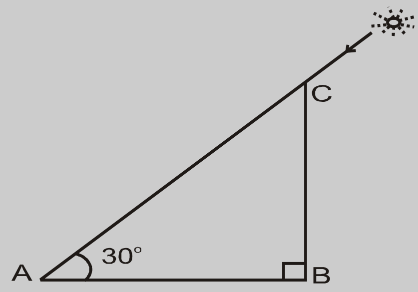

Fig. 10.20

Example 16. The shadow of a tower, standing on level ground, is found to be 45m longer when the sun's altitude is 30° than when it was at 60°. Find the height of the tower.

Solution : Let AB be the tower and BC, BD be the shadows when the altitude of the sun are 60^o and 30^o respectively,

Then $\angle BCA = 60^o$ and $\angle BDA = 30^o$ and CD = 45

Let BC = x m.

In ΔABC, $\tan C = \frac{AB}{BC}$

i.e. $AB = BC \tan 60^o$

$= BC \sqrt{3}$

$= \sqrt{3}x$ ....(i)

In ΔABD, $\tan D = \frac{AB}{BD}$

$\Rightarrow \tan 30^o = \frac{AB}{BC+CD}$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{AB}{x+45}$

$\Rightarrow x + 45 = \sqrt{3}AB$

$= \sqrt{3}x.\sqrt{3}x$ by (i)

$= 3x$

$\Rightarrow 45 = 2x.$

$\therefore x = \frac{45}{2} = 22.5$ m

$\therefore$ AB = height of the tower $= \sqrt{3}x = \sqrt{3} \times 22.5$ m

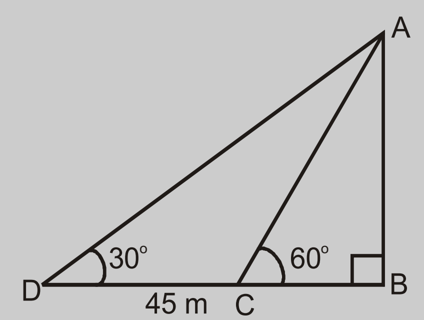

Fig. 10.21

Example 17. The angles of depression of the top and the bottom of a 7m tall tree from the top of a tower are 45° and 60° respectively. Find the height of the tower.

Solution : Let AB represent the tower and CD the tree .
Draw DE parallel to CB.
Then

$\angle XAD = 45^\circ$

and $\angle XAC = 60^\circ$ (Fig 10.22)

$\therefore \angle ADE = 45^\circ$ and $\angle ACB = 60^\circ$

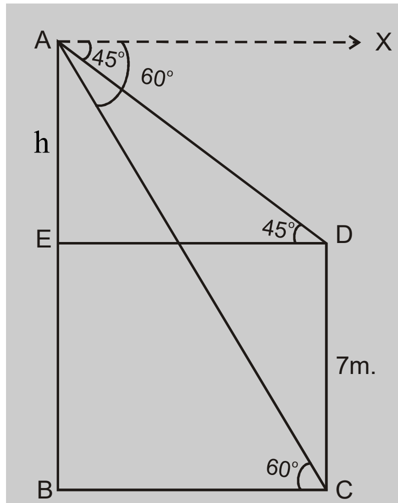

Fig. 10.22

In right ΔABC, $\dfrac{AB}{BC} = \tan 60^\circ$

$\Rightarrow AE+EB=BC.\sqrt{3}$

$\Rightarrow h+DC=BC.\sqrt{3}$

($\because$ EB=DC and if AE=h)

$\Rightarrow h+7=\sqrt{3}BC$ ....(1)

In right ΔADE , $\dfrac{AE}{ED} = \tan 45^\circ = 1$

$\Rightarrow AE=ED$

$\Rightarrow h=BC$ ....(2)

Using (2) in (1)

$h+7=\sqrt{3}h$

$\Rightarrow \sqrt{3}h-h=7$

$\Rightarrow (\sqrt{3}-1)h=7$

$\therefore h=\dfrac{7}{\sqrt{3}-1}=\dfrac{7(\sqrt{3}+1)}{2}$.

$\therefore$ AB = height of the tower = AE + EB = AE +DC

$= h + 7$

$= \left[\dfrac{7}{2}(\sqrt{3}+1)+7\right]$

$= \dfrac{7}{2}(3+\sqrt{3})m$

EXERCISE 10.4

1. **In a right ΔABC, $\angle B = 90^o$.**
   (i) Find BC, if AC = 15m, $\angle CAB = 30^o$
   (ii) Find AB, if AC = 40m, $\angle CAB = 45^o$
   (iii) Find BC if AC = 30m, $\angle CAB = 60^o$

2. Inclination of a ladder leaning against a wall is 30^o and the foot of the ladder is 2m away from the wall. Find the length of ladder.

3. The string of a kite is 100m long and it makes an angle of 60^o with the horizontal. Find the height of the kite, assuming that there is no slack in the string.

4. A ladder 10 m long is leaning against a wall. Its foot is 5 m away from the wall. Find the angle it makes with the wall.

5. A vertical post 9m high casts a shadow $3\sqrt{3}$ m long. Find the altitude of the Sun.

6. From the top of a light house the angles of depression of two ships, one directly behind the other, are 45^o and 30^o respectively. If the ships are 200m apart, find the height of the light house.

7. The angle of elevation of the top of a tower from a point A on the ground, is 30^o. On moving a distance of 20 m. towards the foot of the tower to a point B, the angle of elevation increases to 60^o. Find the height of the tower and the distance of the tower from the point A.

8. Two men, on opposite sides of a temple 75m high, observe the angles of elevation of the top of the temple to be 30^o and 60^o respectively. Find the distance between the two men.

9. A pole 5m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point A on the ground is 60^o and the angle of depression of the point A from the top of the tower is 45^o. Find the height of the tower.

10. From the top of a tower 50 m high the angles of depression of the top and the bottom of a pole are observed to be 45^o and 60^o respectively. Find the height of the pole, if the pole and the tower stand on the same horizontal plane.

11. From the top of a tree 9 m. high, the angle of elevation of the top of a building and the angle of depression of the foot of the building are respectively 45^o and 60^o. What is the height of the building?

12. A vertical stick 10cm. long casts a shadow 8 cm. long. At the same time, a tree casts a shadow 3 m long. Find the height of the tree.

13. A pole is partly broken by the wind and its top touches the ground at an angle of 30^o and at a distance of 8m. from the foot of the pole. Find the whole length of the pole.

14. From a light house the angles of depression of two ships on opposite sides of the light house are observed to be 30^o and 45^o. If the height of the light house be 100 metres, find the distance between the ships if the line joining them passes through the foot of the light house.

15. The angle of elevation of a bird from the eye of a man on the bank of a pond is 30^o and the angle of depression of its reflection in the pond is 60^o. Find the height of the bird above the pond if the distance of the eye from the foot of the man is 1.5 metres.

16. A bridge over a river makes an angle of 45^o with the river bank. If the length of the bridge over the river is 150m. what is the width of the river?

17. From a point on the ground 40 m. away from the foot of a tower, the angle of elevation of the top of the tower is 30^o and the angle of elevation of the top of a water tank (on the top of the tower) is 45^o. Find the (i) height of the tower and (ii) depth of the tank.

18. A straight highway leads to the foot of a tower of height 50m. From the top of the tower, the angles of depression of two cars standing on the highway are 30^o and 60^o. What is the distance between the two cars and how far is each car from the tower ?

19. As observed from the top of a light house 100 m. above the water level, the angles of depression of two ships are 30^o and 45^o. If one ship is to the north and the other to the east of the light house, find the distance between the two ships.

20. The angles of elevation of the top of a tower from two points at distances *a* and *b* from the base and in the same straight line with it are complementary. Prove that height of the tower is $\sqrt{ab}$.

21. A tower subtends an angle α at a point on the same level as the foot of the tower and at a second point h metres above the first, the depression of the foot of the tower is β. Show that the height of the tower is $h \tan\alpha \cot\beta$

22. A vertical tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. . At a point on the plane, the angle of elevation of the bottom of the flagstaff is α and that of the top of the flagstaff is β.Prove that the height of the tower is $\dfrac{h \tan\alpha}{\tan\beta - \tan\alpha}$.

ANSWER

1. (i) BC=$\frac{15}{2}$cm (ii) AB = $20\sqrt{2}$ cm (iii) BC= $15\sqrt{3}$cm
2. $\frac{4\sqrt{3}}{3}$ m.
3. $50\sqrt{3}$
4. 30°
5. 60°
6. 273.2m
7. height of tower = 17.3m
   distance of tower from the point A = 30m.
8. $100\sqrt{3}$ m.
9. 6.83 m.
10. 21.13 m.
11. $(9+3\sqrt{3})$m.
12. 3.75m.
13. $8\sqrt{3}$ m. 14. $100(1+\sqrt{3})$m. 15. 3 metres 16. $75\sqrt{2}$ m.
17. $\frac{40\sqrt{3}}{3}$ m., $\frac{40(3-\sqrt{3})}{3}$ m. 18. $\frac{100\sqrt{3}}{3}$ m.; $50\sqrt{3}$ m. and $\frac{50\sqrt{3}}{3}$ m.
19. 200 m.

SUMMARY

In this chapter, you have learnt the following points:

1. In a right ΔABC, right angled at B,

$$\sin A = \frac{\text{side opposite to } \angle A}{\text{hypotenuse}}, \quad \cos A = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{side oppsite to } \angle A}{\text{side adjacent to } \angle A}, \quad \cot A = \frac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A}$$

$\sec A = \dfrac{\text{hypotenuse}}{\text{side adjacent to } \angle A}$, $\quad$ $\text{cosec} A = \dfrac{\text{hypotenuse}}{\text{side opposite to } \angle A}$

2. The values of trigonometric ratios for angles 0^o, 30^o, 45^o, 60^o, and 90^o.

3. $\text{cosec}A = \dfrac{1}{\sin A}$, $\sec A = \dfrac{1}{\cos A}$, $\cot A = \dfrac{1}{\tan A}$, $\tan A = \dfrac{\sin A}{\cos A}$, $\cot A = \dfrac{\cos A}{\sin A}$,

$\sin^2 A + \cos^2 A = 1$, $\sec^2 A = 1 + \tan^2 A$, $\text{cosec}^2 A = 1 + \cot^2 A$

4. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be easily determined.

5. The values of sinA or cosA never exceeds 1, whereas the value of secA or cosecA is always greater than or equal to 1.

6. $\sin(90^o - A) = \cos A$, $\quad \cos(90^o - A) = \sin A$

$\tan(90^o - A) = \cot A$, $\quad \cot(90^o - A) = \tan A$

$\sec(90^o - A) = \text{cosec} A$, $\text{cosec}(90^o - A) = \sec A$.

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CHAPTER

11 COORDINATE GEOMETRY

11.1 Introduction

In class IX, you have seen that to locate the position of a point on a plane, a pair of mutually perpendicular lines known as the coordinate axes are required. Usually, one of the axes is taken along a horizontal line and the other along a vertical line. The horizontal line is known as the x-axis and the vertical line is known as the y-axis. The point of intersection of the x-axis and the y-axis is known as the origin. The distance of a point from the y-axis is called its x-coordinate or abscissa and the distance of a point from the x-axis is called its y-coordinate or ordinate. You have seen that the coordinates of a point on the plane are of the form (x, y) where x is the abscissa and y is the ordinate. The coordinates of a point on the x-axis are of the form $(x,0)$ and those of a point on the y-axis are of the form $(0,y)$. You have also learnt about plotting of points in a plane when their coordinates are given. Further, you have learnt that the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and in particular, the distance of the point $R(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

In this chapter, you will see how the co-ordinates of the point, which divides the line segment joining two given points in a given ratio, can be found. Next, the area of a triangle will be found in terms of the coordinates of its vertices. Henceforth, we shall write, $(x_1, y_1) = (x_2, y_2)$ whenever the points, whose coordinates are (x_1, y_1) and (x_2, y_2) ,are coincident.

11.2 SECTION FORMULA

Let P and Q be two points in a plane say, the plane of the paper (Fig11.1). Let us take a ratio say, 2:1, at which the line PQ is to be divided. Let R be the point on PQ such that PR=2RQ (Fig 11.1). Then PR:RQ=2:1. Therefore, the point R divides PQ in the ratio 2:1 and observe that R lies on PQ.

Fig. 11.1

Again, let R' be the point on PQ produced (Fig 11.2) such that PQ=QR'. Then PR'=PQ+QR'=2QR' ie. PR':R'Q=2:1

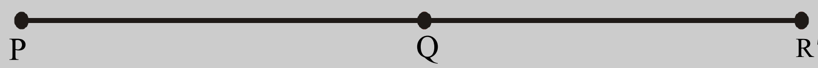


Fig. 11.2

Therefore, the point R' also divides PQ in the same ratio 2:1 and observe that R' lies on PQ produced. We have seen that R and R' divide PQ in the same ratio 2:1. We have also observed that R lies on PQ whereas R' lies on PQ produced. Here R is said to divide PQ internally and R' is said to divide PQ externally in the ratio 2:1. In other words, R is the point of internal division of PQ in the ratio 2:1 and R' is the point of external division of PQ in the same ratio 2:1.

Thus, we have seen that, for a given ratio, there are two points of division of a line joining two points: one, the point of internal division and the other, the point of external division.Here, we shall consider the case of internal division only.

In this section, we will find a formula, known as section formula, for finding the coordinates of the point R in terms of the coordinates of P and Q when R divides PQ internally in the ratio m:n.

Let $X'OX$, $Y'OY$ be the coordinate axes so that O is the origin. Let (x_1, y_1) and (x_2, y_2) be the coordinates of P and Q respectively. Let (x, y) be the coordinates of the point R which divides PQ internally in the ratio m:n (Fig 11.3).

Draw PA, QB and RC perpendicular to the x-axis and draw PD, RE perpendicular to RC, QB respectively as in Fig 11.3.

Then, in ΔPDR,

$$PD = AC = OC - OA = x - x_1$$

$$RD = CR - CD = y - y_1$$

In ΔREQ

$$RE = CB = OB - OC = x_2 - x$$

$$QE = BQ - BE = BQ - CR = y_2 - y$$

In Δ^sPDR and REQ,

$\angle PDR = \angle REQ = 90^\circ$ (why ?)

$\angle RPD = \angle QRE$ (corresponding angles)

$\therefore$ by AA similarity

$\Delta PDR \sim \Delta REQ$

$$\therefore \frac{PR}{RQ} = \frac{PD}{RE} = \frac{RD}{QE}$$

i.e. $\frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$

Considering $\frac{m}{n} = \frac{x - x_1}{x_2 - x}$,

we get $x = \frac{mx_2 + nx_1}{m + n}$

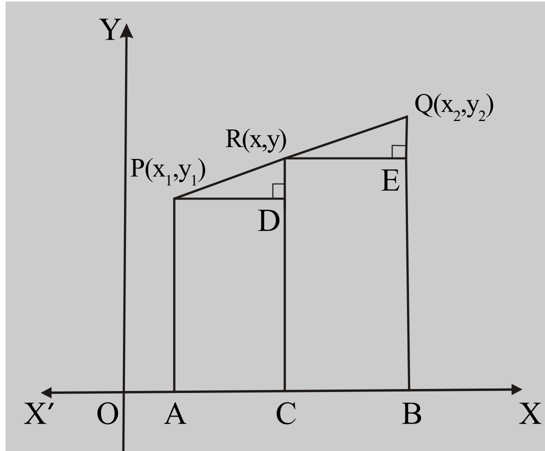


Fig .11.3

Again, considering $\frac{m}{n} = \frac{y - y_1}{y_2 - y}$, we get $y = \frac{my_2 + ny_1}{m + n}$

$$\therefore (x, y) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$$

Thus, the coordinates of the point R which divides the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio m:n are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$. It is known as the section formula.

Note: To find the abscissa of the point of division, m multiplies to x_2 and n to x_1 ie. cross wise. Similar is the case for finding the ordinate also.

Deduction: The mid-point of a line segment is the point which divides the line segment in the ratio 1:1. Thus, the coordinates of the mid-point of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ are

$$\left(\frac{1.x_2 + 1.x_1}{1 + 1}, \frac{1.y_2 + 1.y_1}{1 + 1} \right) \text{i.e.} \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$$

Example 1. Find the coordinates of the point which divides the line segment joining the points (–1, 3) and (4, –7) internally in the ratio 3:4.

Solution : Let P(x,y) be the required point. Then, by Section formula,

$$x = \frac{3.4 + 4.(-1)}{3+4} = \frac{8}{7},\ y = \frac{3.(-7) + 4.3}{3+4} = \frac{-9}{7}$$

$\therefore$ the required point is $\left(\frac{8}{7}, \frac{-9}{7}\right)$

Example 2. Find the points of trisection of the line segment joining the points A(3, –2) and B(–3,– 4).

Solution: Let P and Q be the points of trisection of AB

Then, P divides AB in the ratio 1:2

$\therefore$ The co-ordinates of P are $\left(\frac{1\times(-3)+2\times3}{1+2}, \frac{1\times(-4)+2\times(-2)}{1+2}\right)$

i.e. $\left(1, \frac{-8}{3}\right)$

Again, Q divides AB in the ratio 2:1

$\therefore$ the coordinates of Q are $\left(\frac{2\times(-3)+1\times3}{2+1}, \frac{2\times(-4)+1\times(-2)}{2+1}\right)$

i.e. $\left(-1, \frac{-10}{3}\right)$

$\therefore$ the required points of trisection of AB are $\left(1, \frac{-8}{3}\right)$ and $\left(-1, \frac{-10}{3}\right)$

Example 3. In what ratio, is the line joining (–3, –1) and (–8, –9) divided at the point $\left(-5, \frac{-21}{5}\right)$?

Solution : Let the required ratio be m:n

Then, $\left(-5, \frac{-21}{5}\right) = \left(\frac{m.(-8) + n.(-3)}{m+n}, \frac{m.(-9) + n.(-1)}{m+n}\right)$

i.e. $-5 = \frac{-8m - 3n}{m+n}, \frac{-21}{5} = \frac{-9m - n}{m+n}$

From the 1st eq^n,

$$-5(m+n) = -8m - 3n$$

i.e. $8m - 5m = 5n - 3n$

i.e. $3m = 2n$

i.e. $\frac{m}{n} = \frac{2}{3}$

i.e. $m:n = 2:3$

$\therefore$ The required ratio is 2:3

Note: The ratio m:n could have been solved from the 2nd eqn $-\frac{21}{5} = \frac{-9m - n}{m+n}$, as well.

Example 4. Find the ratio in which the line segment joining A(2, – 4) and B(–3,6) is divided by the x-axis. Find the point of division also.

Solution : Let m:n be the required ratio. Then, the coordinates of the point dividing AB in the ratio m:n are $\left(\frac{-3m+2n}{m+n}, \frac{6m-4n}{m+n}\right)$. But this point lies on the x-axis and so, the ordinate is zero,

i.e. $\frac{6m-4n}{m+n} = 0$

i.e. $6m - 4n = 0$

i.e. $6m = 4n$

i.e. $\frac{m}{n} = \frac{2}{3}$

i.e. $m:n = 2:3$

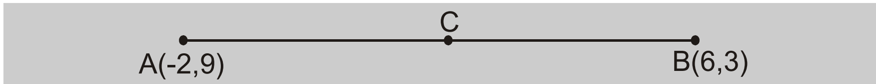
$\therefore$ The required ratio is 2:3

Again, the point of division is $\left(\frac{-3m+2n}{m+n}, \frac{6m-4n}{m+n}\right)$

i.e, $\left(\frac{-3\frac{m}{n}+2}{\frac{m}{n}+1}, 0\right)$ i.e $\left(\frac{-3\times\frac{2}{3}+2}{\frac{2}{3}+1}, 0\right)$ i.e.(0,0)

Example 5. The line segment joining A(-2,9) and B(6,3) is a diameter of a circle with centre C. Find the coordinates of C.

Solution :



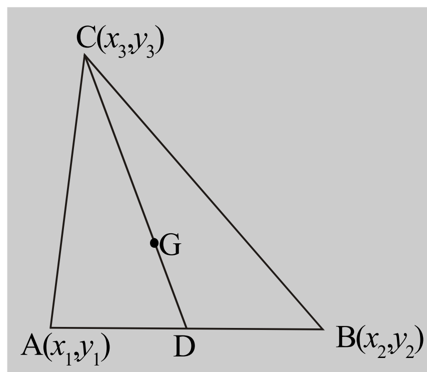
Since the centre C is the mid-point of the diameter AB, the coordinates of C are $\left(\frac{-2+6}{2},\frac{9+3}{2}\right)$ i.e $(2,6)$.

Example 6. Find the coordinates of the centroid of the triangle with vertices A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3).

Solution : Let D be the mid point of the side AB.

Then the co-ordinates of D are $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$. Join CD and let G be the centroid of ΔABC. Then G divides CD internally in the ratio 2:1.

$\therefore$ the coordinates of G are



$$\left(\frac{2.\frac{x_1+x_2}{2}+1.x_3}{2+1},\frac{2.\frac{y_1+y_2}{2}+1.y_3}{2+1}\right)$$

i.e. $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

$\therefore$ the centroid ΔABC is $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

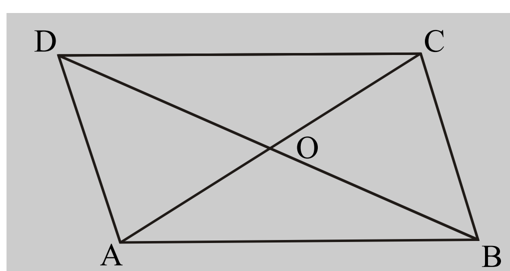
Example 7. Show that the points A (3,1) , B(0, –2) , C(1,1) and D (4,4) are the vertices of a par[m] ABCD.

Solution: Mid-points of AC

$$=\left(\frac{3+1}{2},\frac{1+1}{2}\right)=(2,1)$$

Again, mid-point of BD

$$=\left(\frac{0+4}{2},\frac{-2+4}{2}\right)=(2,1)$$



$\therefore$ the mid-point of AC coincides with the mid-point of BD i.e., the diagonals of ABCD bisect each other.

$\therefore$ ABCD is a parallelogram.

Example 8. If three cosecutive vertices of a par^{m} are A(1,–2), B(3,6) and C(5,10), find its fourth vertex.

Solution : Let D (x,y) be the fourth vertex of the par^{m}.

Then, the diagonals AC and BD bisect each other.

$\therefore$ mid-point of AC = mid-point of BD

i.e. $\left(\frac{1+5}{2}, \frac{-2+10}{2}\right) = \left(\frac{3+x}{2}, \frac{6+y}{2}\right)$

i.e. $(3,4) = \left(\frac{3+x}{2}, \frac{6+y}{2}\right)$

i.e. $3 = \frac{3+x}{2}, 4 = \frac{6+y}{2}$

i.e. $x = 3, y = 2$.

$\therefore$ The fourth vertex is (3,2)

EXERCISE 11.1

1. Find the coordinates of the point which divides the line segment joining the points (6,3) and (–4,5) in the ratio 3:2 internally.

2. Find the coordinates of the point which divides the line joining (1,–2) and (4,7) internally in the ratio 1:2.

3. Find the coordinates of the mid-point of the line joining the points (4,7) and (6,9).

4. Find the co-ordinates of the point which divides internally the line joining the points (p,q) and (q,p) in the ratio p–q : p+q.

5. Let A(–1,5) and B(6,–2) be two given points. Find the coordinates of the point P such that $AP = \frac{3}{7}AB$ and P lies on AB.

6. The mid-point of the line segment joining A($2a$,4) and B(–2,$3b$) is M(1,$2a$+1). Find the values of a and b.

7. AB is a diameter of a circle with centre C(–1,6). If the coordinates of A are (–7,3), find the coordinates of B.

8. Find the coordinates of the point where the diagonals of the parallelogram ABCD formed by joining the points A(–2,–1), B(1,0), C(4,3) and D(1,2) meet.

9. Find the ratio in which the line segment joining (-2,-3) and (5,6) is divided by x-axis. Also, find the coordinates of the point of division.

10. In what ratio is the line segment joining the points (–2,–3) and (3,7) divided by the y-axis ? Also, find the co-ordinates of the point of division.

11. If a vertex of a triangle be (1,1) and the middle points of the sides through it be (–2,3) and (5,2), find the other vertices.

12. Show that the points (3,–2), (4,0),(6,–3) and (5,–5) are the vertices of a parllelogram taken in order.

13. Show that the four points (1,2), (3,0) , (7,4) and (5,6) taken in order are the angular points of a rectangle.

14. Determine the ratio in which the point P(10,*m*) divides the join of A(5,2) and B(17,14). Also, find the value of m.

15. Find the centroid of ΔABC whose vertices are A(-1,0),B(5, –2) and C(8,2).

16. If the points A(a, –11), B(5,b), C(2,15) and D(1,1) taken in order are the vertices of a parallelogram ABCD, find the values of a and b.

17. Three vertices of a rectangle are the points (3,4),(–1,2) and (2, –4); what are the co-ordinates of the fourth vertex.

ANSWER

1.	$\left(0, \frac{21}{5}\right)$	**2.** (2,1)		**3.** (5,8)		**4.** $\left(\frac{p^2-q^2+2pq}{2p}, \frac{p^2+q^2}{2p}\right)$	
5.	**(2,2)**	**6. a=2, b=2**		**7. (5,9)**		**8. (1,1)**	
9.	1:2; $\left(\frac{1}{3}, 0\right)$	**10. 2:3 ; (0,1)**		**11. (-5,5),(9,3)**		**14. 5:7 ; m = 7**	
15.	**(4,0)**	**16. a = 4, b = 3**		**17. (6, –2)**			

11.3 AREA OF A TRIANGLE

In earliar classes, the area of a triangle have been computed by using the formula

Area of a triangle = $\frac{1}{2} \times$ Base $\times$ Altitude

In class IX, you have used Heron's formula to find the area of a triangle when the length of its sides are given. When the coordinates of the vertices of a triangle are given, you can find the lengths of the three sides of the triangle by using distance

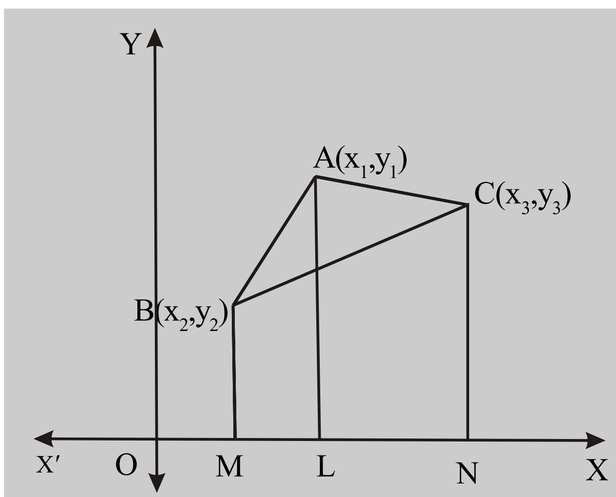


Fig. 11.4

formula and hence the area of the triangle by using the Heron's formula. But, when the lengths of the sides are irrational numbers, this process becomes tedious. So, it is prefered to compute the area in terms of the coordinates of the vertices of the triangle as follows:

Let A (x_1, y_1), B(x_2,y_2) and C(x_3, y_3) be the vertices of a ΔABC (fig 11.4).

Draw AL, BM and CN perpendicular to the x-axis as in the adjoining figure.From the figure

Area of ΔABC = Area of trapezium ABML + Area of trapezium ALNC

–Area of trapezium BMNC

But, we know that area of trapezium

$= \frac{1}{2} \times$ sum of parallel sides × perp. distance between them

$\therefore$ Area of ΔABC

$$= \frac{1}{2}(AL+BM)ML + \frac{1}{2}(AL+CN)LN - \frac{1}{2}(BM+CN)MN$$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1\{(y_1 + y_2) - (y_1 + y_3)\} + x_2\{(y_2 + y_3) - (y_1 + y_2)\} + x_3\{(y_1 + y_3) - (y_2 + y_3)\}]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$\therefore$ Area of a triangle $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Sign of Area : If the three points are taken in the anticlockwise sense, then the area calculated of the ΔABC will be positive, while if the points are taken in clockwise sense, then the area calculated will be negative. In case, the area calculated is negative, the negative sign will be ignored i.e. the absolute value of the area is taken only because negative area is meaningless.

Note: (1) If the area of a ΔABC is zero, then the three vertices A, B, C are collinear (ie, lie on the same line) and conversely.

(2) To find the area of a polygon, we divide it into triangles having no common area and take the numerical value of the area of each of the triangles and add.

Example 1. Find the area of a triangle whose vertices are (6,3),(–3,4) and (4,–2).

Solution : Let $A(x_1,y_1) = (6,3)$, $B(x_2, y_2) = (-3,4)$ and $C(x_3, y_3) = (4,-2)$

Then,

$$\text{area of } \Delta ABC = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[6(4+2) + (-3)(-2-3) + 4(3-4)]$$

$$= \frac{1}{2}[36+15-4]$$

$$= \frac{1}{2} \times 47$$

$$= \frac{47}{2}$$

$\therefore$ the required area of the triangle is $\frac{47}{2}$ sq. units.

Example 2. Find the area of the triangle formed by the points (a, c + a), (a,c) and (–a, c–a).

Solution : Let $A(x_1, y_1) = (a, c+a)$, $B(x_2, y_2) = (a, c)$ and $C(x_3, y_3) = (-a, c-a)$.

$$\text{Then, area of } \Delta ABC = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[a(c-c+a)+(c-a-c-a)+(-a)(c+a-c)]$$

$$= \frac{1}{2}[a^2 + a(-2a) - a(a)]$$

$$= \frac{1}{2}[a^2 - 3a^2]$$

$$= -a^2$$

Neglecting the –ve sign, area of $\Delta ABC = a^2$ sq. units.

Example 3. Show that the points (–5, 1), (5,5) and (10,7) are collinear.

Solution : Let $A(x_1,y_1) = (-5,1)$, $B(x_2, y_2) = (5,5)$ and $C(x_3,y_3) = (10,7)$

$$\text{Then, area of } \Delta ABC = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}[(-5)(5-7)+5(7-1)+10(1-5)]$$
$$= \frac{1}{2}[10+30-40]$$
$$= \frac{1}{2}\times 0$$
$$= 0$$

$\therefore$ the points A, B, C i.e. the given points are collinear.

Example 4. Find the value of p for which the points (–5,1), (1,p) and (4,–2) are collinear.

Solution : Let A(x_1, y_1) = (–5,1), B(x_2, y_2) = (1, p) and C(x_3, y_3) = (4,–2).

Then, since A, B,C are collinear, area of $\Delta ABC = 0$

$$\therefore \quad \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0$$

i.e. $(-5)(p+2)+1(-2-1)+4(1-p)=0$

i.e. $-5p-10-3+4-4-4=0$

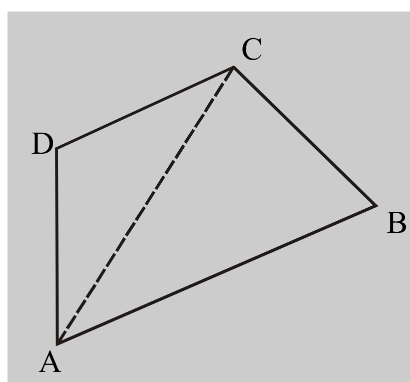
i.e. $9p=-9$

Hence, p = –1.

Example 5. Find the area of the quadrilateral whose vertices are (1,1),(3,4), (5,–2) and (4,–7) taken in order.

Solution: Let A(1,1), B(3,4), C(5,–2), D(4,–7) be the vertices.
Then, area of quadrilateral ABCD

= area of ΔABC + area of ΔACD



$$= \frac{1}{2}\left|1(4+2)+3(-2-1)+5(1-4)\right|$$
$$+\frac{1}{2}\left|1(-2+7)+5(-7-1)+4(1+2)\right|$$
$$= \frac{1}{2}\left|6-9-15\right|+\frac{1}{2}\left|5-40+12\right|$$
$$= \frac{1}{2}\left|-18\right|+\frac{1}{2}\left|-23\right|$$
$$= \left(\frac{1}{2}\times 18\right)+\left(\frac{1}{2}\times 23\right)$$

$= \frac{41}{2}$

$\therefore$ area of the quadrilateral is $\frac{41}{2}$ sq. units.

EXERCISE 11.2

1. Find the area of a triangle whose vertices are

(i) $(-5,-3),(-4,-6),(2,-1)$

(ii) $\left(at_1, \frac{a}{t_1}\right), \left(at_2, \frac{a}{t_2}\right), \left(at_3, \frac{a}{t_3}\right)$

(iii) $(5,2),(4,7),(7,-4)$

2. Find the area of a quadrilateral whose vertices are $(-4,-2),(-3,-5),(3,-2)$ and $(2,3)$ taken in order.

3. Show that the following points are collinear.

(i) $(-1,1),(5,7),(8,10)$

(ii) $(3a, 0), (0,3b), (a,2b)$

(iii) $(a, b+c), (b, c+a), (c, a+b)$

4. In each of the following, find the value of 'x' for which the points are collinear.

(i) $(8,1), (x, -4), (2, -5)$

(ii) $\left(-\frac{1}{2}, 3\right)$, $(-5,6), (-8, x)$

(iii) $(x,5),(2,-5),(4,-9)$

5. If the points (x, y), $(-5,7)$ and $(-4,5)$ are collinear, then show that $2x+y+3=0$.

6. Prove that the points $(a,0)$, $(0,b)$ and $(1,1)$ are collinear, if $\frac{1}{a}+\frac{1}{b}=1$.

7. If the points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on the same line, prove that

$$\frac{y_2-y_3}{x_2x_3}+\frac{y_3-y_1}{x_3x_1}+\frac{y_1-y_2}{x_1x_2}=0$$

8. If a, b, c are distinct, prove that the points (a, a^2), (b, b^2), (c, c^2) can never be collinear.

9. Find the area of the quadrilateral whose vertices are (1,1), (7, –3), (12, 2) and (7, 21) taken in order.

10. Find the area of the rhombus whose vertices, taken in order, are (3,0), (4,5), (–1,4), (–2, –1).

ANSWER

1. **(i)** $\frac{23}{2}$ **sq. units** **(ii)** $\frac{a^2}{2t_1t_2t_3}(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ **(iii) 2 sqr. units**

2. **28 sq. units** **4. (i) 3 (ii) 8 (iii) -3** **9. 132 sq. units**

10. **24 sq. units.**

SUMMARY

In this chapter, the following points have been studied :

1. The co-ordinates of the point R which divides the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio m : n are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

2. The mid-point of the line segment joining (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

3. Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

****

CHAPTER

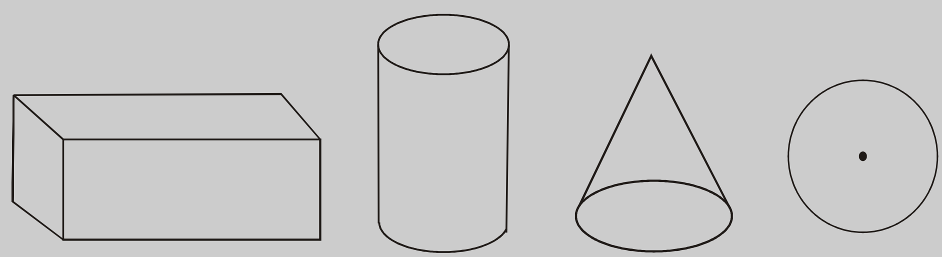
12

MENSURATION

12.1 Introduction

In the earlier classes , you have learnt some methods to find perimeters and areas of simple plane figures like triangles, squares, rectangles, parallelograms and circles. In our daily life, we observe many objects in circular form such as cycle wheels, barrow wheel, round cake, face of a full moon etc. The study of finding perimeter and area of such circular objects plays an important role in our daily life. In this chapter we shall recall the formulas for finding the perimeter and the area of a circle and the process of deriving these formulas. Then we shall derive the area of sectors and segments of a circle and also area of combination of plane figures involving circles or parts of circles.

In your previous classes, you have learnt about the surface areas and volumes of a number of solids such as cuboid, right circular cylinder, right circular cone, sphere etc. [see Fig 12.1]. You are also familiar with the shapes of these solids and the methods to find their surface areas and volumes.



(i) *(ii)* *(iii)* *(iv)*

Fig. 12.1

In our daily life, we use a number of solids which are parts or combinations of the above familiar solids. For example, a test tube used in science laboratory, is a combination of a cylinder and a part of a sphere. A conical circus-tent is a right circular cone surmounted on a circular cylinder. An ice-cream is a combination of a hemisphere and a right circular cone [see Fig 12.2]

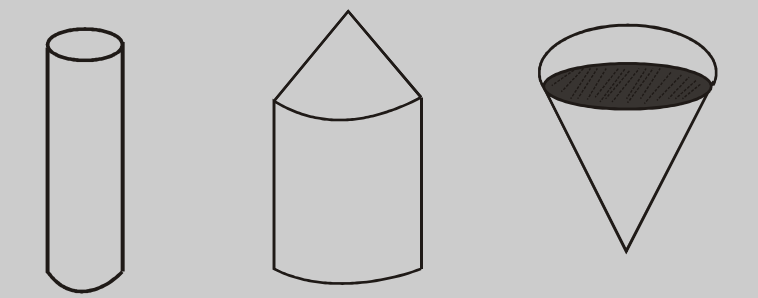


Fig. 12.2

In the subsequent sections, we shall discuss the methods of finding surface area and volume of such combinations.

12.2 Perimeter and Area of a Circle

In class (VIII), you have learnt about the perimeter and the area of a circle. Recall that the perimeter of a circle is the measure of its circumference and the circumference of a circle bears a constant ratio to its diameter. This constant ratio is denoted by the Greek letter π (read as pi). In other words,

$$\frac{\text{Circumference}}{\text{Diameter}} = \pi$$

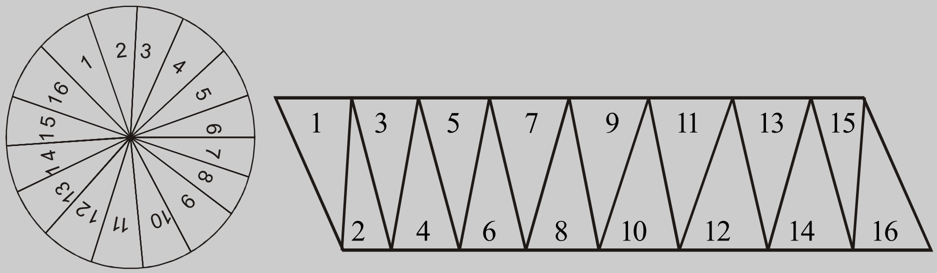
$$\therefore \text{ Circumference} = \pi \times \text{Diameter}$$

$= \pi \times d$, where 'd' denotes diameter

$= \pi \times 2r$, where 'r' denotes radius

$= 2\pi r$

The value of π is generally taken as $\frac{22}{7}$ or 3.14. Both these are approximate values. Further, you may recall that the area of a circle is πr^2, where 'r' is the radius of the circle. You have also verified this fact in class (viii) , by cutting a circle into a number of sectors and rearranging them [see Fig. 12.3]. Here the shape in Fig. 12.3(ii) approaches a rectangle of length $\frac{1}{2} \times 2\pi r$ and breadth r , giving that the area of the circle $= \frac{1}{2} \times 2\pi r \times r = \pi r^2$.



(i) *Fig. 12.3* (ii)

Example 1. **The wheels of an Alto car are of diameter 49 cm. If the car is travelling at the speed of 44 km per hour, find the number of complete revolutions that each wheel makes in 21 minutes (use $\pi = \frac{22}{7}$).**

Solution : Here, the diameter of the car wheel = 49 cm i.e. $2r = 49$.

The distance covered in one revolution $= 2\pi r$

$= \pi(2r)$

$= 49\pi = 154$ cm

The distance covered by the car in 1 hr $= 44$km

$= 44 \times 1000 \times 100\ cm$

$\therefore$ the distance covered in 21 minutes $= \dfrac{4400000 \times 21}{60}$

$= 1540000$ cm

$\therefore$ the required number of revolutions $= \dfrac{1540000}{154}$

$= 10000$

Example 2. **The cost of fencing around a circular track at ₹ 10.50 per metre is Rs. 6468. The track has a uniform width of 7m. Find the area of the track $\left(\text{take } \pi = \frac{22}{7}\right)$.**

Solution : The outer circumference of the circular track $= \dfrac{6468}{10.50}$

$= 616\ m.$

Let R be the outer radius of the track. Then

$$2\pi R = 616$$

$$\Rightarrow R = \frac{616}{2\pi}$$

$$= \frac{616 \times 7}{2 \times 22}$$

$$= 98 \text{ m}$$

Also, the width of the track = 7 m

$\therefore$ the inner radius of the track, r = (98–7)
= 91 m

and the area of the track $= \pi(R^2 - r^2)$

$$= \frac{22}{7}(98 + 91)(98 - 91)$$

$$= \frac{22}{7} \times 189 \times 7$$

$$= 4158 \text{ m}^2$$

$\therefore$ the required area of the track is 4158 m^2

EXERCISE 12.1

$\left(\text{Unless otherwise stated, use } \pi = \frac{22}{7}\right)$

1. Find the area of a circle whose radius is 14 cm.

2. Find the area of a quadrant of a circle whose circumference is 44 cm.

3. The circumference of a circle exceeds the diameter by 21 cm. Find the area of the circle.

4. The area of a circle is 1386 cm^2; find its circumference.

5. The sum of the radii of two circles is 35 cm and the difference of their areas is 770 cm^2. Find their circumferences.

6. The diameter of each wheel of a vehicle is 42 cm. Each wheel makes 20 revolutions in 12 seconds. Find the speed of the vehicle in metre per minute.

7. Find the radius of a circle whose area is equal to the difference of the areas of two circles of radii 6 cm and 10 cm respectively.

8. The difference between the perimeter and the radius of a circle is 37 cm. Find the area of the circle.

9. In Fig. 12.4, OABC is a rhombus, three of whose vertices lie on a circle with centre O. If the area of the rhombus is $18\sqrt{3}\ cm^2$, find the perimeter and area of the circle [take $\pi = 3.14$].

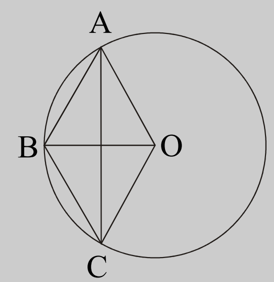


Fig. 12.4

10. If the areas of two concentric circles are 154 cm^2 and 616 cm^2 respectively, then find the width and area of the circular path between them.

11. Find the area of the largest circle that can be drawn inside a square of side 21 cm.

12. The cost of levelling a path of uniform width which surrounds a circular park of diameter 320 m at the rate of ₹ 7.00 per square metre is ₹ 35750. Find the width of the path.

13. The cost of fencing a circular field at the rate of ₹ 18.00 per metre is ₹ 3960. The field is to be levelled at the rate of ₹ 6.00 per square metre. Find the cost of levelling.

14. Grass is planted on a circular strip of width 7m. The radius of the inner edge of the grass strip is 21m. If the cost of planting is ₹ 4.00 per square metre, then find the total cost of planting grass.

ANSWER

1. 616 cm^2	**2. 38.5 cm^2**	**3. 75.46 cm^2**	**4. 132 cm**
5. 132 cm, 88 cm	**6. 132 m/min**	**7. 8 cm**	**8. 154 cm^2**
9. 37.68 cm, 113.04cm^2	**10. 7 cm, 462 cm^2**	**11. 346.5 cm^2**	**12. 5 m**
13. ₹ 23100	**14. ₹ 4312**		

12.3 Area of Sectors and Segments of a Circle

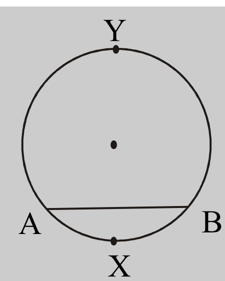


Fig. 12.5

In earliar classes, you have learnt about the terms chord, arc, sector, segment etc. of a circle. In this section, we shall find the area of a sector assuming the fact that the area of a sector of a circle is proportional to the sectorial angle and then deduce the area of a segment.

Recall :

The line segment joining any two points of a circle is called a **Chord**. In Fig. 12.5, $\overline{AB}$ is a chord of the circle.

A part of a circle between two distinct points on it, is called an **arc** of the circle. In fig. 12.5, $\overset{\frown}{AXB}$ is a **minor** arc and $\overset{\frown}{BYA}$ is a **major** arc.

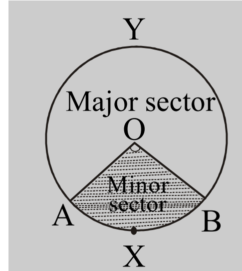


Fig. 12.6

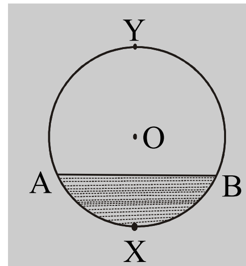


Fig. 12.7

The part (or portion) of the circular region enclosed by two radii (say) OA, OB and the corresponding arc $\overset{\frown}{AB}$ of a circle is called a sector of the circle. [Fig. 12.6] Here, for the sector OAXB, $\angle AOB$ is called sectorial angle. In Fig. 12.6, the shaded region OAXB is a minor sector and the unshaded region OAYB is the corresponding major sector.

The part (or portion) of the circular region enclosed by a chord and the corresponding arc of a circle is called a segment of a circle. In Fig. 12.7, the shaded region $\overset{\frown}{AXB}$ is the minor segment and the unshaded region AYB is the major segment.

Let us now find the area of a sector. In Fig. 12.6, OAXB is a sector of a circle with centre O and radius r.

Let $\angle AOB = \theta$ (in degrees).

Here, you recall that the area of a circle is πr^2, where r is the radius of the circle. We may consider the whole circular region as a sector of sectorial angle 360°.

When the sectorial angle is 360°, the area of the sector is πr^2

$\therefore$ when the sectorial angle is 1° , the area of the sector is $\dfrac{\pi r^2}{360}$

$\therefore$ when the setorial angle is θ° , the area of the sector is $\dfrac{\pi r^2\theta}{360}$

Thus, the area of a sector of sectorial angle θ, is $\dfrac{\pi r^2\theta}{360}$.

From Fig. 12.6 we may observe that

area of the major sector OAYB = area of circle – area of the minor sector OAXB

$$= \pi r^2 - \frac{\pi r^2\theta}{360}$$

$$= \left(\frac{360-\theta}{360}\right)\pi r^2$$

Let us express the area of a sector in terms of the radius and the arc length. You know that

$$s = r\theta$$

where **s** is the arc length, **r** is the radius (measured in the same unit) and θ is the sectorial angle measured in radians.

Further, we know that $\theta^\circ = \frac{\pi\theta}{180}$ radians

It follows that $s = r \times \frac{\pi\theta}{180}$, where θ is measured in degrees

Now, the area of the sector $= \frac{\pi r^2\theta}{360}$

$$= \frac{1}{2} r \times \frac{\pi r\theta}{180}$$

$$= \frac{1}{2} rs$$

$$= \frac{1}{2} (\text{radius}) \times (\text{arc length})$$

Let us now find the area of a segment.

Let AB be a chord of a circle with centre O and radius r (Fig 12.8). Then AB divides the circular region into two segments AXB and AYB.

Here, let us find the area of the minor segment AXB (shown shaded). Join OA, OB and draw BL perpendicular to OA. Let $\angle AOB = \theta$.

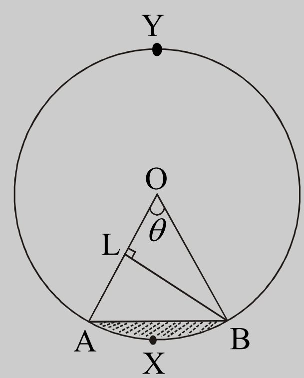


Fig. 12.8

From the right ΔOBL,

$$BL = OB \times \frac{BL}{OB}$$

$$= OB \sin\theta \qquad ...(1)$$

Area of the minor segment AXB = area of sector OAXB–area of ΔAOB

$$= \frac{\pi r^2\theta}{360} - \frac{1}{2} OA.BL$$

$$= \frac{\pi r^2\theta}{360} - \frac{1}{2} OA.OB \sin\theta \qquad \text{[by (1)]}$$

$$= \frac{\pi r^2\theta}{360} - \frac{1}{2} r.r \sin\theta \qquad [\because OA = OB = r]$$

$$=\frac{r^2}{2}\left(\frac{\theta\pi}{180}-\sin\theta\right)$$

And area of the major segment OAYB = area of the circle - area of the minor segment

$$=\pi r^2-\frac{r^2}{2}\left[\frac{\theta\pi}{180}-\sin\theta\right]$$

$$=r^2\left[\pi\left(1-\frac{\theta}{360}\right)+\frac{\sin\theta}{2}\right]$$

$$=r^2\left[\frac{\pi(360-\theta)}{360}+\frac{\sin\theta}{2}\right]$$

Henceforth 'sector' and 'segment' will mean the 'minor sector' and the 'minor segment' respectively, unless otherwise stated.

Example 3. If a chord of a circle of radius 12 cm subtends an angle of 60° at the centre, find the area of the corresponding (i) minor sector (ii) major sector and (iii) major segment (take π = 3.14 and $\sqrt{3}=1.73$).

Solution : Here,

$\theta = 60^o$, r = 12 cm

Then

(i) the area of the minor sector $=\frac{\pi r^2\theta}{360}$

$$=3.14\times 12^2\times\frac{60}{360}$$

$$=3.14\times 12\times 2$$

$$=75.36 \text{ cm}^2$$

(ii) the area of the corresponding major sector

$$=\pi r^2-\text{area of the minor sector}$$

$$=3.14\times 12\times 12-75.36$$

$$=452.16-75.36$$

$$=376.8 \text{ cm}^2$$

(iii) we know that the area of the major segment

$$=\pi r^2-\text{area of the minor segment} \;.....(1)$$

[For obtaining the area of the minor segment refer Fig 12.9.]

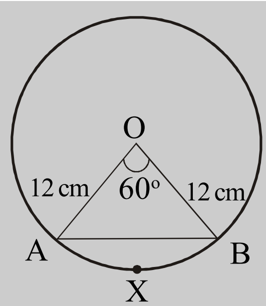


Fig. 12.9

And, the area of the minor segment $=$ area of sector OAXB $-$ area of ΔOAB

$$= 75.36 - \frac{1}{2} \times 12 \times 12 \sin 60^\circ$$
$$= 75.36 - 36\sqrt{3}$$
$$= 75.36 - 62.28$$
$$= 13.08 \text{ cm}^2$$

$\therefore$ the area of the major segment $= 452.16 - 13.08$

$$= 439.08 \text{ cm}^2$$

Example 4. Find the sectorial angle and area of the sector of a circle with radius 3.5 cm if the length of the corresponding arc is 5.5 cm.

Solution : Here,

$r = 3.5$ cm, and $s = 5.5$ cm.

Let the required angle of the sector be θ°

We know that, $s = \frac{\theta}{360} \times 2\pi r$

$$\Rightarrow 5.5 = \frac{\theta}{360} \times 2 \times \frac{22}{7} \times 3.5$$
$$\Rightarrow 5.5 = \frac{11\theta}{180}$$
$$\Rightarrow \theta = \frac{180 \times 5.5}{11}$$
$$= 90$$

$\therefore$ the sectorial angle is 90^0.

And the area of the sector $= \frac{1}{2} r.s$

$$= \frac{1}{2} \times 3.5 \times 5.5$$
$$= 9.63 \text{ cm}^2$$

[Alternatively, the area of the sector $= \frac{\pi r^2 \theta}{360}$

$$= \frac{22}{7} \times (3.5)^2 \times \frac{90}{360}$$
$$= \frac{11 \times 3.5 \times 0.5}{2}$$
$$= 9.63 \text{ cm}^2]$$

Example 5. **A chord of a circle of radius 4.2 cm subtends 120° at the centre. Find the area of the corresponding minor segment of the circle.**

(take $\pi = \frac{22}{7}$ and $\sqrt{3} = 1.73$)

Solution : We have

$r = 4.2$ cm

$\theta = 120^\circ$

For finding the minor segment refer Fig. 12.10. In ΔOAB, draw $OM \perp AB$ and OA=OB.

$\therefore$ by RHS congruence, $\Delta AMO \cong \Delta BMO$.

So, M is the mid-point of AB

and $\angle AOM = \angle BOM$

$$= \frac{1}{2} \times 120^\circ$$

$$= 60^\circ$$

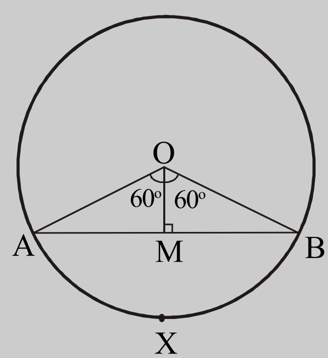


Fig. 12.10

From ΔOMA, we have

$$\frac{OM}{OA} = \cos 60^\circ = \frac{1}{2}$$

$$\Rightarrow OM = \frac{1}{2} \times OA$$

And $$\frac{AM}{OA} = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore AM = \frac{\sqrt{3}}{2} \times OA$$

$\therefore$ area of the $\Delta OAB = \frac{1}{2} \times AB \times OM$

$$= AM \times OM$$

$$= \frac{\sqrt{3}}{2} \times OA \times \frac{1}{2} \times OA$$

$$= \frac{\sqrt{3}}{4} \times (4.2)^2$$

$$= 7.63 \text{ cm}^2$$

$\therefore$ the area of the corresponding minor segment AXB

= area of sector OAXB – area of ΔOAB [refer fig. 12.10]

$$=\frac{120}{360}\times\frac{22}{7}\times(4.2)(4.2)-7.63$$

$$=22\times(0.84)-7.63$$

$$=18.48-7.63$$

$$=10.85\ \text{cm}^2$$

EXERCISE 12.2

$\left(\text{Unless otherwise stated, use } \pi=\frac{22}{7}\right)$

1. The perimeter of a certain sector of a circle of radius 6.5 cm is 21 cm. Find the area of the sector.

2. The radius of a circle is 6 cm and the area of a sector of this circle is $18\frac{6}{7}\text{cm}^2$. What is the angle of the sector?

3. The circumference of a circle is 176 cm. Find the area of a sector whose sectorial angle is 45^o.

4. The area of a sector of a circle of radius 21cm is 462 cm^2. Find the length of the arc of the sector.

5. Find the sectorial angle and area of the sector of a circle if the arc length of the corresponding sector is 8.8 cm and the radius of the circle is 5.6 cm.

6. The minute hand of a wall clock is 4.2 cm long. Find the area swept by it in 20 minutes.

7. Find the area of a sector of a circle of radius 14 cm when the sectorial angle is

(i) 30^o **(ii)** 45^o **(iii)** 60^o **(iv)** 90^o **(v)** 120^o

8. Two concentric circles have radii 6 cm and 4 cm. A sector of 60^o is drawn in each circle. Find the difference between the areas of the two sectors, (use $\pi = 3.14$)

9. In a circle of radius 6.3 cm, an arc AB subtends an angle of 90^o at the centre O of the circle. Find

(i) the length of the arc AB

(ii) the area of the sector AOB

(iii) the perimeter of the sector AOB

(iv) the area of the minor segment and the major segment formed by the chord AB.

10. A chord of a circle of radius 12 cm subtends a right angle at the centre. Find the area of the corresponding (i) major sector and (ii) minor segment (use $\pi = 3.14$).

11. A chord of a circle of radius 18 cm subtends an angle of 120^o at the centre. Find the area of the corresponding (i) minor sector and (ii) major segment (use $\pi=3.14$ and $\sqrt{3}=1.73$).

12. A chord of a circle of radius 6 cm subtends an angle of 60^o at the centre. Find the area of the corresponding (i) minor segment and (ii) major segment.(take $\pi=3.14$ and $\sqrt{3}=1.73$).

13. The radius of a circle is 5 cm. A chord of length $5\sqrt{2}$ cm is drawn in the circle. Find the area of the major segment (take $\pi=3.14$).

14. A chord of a circle of diameter 12 cm makes an angle of 30^o at the centre. Find the area of the minor segment of the circle.

15. A chord of a circle subtends an angle of 60^o at the centre . If the length of the chord is 12 cm, find the area of the two segments into which the chord divides the circle (take $\pi=3.14$ and $\sqrt{3}=1.73$)

16. A grass field is in the form of an equilateral triangle of sides 48 m. A cow is tethered at a vertex of the field. If the cow can reach upto 21 m from the vertex, find

(i) the area of the part of the field in which the cow can graze.

(ii) how much area remains ungrazed.

(iii) the increase in grazing area if the length of the rope be increased to 35 m (use $\pi=3.14$).

ANSWER

1. 26 cm^2 **2.** 60^o **3.** 308 cm^2 **4.** 44 cm

5. 90^o, 24.64 cm^2 **6.** 18.48 cm^2 **7.** (i) 51.33 cm^2 (ii) 77 cm^2

(iii) 102.67 cm^2 (iv) 154 cm^2 (v) 205.33 cm^2 **8.** 10.47 cm^2

9. (i) 9.9 (ii) 31.19 cm^2 (iii) 22.5 cm (iv) 11.34 cm^2, 113.40 cm^2

10. (i) 339.12 cm^2, (ii) 41.04 cm^2 **11.** 339.12 cm^2, 818.37 cm^2

12. (i) 3.27 cm^2 (ii) 109.77 cm^2 **13.** 71.38 cm^2

14. 0.43 cm^2 **15.** 13.08 cm^2, 439.08 cm^2 **16.** (i) 230.79 m^2

(ii) 765.69 m^2 (iii) 410.29 m^2

12.4. Area of combinations of plane figures.

So far, you have learnt some methods to find the areas of different figures seperately. In this section, you will learn some methods of finding the areas of combinations of plane figures. In our daily life, we come across many types of plane figures such as flowerbeds, drain covers, window designs, designs on table covers etc. We shall now illustrate the process of finding areas of such figures through some examples.

Example 6. Four equal coins, each of radius 2.5 cm touch each other as shown in the adjoining Fig 12.11. Find the area of the shaded region (take $\pi = 3.14$).

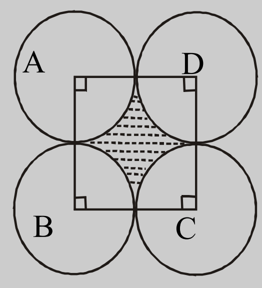


Fig. 12.11

Solution: Here, $AB = BC = CD = DA = 2.5cm + 2.5cm = 5cm$.

$\therefore$ ABCM is a square and its area $= 5 \times 5 = 25$ cm^2

$$\text{Area of a sector} = \frac{\pi r^2 \theta}{360}$$

$$= \frac{90}{360} \times 3.14 \times 5^2 \text{, where } \theta = 90^{o}, r = 2.5 \text{ cm}$$

$$= 4.91 \text{ cm}^2$$

$\therefore$ the required area of the shaded region

= area of ABCD $-$ area of the four sectors

$= 5 \times 5 - 4 \times 4.91$

$= 25 - 19.64$

$= 5.36$ cm^2

Example 7. The area of an equilateral triangle is $49\sqrt{3}$ cm^2. With each vertex of the triangle as centre, a circle is drawn with radius equal to half the length of the side of the triangle. Find the area of the portion of the triangle lying outside the circles.

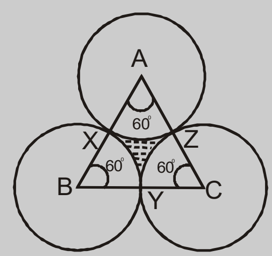


Fig. 12.12

Solution : Let 'a' cm be the side of the equilateral ΔABC

$\because$ area of $\Delta ABC = \frac{\sqrt{3}}{4} a^2$ cm^2

and area of $\Delta ABC = 49\sqrt{3}$ cm^2

$$\therefore \frac{\sqrt{3}}{4} a^2 = 49\sqrt{3}$$

$$\Rightarrow a^2 = 4 \times 49$$

$$\Rightarrow a = 14$$

$\therefore$ length of the radius of each circle = 7 cm.

Now, area of the sector AXZ $= \dfrac{60}{360}\pi.7^2$

$= \dfrac{49\pi}{6}\text{ cm}^2$

Similarly, area of the sector BYX $= \dfrac{49\pi}{6}\text{cm}^2$

and area of the sector CZY $= \dfrac{49\pi}{6}\text{cm}^2$

$\therefore$ the total area of the three sectors $= 3.\dfrac{49\pi}{6}$

$= \dfrac{49\pi}{2}\text{cm}^2$

$\therefore$ the required area regions

= Area of $\Delta ABC -$ total area of the three sectors

$= 49\sqrt{3} - \dfrac{49\pi}{2}$

$= 49\left(\sqrt{3} - \dfrac{\pi}{2}\right)\text{cm}^2$

Example 8. **In Fig 12.13, AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OA is the diameter of the smaller circle. If OA = 6 cm, If find the total area of the shaded regions.**

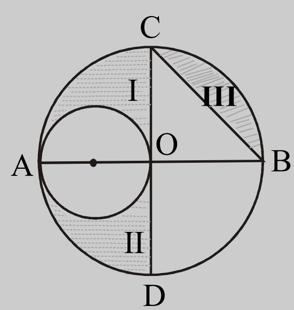


Fig. 12.13

Solution : Here OA = OB = OC = OD = 6 cm

Now, area of the shaded region I

= area of the quadrant AOC $-$semi circle of smaller circle

$= \dfrac{1}{4}\times\pi 6^2 - \dfrac{1}{2}\times\pi\times 3^2$

$= 9\pi - \dfrac{9\pi}{2}$

$= \dfrac{9}{2}\pi\ cm^2$

Similarly, area of the shaded region II = $\dfrac{9}{2}\pi\ cm$.

Also, area of the quadrant BOC $= \frac{1}{4}.\pi \times 6^2$

$= 9\pi\ cm^2$

and area of the $\Delta BOC = \frac{1}{2} \times 6 \times 6$

$= 18\ cm^2$

$\therefore$ area of the shaded region III = Area of the quadrant BOC − area of ΔBOC

$= (9\pi - 18) cm^2$

$\therefore$ the total area of the shaded regions I, II, and III

$$= \frac{9\pi}{2} + \frac{9\pi}{2} + 9\pi - 18$$

$$= 9\pi + 9\pi - 18$$

$$= 18(\pi - 1)\ cm^2$$

EXERCISE 12.3

$\left(\text{Unless otherwise stated, use } \pi = \frac{22}{7}\right)$

1. A circle circumscribes a rectangle of sides 8 cm and 6 cm. Find the area enclosed between the circle and the rectangle.

2. An equilateral triangle is inscribed in a circle of radius 14 cm. Find the area between the circle and the triangle (use $\sqrt{3} = 1.73$).

3. A circle is inscribed in an equilateral triangle of sides 18 cm. Find the area between the triangle and the circle. (take $\sqrt{3} = 1.73$).

4. ABC is an equilateral triangle of side 12 cm in which three circular arcs, each of radius 6 cm, are drawn with centre A, B, C. Find the area of the region enclosed by the circular arcs (take $\sqrt{3} = 1.73$).

5. Four cows are tethered at the four corners of a square field of side 56 m so that consecutive cows can just reach each other. What area of the field will remain ungrazed?

6. ABC is an equilateral triangle inscribed in a circle of radius 7 cm. Find the area of the minor segment of the chord BC (take $\sqrt{3} = 1.73$).

7. From an equilateral triangle of sides 14 cm, a sector of radius 7 cm with centre at a vertex and enclosed by two sides, is cut off. Find the area of the remaining portion (take $\sqrt{3} = 1.73$).

8. ABC is an isosceles right angled triangle, right angled at A where AB = AC= 8 cm. A sector of radius 3.5 cm with centre A and enclosed by AB, AC is cut off from the triangle. Find the area of the remaining portion.

9. BC is the arc of a quadrant of a circle of radius 8cm. A semi - circle is described on the chord BC on the side opposite to the centre of the quadrant. Find the area enclosed between arc BC and the semi circle.

10. In a 400 m track the length of each of the straight portions is 90 m and the diameter of each of the inner semi-circular ends is 70 m. If the uniform width of the track is 7m, find the cost of turfing the track at the rate of ₹ 250 per square metre.

11. Two flower beds are in the form of segments (minor) of the circle circumscribing a square field of side 42 m, with two opposite sides of square as the corresponding chords. Find the cost of planting flowers at the rate of ₹ 200 per square metre.

12. On a rectangular window of dimension, $84 \text{ cm} \times 56$ cm, twentyfour nonoverlapping circular designs each of radius 7 cm are made at the rate of ₹ 25 per 20 cm^2. Find the cost of making the designs. Also find the area of the remaining portion.

13. In a round table of diameter 112 cm, there is a portion in the form of a regular hexagon inscribed in it. Designs are made on the remaining portion of the table top at the rate of ₹ 8 per 15 cm^2. Find the cost of designing (use $\sqrt{3} = 1.73$).

14. Find the area and perimeter of the shaded region in the given Fig.12.14, having given that OA =OB = 28cm.

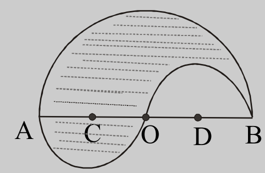


Fig. 12.14

15. In the given Fig. 12.15, AC = CB = 18cm. Find the area of the shaded region using the fact that $OP = \frac{1}{6} AB$.

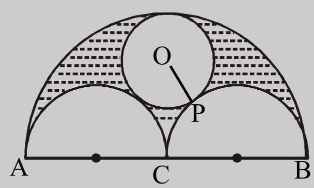


Fig. 12.15

16. In the given fig. 12.16, O is the centre and AOB is the diameter of the circle. Find the area of the shaded region.

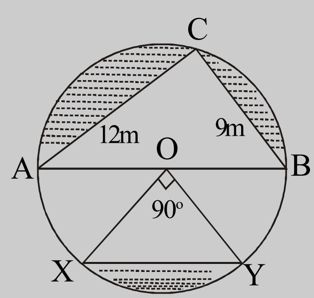


Fig. 12.16

17. ABCD is a square of side 7 cm and semi-circles are drawn with each side of the square as diameter, thereby forming four similar petals as shown in the adjoining figure. Find the cost of making design on the petals at the rate of ₹ 12 per square centimetre.

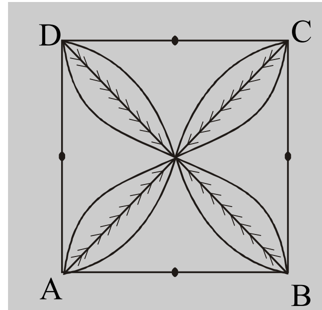


Fig. 12.17

18. Semi-circles are described on the sides AB and CD of a rectangle ABCD as diameters, both inside the rectangle. If AB = 14 cm and BC = 7 cm, find (i) the area enclosed by the semi circles (ii) the area of the portion of the rectangle lying outside the semi -circles.

ANSWER

1. 30.57 cm^2	**2. 361.69 cm^2**	**3. 55.27 cm^2**
4. 5.72 cm^2	**5. 672 m^2**	**6. 30.14 cm^2**
7. 59.10 cm^2	**8. 22.4 cm^2**	**9. 32 cm^2**
10. ₹ 7,38,500	**11. ₹ 100,800**	**12. ₹ 4620,1008 cm^2**
13. ₹ 916.30	**14. 1232 cm^2; 176 cm**	**15. 141.43 cm^2**
16. 50.46 cm^2	**17. ₹ 336**	**18. (i) 60.94 cm^2 (ii) 4.29 cm^2**

12.5 Surface Area and Volume of Combination of Solids

In this section, you will learn the methods to find surface area and volume of combination of two or more basic solids. Some examples of such combinations are already mentioned in the introduction.

Now, you recall the following facts :

If the length of a side of a cube is l, then its surface area is $6l^2$ and its volume is l^3.

If l, b, h are the length, breath, height respectively of a cuboid, then its surface area is $2[lb + bh + hl]$ and its volume is lbh.

If r and h are respectively the radius of the base and height of a right circular cylinder, then the total surface area of the cylinder

$= $ curved surface area + area of two circular ends

$= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

and the volume = area of the base× height

$= \pi r^2 \times h$

$= \pi r^2 h$

If l, r and h are respectively the slant height, radius of the base and height of a right circular cone, then $l^2 = r^2 + h^2$. Further, the total surface area

$= $ curved surface area + area of base

$= \pi rl + \pi r^2$

$= \pi r(l + r)$

and the volume $= \frac{1}{3}\pi r^2 h$

If r be the radius of sphere, then its surface area is $4\pi r^2$ and its volume is $\frac{4}{3}\pi r^3$.

If r is the radius of a hemisphere, then its total surface area

$= $ curved surface area + area of circular face

$= 2\pi r^2 + \pi r^2$

$= 3\pi r^2$

and volume $= \frac{2}{3}\pi r^3$

Henceforth, cylinder and cone will mean right circular cylinder and right circular cone respectively, unless otherwise stated.

Example 9. A vessel is in the form of a hollow cylinder surmounted on a hemispherical bowl of the same radius. The diameter of the hemisphere is 12cm and the total height of the vessel is 14 cm. Find

(i) the total surface area of the vessel

(ii) the volume of the vessel

(iii) the cost of painting the outer surface of the vessel at the rate of ₹ 5 per square cm. (take $\pi = 3.14$).

Solution : Here, radius of the cylinder (r) = radius of the hemisphere

= 6 cm.

and height of the cylinder (h) = 14–6

= 8 cm

(i) Surface area of the cylinder $= 2\pi rh$

$= 2\times3.14\times6\times8$

$= 301.44 \text{ cm}^2$

and surface area of the hemisphere $= 2\pi r^2$

$= 2\times3.14\times6^2$

$= 226.08 \text{ cm}^2$

$\therefore$ total surface area of the vessel $= 301.44 + 226.08$

$= 527.52 \text{ cm}^2$.

(ii) Volume of the cylinder $= \pi r^2 h$

$= 3.14\times6^2\times8$

$= 904.32 \text{ cm}^2$.

and volume of the hemisphere $= \frac{2}{3}\pi r^3$

$= \frac{2}{3}\times3.14\times 6^3$

$= 452.16 \text{ cm}^2$.

$\therefore$ total volume of the vessel $= 904.32 + 452.16$

$= 1356.48 \text{ cm}^3$

(iii) Required cost of painting = Rs. (527.32×5)

= Rs. 2637.60

Example 10. A solid toy is in the form of a right circular cone surmounted on a hemisphere of the same radius. If the height of the cone is 4 cm and the diameter of the base 6 cm, find

(i) the volume of the toy

(ii) the total surface area of the toy (use $\pi = 3.14$**)**

Solution : For the right circular cone,

radius of the base (r) = 3cm, height (h)= 4 cm

and slant height $(l) = \sqrt{3^2+4^2}$

= 5 cm.

$\therefore$ volume of the right circular cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times 3.14 \times 3^2 .4$$

= 37.68 cm^2.

and surface area of the right circular cone = πrl

= 3.14× 3.5

= 47.1 cm^2.

Also, volume of the hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times 3.14 \times 3^3$$

$= 56.52$ cm^3

and surface area of the hemisphere $= 2\pi r^2$

$= 2 \times 3.14 \times 32$

= 56.52 cm^2

(i) The required volume of the toy

= volume of the cone + volume of the hemisphere

= 37.68 + 56.52 = 94.2 cm^3

(ii) The required surface area of the toy

= surface area of the cone + surface area of the heisphere

= 47.1+ 56.52= 103 .62 cm^2.

Example 11. A canvas-tent is in the form of a cylinder of diameter 16 m and height 5 m surmounted by a cone of equal base and height 6 m. Find the capacity of the tent and the cost of the canvas at Rs 150 per square metre [use $\pi = 3.14$ **].**

Solution : Here,

radius of the base of the cone (r)

= radius of the cylinder (r)

$= \frac{16}{2} = 8$ m.

and height (h) = 6 m

$\therefore$ slant height $(l) = \sqrt{r^2 + h^2}$

$= \sqrt{8^2 + 6^2}$

$= \sqrt{64+36}$

$= 10$ m

$\therefore$ curved surface area of the cone $= \pi rl$

$= 3.14 \times 8 \times 10$

$= 251.2$ m^2.

and volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times 3.14 \times 8^2 \times 6$

$= 401.92$ m^3.

The height (h) of the cylinder = 5 m.

$\therefore$ curved surface area of the cylinder $= 2\pi rh$

$= 2 \times 3.14 \times 8 \times 5$

$= 251.2$ m^2.

and volume of the cylinder $= \pi r^2 h$

$= 3.14 \times 8^2.5$

$= 1004.8$ m^3.

$\therefore$ the required capacity of the tent = volume of the cone+ volume of the cylinder

$= 401.92 + 1004.8$

$= 1406.72$ m^3

and total curved surface area of the tent = curved surface area of the cone + curved surface area of the cylinder

$= 251.2 + 251.2$

$= 502.4$ m^2.

$\therefore$ cost of the canvas at the rate of ₹ 150 per square metre

$=$ Rs. (502.4×150)

$=$ Rs.75360.

Example 12. A cylindrical boiler of diameter 4.2 m has a hemispherical end on one side. If the total length of the boiler is 6.1m, find

(i) the total curved surface area and

(ii) the total capacity of the boiler , (use $\pi = \frac{22}{7}$)

Solution : Here, diameter = 4.2 m

$\therefore$ radius (r) $= \frac{4.2}{2} = 2.1$ m

and height of the cylinder (h) = 6.1 - 2.1

= 4 m.

$\therefore$ curved surface area of the cylinder $= 2\pi rh$

$$= 2.\frac{22}{7}\times 2.1\times 4$$

$$= 52.8 \text{ m}^2.$$

and volume of the cylinder= $\pi r^2 h$

$$= \frac{22}{7}\times 2.1\times 4$$

$$= 55.44 \text{ m}^3.$$

and curved surface area of the hemispherical end $= 2\pi r^2$

$$= 2\times\frac{22}{7}\times(2.1)^2$$

$$= 27.72 \text{ m}^2$$

and volume of the hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$$= \frac{2}{3}\times\frac{22}{7}\times(2.1)^3$$

$$= 19.40 \text{ m}^3$$

$\therefore$ the required total curved surface area = 52.8 + 27.72

= 80.52 m^2.

and the required capacity of the boiler = 55.44+19.40

= 74.84 m^3.

EXERCISE 12.4

$\left(\text{Unless otherwise stated , use } \pi = \frac{22}{7}\right)$

1. A solid wooden toy is in the shape of a cone surmounted on a hemisphere of the same radius 2.1 cm. If the total height of the wooden toy is 8.1cm, find its total surface area and volume.

2. A solid is in the form of a cylinder of diameter 7 cm with hemispherical ends. If the total length of the solid is 42 cm, find the total surface area and the volume of the solid.

3. An ice-cream is in the form of an inverted cone surmounted by a hemisphere of the same-radius. If the radius and the height of the cone are respectively 2.1 cm and 8cm, find the volume of the ice-cream.

4. A cylindrical container of radius 3 cm and height 14 cm is filled with ice-cream. The ice-cream is to be distributed among 16 children in equal cones with hemispherical tops. If the height of the conical portion is 5 times the radius of its base, find the volume of each iec-cream cone.

5. A cylindrical vessel of radius 7 cm and height 30 cm is full of ice cream. How many ice-cream cones each of radius 3 cm and height 8 cm with hemispherical tops, can be formed with ice-cream from the vessel ?

6. A circus tent is in the shape of a cylinder of diameter 24 m and height 4 m surmounted by a cone of the same radius and height 5m. Find the capacity of the tent and the cost of the convas at the rate of ₹ 120 per square metre.

7. A circus tent of height 15 m is in the form of cylinder of diameter 32 m and height 3m, surmounted by a cone of the same radius. Find the volume of the tent and the cost of the canvas at the rate of ₹ 110 per square metre.

8. A hall is in the form of a cylinder of diameter 12 m and height 4m, surmounted by a cone whose vertical angle is a right angle. Find the outer surface area of the hall and volume of the air inside the hall . (use $\sqrt{2}=1.41$)

9. A geyser (water boiler) is in the form of a cylinder with hemispherical ends. If the length of the cylinder is 56 cm and the diameter of each end is 18 cm, find the capacity of the geyser in litres.

10. A container is formed of a hollow cylinder fitted with a hemispherical bottom of radius 3 cm. The depth of the cylinder is 14 cm and the diameter of the hemisphere is 6 cm. Find the volume and the internal surface area of the container.

11. From a solid cylinder of height 28 cm and radius 12 cm, a cone of the same height and same radius is removed. Find the volume of the remaining solid.

12. A cylindrical container of radius 10 cm and height 35 cm is fixed co-axially inside another cylindrical container of radius 14 cm and height 35 cm. The total space between the two containers is filled with cork for insulation purposes. Find the volume of the cork required.

13. A solid is in the shape of a hemisphere surmounted by a cone of the same radius. The diameter of the cone is 18 cm and the height of the cone is 14 cm. The solid is completely immersed in a cylindrical tub, full of water. If the diameter of the tub is 26 cm and its height is 21 cm, find the quantity of water left in the cylindrical tub in litres.

14. A solid toy is in the form of a cone surmounted on a hemisphere of the same radius. The height of the cone is 6 cm and the radius of the base is 2.8 cm. If a cylinder circumscribes the solid, find how much more space it will cover.

15. A right triangle, with legs 9 cm and 12 cm, is made to revolve about its hypotenuse. Find the volume and the surface area of the double cone so formed.

16. A godown is formed of a cuboid of dimensions 63 m $\times$ 28 m $\times$ 5 m covered by a half cylindrical roof. If the length and breadth of the cuboid are 63 m and 28 m respectively, find the volume of air inside the godown. Also find the cost of roofing at the rate of ₹ 500 per square metre.

17. A solid is in the form of a cylinder surmounted by a cone of the same radius. If the radius of the base and the height of the cone are 'r' cm and 'h' cm respectively and the total height of the solid is $3h$, prove that the volume of the solid is $\frac{7}{3}\pi hr^2$.

ANSWERS

1. 69.63 cm^2, 47.12 cm^3 **2.** 924 cm^2, 1527.16 cm^3 **3.** 56.36 cm^3
4. 24.75 cm^3 **5.** 35 **6.** 2564.58m^3, ₹ 9,50,40
7. 5632 m^3, ₹ 1,43,817.30 **8.** 678.86 m^3, 310.39 m^2
9. 17.31 litres **10.** 452.57 cm^3, 320.57 cm^2 **11.** 8448 cm^3
12. 10560 cm^3 **13.** 8.44 litres **14.** 121.56 cm^3
15. 814.63 cm^3, 475.2 cm^2 **16.** 28224 m^3, ₹ 13,86,000

12.6 Conversion of solid from one shape to another

In this section, we shall discuss about the conversion of metallic solid into one or more solids of similar shape. For example, a solid metallic sphere is melted and recast into smaller spherical balls or recast into a wire; a solid metallic cone is melted and recast into the form of a solid cylinder etc. The calculation of surface areas and volumes in such cases will now be illustrated through examples.

We may note the following facts :

If r_1 and r_2 are respectively the inner and outer radii of a spherical shell (hollow sphere), then the volume of the material in the shell is $\frac{4}{3}\pi(r_2^3 - r_1^3)$ cubic units

If r_1 and r_2 are respectively the internal and external radii of a cylindrical shell (hollow cylinder) of height h, then the volume of the material in the shell = $\pi h(r_2^2 - r_1^2)$ cubic units.

Example 13. How many 5 cm solid cubes can be cut from a solid metallic cuboid measuring 10 cm $\times$ 15 cm $\times$ 25 cm ?

Solution : Here,

volume of the cuboid = $10 \times 15 \times 25$ cm^3

and volume of a cube of side 5 cm = 5^3

= 125 cm^3

Let n be the required number of cubes.

Then,

total volume of the n cubes = volume of the cuboid.

$$\Rightarrow \quad n \times 125 = 10 \times 15 \times 25$$

$$\Rightarrow \quad n = \frac{10 \times 15 \times 25}{125} = 30$$

i.e. 30 solid cubes can be cut.

Example 14. A metallic sphere of radius 6 cm is melted and recast to form a cylinder of radius 3 cm. Find the curved surface area of the cylinder.

Solution : Here, radius of the sphere = 6 cm

$\therefore$ volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi \times 6^3$$

$$= 288\pi \text{ cm}^3$$

And the radius of the cylinder = 3 cm

Let h cm be the height of the cylinder.

Then the volume of the cylinder = $\pi r^2 h$

$$= \pi \times 3^2 \times h$$

$$= 9\pi h$$

By question

volume of the cylinder = volume of the sphere

$$\therefore \quad 9\pi h = 288\pi$$

$$\Rightarrow \quad h = \frac{288}{9}$$

= 32 cm.

$\therefore$ the curved surface area of the cylinder = $2\pi r h$

$$= 2\pi \times 3 \times 32$$

$$= 192\pi \text{ cm}^2$$

Example 15. Three solid metallic cones of the same height 18 cm and the same base radius 4 cm are melted together and recast into a solid sphere. Find the surface area of the sphere.

Solution : For the solid cone,

$r = 4$ cm , $h = 18$ cm

$\therefore$ volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 4^2 \times 18$

$= 96\pi \text{ cm}^3$

$\therefore$ volume of 3 such cones $= 3 \times 96\pi$

$= 288\pi \text{ cm}^3$

Let R be the radius of the sphere. Then its volume $= \frac{4}{3}\pi R^3$

By question, $\frac{4}{3}\pi R^3 = 288\pi$

$\Rightarrow R^3 = \frac{3 \times 288}{4}$

$= 216$

$\therefore \quad R = 6\ cm$

$\therefore$ the surface area of the sphere $= 4\pi r^2$

$= 4\pi . 6^2$

$= 144\pi\ cm^2$

Example 16. A solid metallic cylinder of height 24 cm and radius 3 cm is melted and recast into a cone of radius 6 cm. Find the height of the cone.

Solution : For the solid cylinder,

$h = 24$ cm, $r = 3$ cm

$\therefore$ volume of the cylinder $= \pi r^2 h$

$= \pi . 3^2 . 24$

$= 216\pi\ cm^3$

For the solid cone,

$r = 6$ cm, $h = ?$

$\therefore$ volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 6^2 \times h$

$= 12\pi h$ cm^3.

By question,

volume of the cone = volume of the cylinder

$\therefore \quad 12\pi h = 216\pi$

$\Rightarrow \quad h = 18$ cm

$\therefore$ the required height of the cone is 18 cm.

Example 17. Rain water from a rectangular roof of dimensions 11m $\times$ 20 m drains into a conical vessel radius 2 m and height 4.2 m. If the vessel is just full, find the rainfall in cm.

Solution : For the conical vessel.

$r = 2$ m,

and $h = 4.2$ m

$\therefore$ volume of the vessel $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 2^2 \times 4.2$

$= 5.6\pi \ m^3$

Let the rainfall be x

Then,

$11 \times 20 \times x = 5.6\pi$

$\Rightarrow 11 \times 20 \times x = \frac{56}{10} \times \frac{22}{7}$

$\Rightarrow x = \frac{8 \times 11}{5 \times 20 \times 11} = \frac{2}{25}$

$\therefore$ the required rainfall is $\frac{2}{25}$ m i.e. 8 cm.

EXERCISE 12.5

1. A solid cube is cut into two cuboids of equal volumes. Find the ratio of the total surface area of the given cube to that of one of the cuboids.
2. A sphere has the same surface area as the curved surface of a cone of height 16 cm and base radius 12 cm. Find the radius of the sphere.
3. A solid metallic cone is 81 cm high and radius of its base is 6 cm. If it is melted and recast into a solid sphere, find the curved surface area of the sphere.
4. A solid metallic sphere of diameter 28 cm is melted and recast into a number of cones, each of diameter 7 cm and height 4 cm. Find the number of cones so formed.
5. The internal and external radii of a hollow metallic sphere are 3 cm and 5cm respectively. If the sphere is melted and recast into a solid cylinder of height $2\frac{2}{3}$ cm, find the curved surface area of the cylinder.
6. A metallic cone of height 28 cm and radius of base 12 cm, is melted and recast into a cylinder of height $9\frac{1}{3}$ cm. Find the curved surface area of the cylinder so formed.
7. 20 circular plates, each of radius 7 cm and thickness 1.5 cm are placed one above another to form a solid cylinder. Find the curved surface area and the volume of the cylinder so formed.
8. Find the volume of the largest sphere that can be carved out of a cube of side 4.2 cm. Also find the ratio of the volume of the cube to that of the sphere.
9. Find the volume of the largest cone that can be carved out of a cube of side 16.8 cm.
10. A solid cone of diameter 14 cm and height 8 cm is melted and recast into a hollow sphere. If the external diameter is 10 cm, find the internal diameter of the sphere.
11. A hemisphere of lead of radius 7 cm is cast into a cone of base radius 3.5 cm. Find the height of the cone.
12. A cylinder is cut lengthwise and flattened, thereby forming a rectangle of dimensions $66\text{cm} \times 28\text{cm}$. Find the curved surface area and volume of the original cylinder.
13. A sector of a circle of radius 6 cm has angle 120°. It is rolled up so that two bounding radii are joined together to form a cone. Find the volume of the cone. (Take $\sqrt{2} = 1.4$)
14. A conical flask of radius x units and height $2x$ units, is full of water. The water is poured into a cylindrical flask of radius $\frac{2x}{3}$ units. Find the height of water in the cylindrical flask.

15. The radii of the bases of two solid metallic cones of same height h are x_1 and x_2. If the two cones are melted together and recast into a cylinder of height h, then show that the radius of the base of the cylinder is $\sqrt{\frac{1}{3}(x_1^2 + x_2^2)}$

16. A solid metallic cylinder and another solid metallic cone have the same height h and the same radius r. If the two solids are melted together and recast into a cylinder of radius $\frac{1}{2}r$, prove that the height of the new cylinder is $\frac{16}{3}h$.

17. Two solid metallic cuboids of dimensions 15 cm $\times$ 8 cm $\times$ 5 cm and 20 cm $\times$ 5 cm $\times$ 4 cm, are melted together and recast into solid cubes each of side 5 cm. Find the number of solid cubes so formed.

18. A sphere of diameter 6 cm is dropped into a cylindrical vessel, partly filled with water. The diameter of the vessel is 12 cm. If the sphere is completely submerged, how high will the level of the water be raised?

19. A solid metallic cylinder of radius 14 cm and height 21 cm is melted and recast into a number of spheres, each of radius 3.5 cm. Find the number of spheres so formed.

20. A spherical shell of lead, whose external diameter is 16 cm, is melted and recast into a cylinder of height $9\frac{1}{3}$ cm and diameter 16 cm. Find the internal diameter of the shell.

21. A vessel is in the form of an inverted cone of height 8 cm and radius 6cm. It is filled with water upto the rim. When lead shots, each of which is a sphere of radius 0.2cm are dropped into the vessel, one -sixth of the water flows out. Find the number of lead shots dropped into the vessel.

22. Water is flowing at the rate of 5 km per hour through a pipe of diameter 14 cm into a rectangular tank of base 30 m $\times$ 22 m. Find the time during which the level of water in the tank rises by 35 cm.

ANSWER

1. 3:2 **2.** $2\sqrt{15}$ cm. **3.** 1018.29 cm^2 **4.** 224

5. $117\frac{1}{3}$ cm^2 **6.** 704 cm^2 **7.** 1320 cm^2, 4620 cm^3

8. 38.81 cm^3, 21:11 **9.** 1241.86 cm^3 **10.** 6 cm **11.** 56 cm

12. 1848 cm^2, 9702 cm^3 **13.** 23.47 cm^3 (approx) **14.** $\frac{3}{2}x$ units

17. 8 **18.** 1 cm **19.** 72. **20.** 8 **21.** 1500 **22.** 3 hrs

12.7 Frustum of a Right Circular Cone

In this section, you will learn to derive the formulas for finding the volume and the surface area of a frustum of a right circular cone. In our daily life, we use a number of solids of such form e.g. a bucket, a glass tumbler etc. The study of the volume and surface area of such type of solids will be very useful.

Recall:

If a right circular cone is cut off by a plane parallel to the base, the portion of the cone between the plane and the base of the cone is called a **frustum** of the cone.

Let us now consider the following figures:-

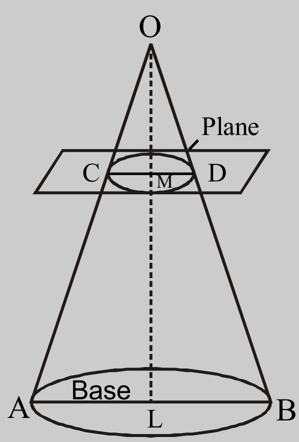


Fig. 12.18 (i)

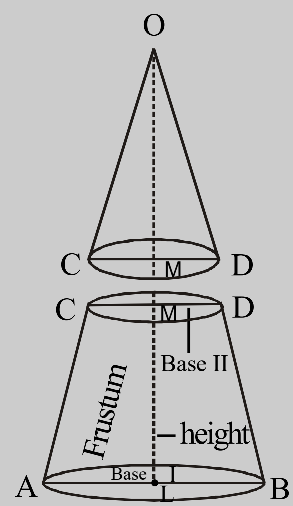


Fig. 12.18 (ii)

Of the two parts shown in Fig 12.18 (ii) the lower one is the frustum of the cone. It has two parallel flat circular bases namely base I and base II and a curved surface between the bases.

A section of the frustum by any plane containing its axis i.e. the line through the centres of the bases, is an isosceles trapezium ABCD in Fig 12.18(iii). The length of any of the pair of non parallel sides AD and BC is the slant height of the frustum.

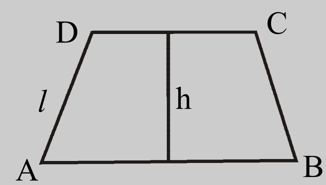


Fig. 12.18(iii)

Also, the slant height of the frustum equals the difference between the slant heights of the cones OAB and OCD [Fig. 12.18(i)].

Also, the height of the frustum is equal to the difference between the heights of the cones OAB and OCD.

12.8 Volume and Surface Area of a Frustum

Let h be the height, l the slant height and r_1 and r_2 the radii of the bases ($r_1 > r_2$) of the frustum of a cone.

The frustum is made from the complete cone OAB by cutting the conical part OCD.

Let the height of the cone OAB be h_1 and its slant height l_1

i.e. OL = h_1 and OA = OB = l_1

Then the height of the cone OCD (say) $h_2 = h_1 - h$

Since the right triangles OMC and OLA are similar (AA similarity), therefore

$$\frac{h_2}{h_1} = \frac{r_2}{r_1}$$

$$\Rightarrow \quad \frac{h_1 - h}{h_1} = \frac{r_2}{r_1}$$

$$\Rightarrow \quad 1 - \frac{h}{h_1} = \frac{r_2}{r_1}$$

$$\Rightarrow \quad \frac{h}{h_1} = 1 - \frac{r_2}{r_1}$$

$$= \frac{r_1 - r_2}{r_1}$$

$$\Rightarrow \quad h_1 = \frac{hr_1}{r_1 - r_2} \quad \text{......................(1)}$$

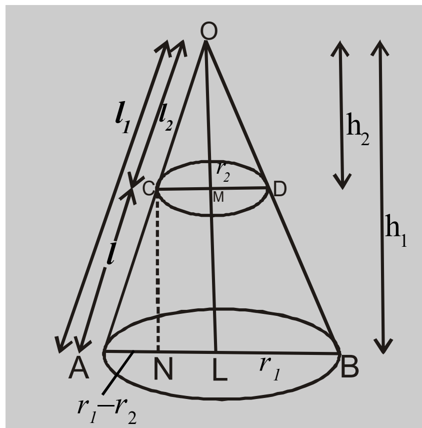


Fig. 12.19

$$\therefore \quad h_2 = h_1 - h$$

$$= \frac{hr_1}{r_1 - r_2} - h$$

$$= \frac{hr_2}{r_1 - r_2} \quad \text{.......................(2)}$$

Now, the volume of the frustum of cone = the volume of the cone OAB − the volume of the cone OCD

$$= \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 h_2$$

$$= \frac{1}{3}\pi r_1^2 \frac{hr_1}{r_1 - r_2} - \frac{1}{3}\pi r_2^2 . \frac{hr_2}{r_1 - r_2} \quad \text{[by (i) and (2)]}$$

$$= \frac{1}{3}\pi h\left(\frac{r_1^3 - r_2^3}{r_1 - r_2}\right)$$

$$= \frac{1}{3}\pi h\frac{(r_1 - r_2)(r_1^2 + r_1r_2 + r_2^2)}{(r_1 - r_2)}$$

$$= \frac{1}{3}\pi h(r_1^2 + r_1r_2 + r_2^2)$$

Thus, the volume V of the frustum of cone is given by

$$V = \frac{1}{3}\pi h(r_1^2 + r_1r_2 + r_2^2)$$

The volume may also be expressed in terms of the area of bases and the height. If A_1 and A_2 ($A_1>A_2$) are the surface areas of the two circular bases, then

$$A_1 = \pi r_1^2$$

and $$A_2 = \pi r_2^2.$$

Thus, the volume of the frustum of cone is given by

$$V = \frac{1}{3}h[\pi r_1^2 + \pi r_1 r_2 + \pi r_2^2]$$

$$= \frac{1}{3}h\left[\pi r_1^2 + \pi r_2^2 + \sqrt{(\pi r_1^2)(\pi r_2^2)}\right]$$

$$= \frac{1}{3}h\left[A_1 + A_2 + \sqrt{A_1A_2}\right]$$

Now from the right ΔCAN, right $\angle$ed at N, we have

$$AN = r_1 - r_2$$

and $$AC^2 = l^2 = h^2 + (r_1 - r_2)^2$$

$$\therefore \text{ slant height } \quad l = \sqrt{h^2 + (r_1 - r_2)^2}$$

Let l_2 be the slant height of the cone OCD

Since $\Delta OMC \sim \Delta OLA$, therefore

$$\frac{l_2}{l_1} = \frac{r_2}{r_1}$$

$$\Rightarrow \frac{l_1 - l}{l_1} = \frac{r_2}{r_1}$$

This gives, $l_1 = \dfrac{lr_1}{r_1 - r_2}$ ....(3)

and $l_2 = l_1 - l$

$$= \frac{lr_1}{r_1 - r_2} - l$$

$$= \frac{lr_2}{r_1 - r_2} \quad \text{.....................(4)}$$

Now, the curved surface area of the frustum

= the curved surface area of the cone OAB

− the curved surface area of the cone OCD

$$= \pi r_1 l_1 - \pi r_1 l_2$$

$$= \pi r_1 . \frac{lr_1}{r_1 - r_2} - \pi r_1 . \frac{lr_2}{r_1 - r_2}$$

$$= \pi l \left(\frac{r_1^2 - r_2^2}{r_1 - r_2} \right)$$

$$= \pi l (r_1 + r_2)$$

Hence, the curved surface area of the frustum is $\pi l(r_1 + r_2)$.

$\therefore$ the total surface area of the frustum = curved surface area

+ area of base I + area of base II

$$= \pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$$

$$= \pi [l(r_1 + r_2) + r_1^2 + r_2^2]$$

However, in case of a bucket which is a frustum with the largest face open, the total surface area

= curved surface area + area of bottom

$$= \pi l(r_1 + r_2) + \pi r_2^2$$

Example 18. **If the radii of the circular faces of a frustum of height 8 cm are 11 cm and 5 cm respectively, find the volume and curved surface area (use $\pi = 3.14$).**

Solution : Here,

r_1 = 11cm, r_2 = 5 cm, h = 8 cm

$\therefore$ volume of the frustum $= \frac{1}{3} \pi h [r_1^2 + r_2^2 + r_1 r_2]$

$$= \frac{1}{3} \pi \times 8 \; [11^2 + 5^2 + 11.5]$$

$$= \frac{1}{3} \times 3.14 \times 8 \times 201$$

$$= 1683.04 \text{ cm}^3$$

Slant height, $l = \sqrt{h^2 + (r_1 - r_2)^2}$

$$= \sqrt{8^2 + 6^2} \quad [\because \ r_1 - r_2 = 11 - 5 = 6]$$

$$= \sqrt{100}$$

$$= 10$$

$\therefore$ curved surface area $= \pi l(r_1 + r_2)$

$$= 3.14 \times 10(11+5)$$

$$= 3.14 \times 10 \times 16$$

$$= 502.4 \text{ cm}^2.$$

Example 19. A bucket is in the form of a frustum of a cone whose top and bottom are of diameters 42 cm and 28 cm respectively. If the height of the bucket is 21 cm, find the capacity of the bucket in litres.

Solution : We have

$r_1 = 21$cm, $r_2 = 14$ cm.

$h = 21$ cm.

$\therefore$ capacity of the bucket $= \frac{1}{3}\pi h[r_1^2 + r_2^2 + r_1 r_2]$

$$= \frac{1}{3} \times \frac{22}{7} \times 21\,[21^2 + 14^2 + 21.14]$$

$$= 22[441 + 196 + 294]$$

$$= 22 \times 931$$

$$= 20482 \text{ cm}^3$$

$$= \frac{20482}{1000} \text{ litres}$$

$$= 20.48 \text{ litres.}$$

Example 20. A container made up of a metal sheet is in the form of a frustum of a cone of height 20 cm with radii of its upper and lower ends as 40 cm and 25 cm respectively. Find the quantity of milk the container can hold . Also find the area of the metal sheet used for making the containter. ($\pi = 3.14$).

Solution : We have

$r_1 = 40$ cm

$r_2 = 25$ cm

and h = 20 cm

$\therefore \quad l = \sqrt{h^2 + (r_1 - r_2)^2}$

$= \sqrt{20^2 + (40-25)^2}$

$= \sqrt{400 + 225}$

$= \sqrt{625} = 25$ cm

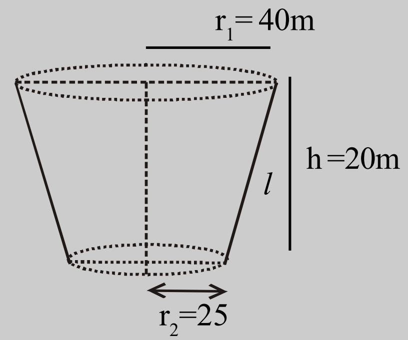


Fig. 12.20

and volume of the container $= \frac{1}{3}\pi h[r_1^2 + r_2^2 + r_1 r_2]$

$= \frac{1}{3} \times 3.14 \times 20[40^2 + 25^2 + 40 \times 25]$

$= \frac{1}{3} \times 3.14 \times 20[1600 + 625 + 1000]$

$= \frac{1}{3} \times 3.14 \times 20 \times 3225$

$= \frac{1}{3} \times 202530$

$= 67510$ cm^3.

$\therefore$ The quantity of the milk in the container $= \frac{67510}{1000}$ litres

= 67.51 litres.

Also, the area of the metal sheet $= \pi l(r_1 + r_2) + \pi r_2^2$

$= 3.14\,[25(40+25) + 25^2]$

$= 3.14 \times 2250 = 7065$ cm^2

Example 21. From a cone of height 18 cm, a smaller cone is cut off by a plane parallel to the base. If the volumes of the cones are in the ratio 1:27, find the height of the resulting frustum.

Solution : Let r_1 and r_2 be the radii of the two cones ($r_1 > r_2$)

Let h_2 be the height of the smaller cone.

Here, the height of the given cone (say) $h_1 = 18$ cm.

From the Fig.12.21, ΔOMD and ΔOLB are similar

$\therefore \quad \frac{OM}{OL} = \frac{MD}{LB}$

$\Rightarrow \frac{h_2}{h_1} = \frac{r_2}{r_1}$ ....(1)

Also,

$$\frac{\text{volume of the smaller cone}}{\text{volume of the given cone}} = \frac{1}{27}$$

$$\Rightarrow \frac{\frac{1}{3}\pi r_2^2 h_2}{\frac{1}{3}\pi r_1^2 h_1} = \frac{1}{27}$$

$$\Rightarrow \left(\frac{r_2}{r_1}\right)^2 \cdot \left(\frac{h_2}{h_1}\right)^2 = \frac{1}{27} \quad \text{[(by (1)]}$$

$$\Rightarrow \left(\frac{h_2}{h_1}\right)^3 = \left(\frac{1}{3}\right)^3$$

$$\Rightarrow \frac{h_2}{h_1} = \frac{1}{3}$$

$$\Rightarrow h_2 = \frac{1}{3} \times h_1$$

$$= \frac{1}{3} \times 18 = 6$$

$\therefore$ the height of the resulting frustum $= h_1 - h_2$

$= 18 – 6 = 12$ cm.

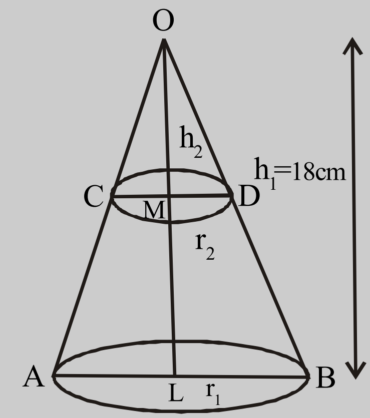


Fig. 12.21

EXERCISE 12.6

$\left(\text{unless otherwise stated, use } \pi = \frac{22}{7}\right)$

1. If the radii of the circular ends of frustum of height 6 cm are 20 cm and 12 cm respectively, find the volume and the curved surface area of the frustum.(Take $\pi = 3.14$)

2. A bucket is in the form of a frustum of a cone. If the height of the bucket is 16 cm and the radii of the upper and lower ends are 18 cm and 6 cm respectively, find

(i) the height of the cone of which the bucket is a part.

(ii) the capacity of the bucket.

(iii) the slant height of the bucket.

(iv) the surface area of the bucket.

3. A bucket is in the form of a frustum of height 21 cm. The diameters of the top and the bottom are 70 cm and 56 cm respectively. Find the capacity of the bucket.

4. The circumference of one plane face of a frustum is 44 cm and that of the other is 66 cm. If the height of the frustum is 12.6 cm, find the volume.

5. A glass tumbler is in the form of a frustrum of height 12 cm, the diameters of the upper and the lower ends being 7 cm and 4.2 cm respectively. Find the capacity of the tumbler (use $\pi = 3.14$).

6. The perimeters of circular ends of a solid frustum are 88 cm and 66 cm and its slant height is 21 cm., find the total surface area of the frustum.

7. A container is in the form of a frustum of height 12 cm with radii of its upper and lower ends as 17 cm and 8 cm respectively. Find the cost of milk the container can hold at the rate of ₹ 20 per litre. Also find the curved surface area of the container (take $\pi = 3.14$).

8. A cone of height h cm, is divided into two parts by a plane through the mid-point of the axis of the cone and parallel to the base. Find the ratio of the volume of the conical part to that of the frustum.

9. A cone is divided by plane parallel to its base into a smaller cone of volume v_1 and a frustum of volume v_2. If $v_1 : v_2 = 8 : 19$, find the ratio of the radius of the smaller cone to that of the given cone.

10. A circular cone has a base of radius 10 cm and height 25 cm . The area of the cross-section of the cone by a plane parallel to its base is 154 cm^2. Find the distance of the plane from the base of the cone.

11. A circular cone is cut by a plane parallel to the base and the conical portion is removed. If the curved surface area of the frustum is $\frac{15}{16}$ of the curved surface area of the whole cone, prove that the height of the frustum is $\frac{3}{4}$ of the height of the whole cone.

12. From a cone of height 24 cm, a frustum is cut off by a plane parallel to the base of the cone. If the volume of the frustum is $\frac{19}{27}$ of the volume of the cone, find the height of the frustum.

13. A cone is cut into three parts by planes through the points of trisection of its altitude and parallel to the base. Prove that the volumes of the parts are in the ratio 1: 7:19.

14. A bucket is in the form of a frustum with a capacity of 45584 cubic cm. If the radii of the top and bottom of the bucket are 28 cm and 21cm respectively, find its height and surface area.

ANSWER

1. 4923.52 cm^3, 1004.8 cm^2

2. 24cm, 7844.57 cm^2, 20 cm, 1621.71cm^2

3. 65.76 litres **4. 3072.3 cm^3** **5. 301.57 cm^3.**

6. 14589.14 cm^2 **7. ₹ 122.80, 1177.5 cm^2**

8. 1: 7 **9. 2:3** **10. 7.5 cm.** **12. 8 cm**

14. 24 cm, 5236 cm^2

SUMMARY

In this chapter, you have studied the following points :

1. Circumference of a circle = $2\pi r$

2. Area of a circle = πr^2

3. Length of an arc of a sector of a circle with radius r and angle θ (measure in degrees) is $\frac{\theta}{360} \times 2\pi r$

4. Area of a sector of circle with radius r and angle θ (measured in degrees) is $\frac{\theta}{360} \times \pi r^2$.

5. Area of segment of a circle = Area of the corresponding sector – Area of the corresponding triangle.

6. To find the surface area and volume of a solid formed by combining any two of the basic solids namely cuboid, cone, cylinder, shpere and hemisphere.

7. Conversion of solid from one shape to another.

8. The volume of a frustum is given by

$$V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$$

9. Total surface area of a frustum of a cone

$= \pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$, where the symbols have usual meaning.

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CHAPTER

13

STATISTICS

13.1 Introduction

In your previous classes, you have seen and studied Statistics as a subject dealing with collection, compilation, analysis and interpretation of data. We can also look upon it as the weighing of evidence, meaning thereby that standard statistical methods give quantified evidence to draw certain conclusions in respect of one or more characteristics of the data (population).

It has also been stated that statistics gives information which are of representative nature and do not pertain to information on particular individual or individuals.

For example, the per capita income of India has statistical importance to represent the income of an average Indian while the income of the richest person of the country has less importance to represent the income of a typical Indian.

The main parameters of a population that a statistician desires to know in order to draw conclusions on it are the measures of central tendency and location, deviation, skewness and kurtosis of the data. These are the five important indicators whose knowledge promote a systematic study of the population.

In fact they are the five most important parameters indicating the state of being of the population in respect of a characteristic or characteristics.

As an analogy, let us consider the case of a medical practitioner. A doctor generally wants to know the blood pressure reading, the number of heart beats per minute , the body mass index etc. of a person in order to study the existing conditions of some of the vital organs of the person. Actually, they are important indicators of the state of health of the person under investigation similar to the parameters of a population under study.

In elementary classes of statistics we shall devote our attention to the first parameter namely, the central tendency of the data and a little to the measures of location.

What we are to study in the following sections of this chapter are the methods and devices which are generally adopted to determine the measures of central tendency of the data and also their representation and interpretation in numerical and graphical forms.

Before we proceed to study the methods and devices for the estimation of certain parameters of a population, a brief discussion on the innate character of the subject Statistics is worth mentioning.

Unlike the cases of deterministic experiments and the consequent formulations in accordance with an underlying idea or theory, statistical formulae are empirical in nature and its experiments are all non-deterministic. This explains why there are cases where one may find more than one formula for a particular parameter.

For example, when we perform an experiment on electricity related to potential difference, current and resistance, we know that they obey Ohm's law viz $V = IR$. Having given any two of the three variables we can determine or predict the value of the third. On the other hand, even though we know that the yield of a field is related to the amount of rainfall it gets, we do not have a formula like $V = IR$ relating yield to rainfall.

Such is the character of statistical variable which are all chance variables called stochastic variables or variates.

Nevertheless, pure statistics expounds and develops conceptual explanations called theoretical distributions and predicts with probability the expected values of the variates. These expected values agree with the observed values to a fair degree of closeness thereby consolidating the validity of the theory.

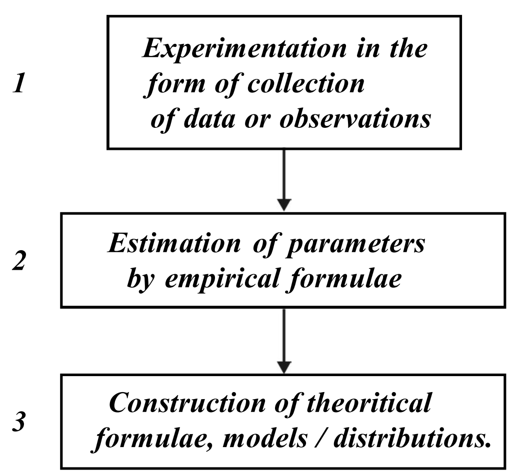
[It is to be noted that in the classical statistical approach, we first collect data in the form of numerical values. This is followed by the estimation of parameters using empirical formulae and finally we develop a theoretical distribution using the empirically estimated parameters.

On the other hand, in other physical sciences, following an underlying theory or idea, formulations are done and experiments are conducted in line with the theoritical indication and observations are made to get data of values.]

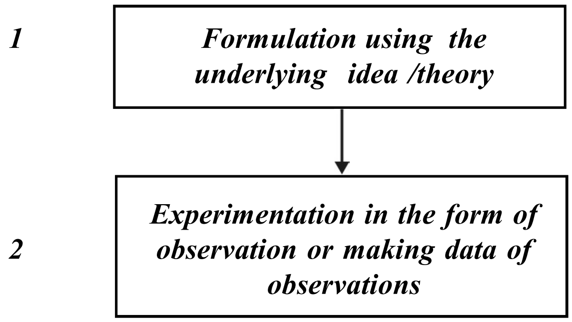
Also you can see the following route diagrams to observe the difference of the classical statistical methods from those of other physical sciences.

Two opposite routes

In Statistics :



In the case of other physical sciences:



Now, you have seen both the routes and must have observed that one is the other way round of the other.

However, it is noteworthy to mark that empirical findings are the forerunners of almost all the conceptually established theories. History of scientific discoveries and inventions are replete with many anecdotes of keen observations leading to profound theories.

One such example is John Kepler's laws of planetary motion derived from purely observational data and the same being profitably used by Sir Isaac Newton leading to the discovery of his famous laws of gravitation.

13.2 Measures of Central Tendency

You have seen that one of the major objectives of the study of data, particularly numerical data is to find the measures of central tendency of the data. However, we should note that there is no unique name for the measure of central tendency. Depending on the nature of the data under study, we may take the mean or the median or the mode to represent the central tendency. In any case, a measure of central tendency is the value of the variate around which the other values are supposed to cluster.

We shall first describe the most commonly used measures of central tendency.

13.3 The Weighted Arithmetic Mean (AM)

For constructing a grouped frequency distribution, we make classes of the variate values into which all the observations are packed piecewise.

For example, when we consider the grouped frequency distribution of marks obtained by 250 students in an examination of 100 maks, suppose we write 10-20 in the column of classes and 35 in the corresponding column of frequencies against the class, we mean that the number of students securing marks 10 and above but less than 20 is 35.

Here, we are making an assumption that each of the 35 students in the class gets $\frac{10+20}{2}$ i.e. 15 marks. This is like the case of taking the per capita income of India (Rs. 22,553 in 2006–2007) to be the annual income of an Indian.

This value which is obtained by using $\frac{\text{lower class limit + upper class limit}}{2}$ is called the class mark or mid value of the class and the number of observations lying in the class is called the frequency of the class. Since, all the frequencies are distributed in different classes, we call this data a grouped frequency distribution.

For a grouped frequency distribution having $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ as mid values of the classes with respective frequencies $f_1, f_2, f_3, \ldots\ldots\ldots, f_n$, the quantity $\bar{x}$ given by

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \ldots\ldots\ldots + x_n f_n}{f_1 + f_2 + \ldots\ldots\ldots + f_n}$$

, is called the weighted Arithmetic Mean, AM in short of the distribution.

Using the summation symbol $\sum$ (*pronounced as sigma, a letter the Greek alphabet)*

we write , $$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

The quantity $\sum_{i=1}^{n} f_i$ is the total number of observations in the data. If $\sum_{i=1}^{n} f_i = N$, then we say that *N* is the size of the sample or the population and it is the sum of all the frequencies of all the classes. We use the notation $(f_i, x_i), i = 1, 2, \ldots\ldots, n$ to denote a grouped frequency distribution.

Now, *n* is the number of classes into which the data has been divided and *N* is the sum of all the frequencies.

It is easy to see that *N*= *n* when the frequency of each class is 1(one).

Note : The derivation of the formula for $\bar{x}$ is simple as seen from the following consideration. Suppose there are f_1 persons each having Rs. x_1, f_n persons each having Rs. x_2,......, f_n persons each having Rs. x_n.

Then the total number of persons = $f_1 + f_2 + ... + f_n = N$ and the total amount with all the persons $= f_1x_1 + f_2x_2 + ... + f_nx_n$.

Hence, the average amount with a person denoted by $\bar{x}$ is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 .... + f_nx_n}{f_1 + f_2 + ... + f_n}$$

or, $$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i$$

Note 2. The term weighted is used because of the analogy of the formula for $\bar{x}$ and that of the distance of the centre of gravity of a system of weights $w_1, w_2, ..., x_n$ at respective distances of $x_1, x_2, ..., x_n$ from the origin along the same line. The distance $\bar{x}$ of the centre of gravity of the system of weights from the origin is given by

$$\bar{x} = \frac{x_1w_1 + x_2w_2 + ... + x_nw_n}{w_1 + w_2 + ... + w_n}$$

The weights are comparable to the frequencies of the various classes.

For a grouped frequency distribution (f_i, x_i), $i = 1, 2, ..., n$, two more means as measures of central tendency are used in statistics. They are the weighted Geometric Mean (GM) and the weighted Harmonic Mean (HM). They are respectively denoted by G and H and are given by the formulae. $G = \left(x_1^{f_1}.x_2^{f_2} .... x_n^{f_n}\right)^{\frac{1}{f_1+f_2+........+f_n}}$

or, $$G = \left(\prod_{i=1}^{n} x_i^{f_i}\right)^{\frac{1}{N}}$$

and $$\frac{(f_1 + f_2 + ... + f_n)}{H} = \frac{f_1}{x_1} + \frac{f_2}{x_2} + ... + \frac{f_n}{x_n}$$

i.e. $$\frac{N}{H} = \sum_{i=1}^{n} \frac{f_i}{x_i}$$

Or, $$\frac{1}{H} = \frac{1}{N}\sum_{i=1}^{n} \frac{f_i}{x_i} \quad ; \quad N = \sum_{i=1}^{n} f_i$$ in both the cases.

However, mainly because of the following two important reasons

(a) inconveniences in calculating the numerical values

(b) lack of versatility of these measures of the mean viz. G and H, generally, $\bar{x}$, the Arithmetic Mean is taken in most cases to represent the mean of the data.

You can also imagine the invalid situation in the definitions of both G and H when one of the variate values i.e. x_i 's becomes 0.

Never the less, we cannot ignore them because, under special conditions G may become the best mean and under other situations H may be the best mean of the data.

In fact, they are all tools to determine the mean of the data. Of them which one is the most appropriate is to be decided by the situation and nature of the data.

13.4 Devices to determine the Arithmetic Mean (AM) of a grouped data

When the given data is such that the mid values of the various classes and the corresponding frequencies are not large, we can use this method called Direct Method.

Example 1. The following is the grouped data of the number of persons of various age groups in an isolated hill village in a border area of Manipur.

Age group	*No. of persons*
0–10	55
10–20	57
20–30	80
30–40	75
40–50	62
50–60	47
60–70	25
70–80	7
80–90	2
90–100	0

Find the mean age of the inhabitants of the village.

Solution : The data is reproduced with the mid values of the classes

Classes	Mid value	Frequency	
	x_i	f_i	$f_i x_i$
0 – 10	5	55	275
10 – 20	15	57	855
20 – 30	25	80	2000
30 – 40	35	75	2625
40 – 50	45	62	2790
50 – 60	55	47	2585
60 – 70	65	25	1625
70 – 80	75	7	525
80 – 90	85	2	170
90 –100	95	0	0
		410=N	$13450=\sum f_i x_i$

(a) By Direct method

The mean age $\bar{x}$ is given by $\bar{x} = \frac{1}{N}\sum_{i=1}^{10} f_i x_i$

$$\therefore \quad \bar{x} = \frac{1}{410} \times 13450 = 32.8 \text{ years (nearly).}$$

Here , $\sum_{i=1}^{10} f_i = 410 = \text{N}$ and,

$$\sum_{i=1}^{10} f_i x_i = 13450.$$

Note: Since the frequency of the last class is 0, we can drop the class and take only 9 classes. This will not affect the value of $\bar{x}$. If 0 frequency occurs anywhere in the table, then also the class can be dropped without affecting the value of the mean . However, retaining it gives the information that there is no variate assuming that value or there is no observation in the corresponding class.

(b) The Assumed mean or Change of origin method

In the direct method, we have to find the products of f_i 's and x_i 's in the column for $f_i x_i$. Some times there may be big figures and it may require more time and mistakes may be committed. To avoid these difficulties, we assume any convenient arbitrary number 'a' as the assumed mean and construct the table as follows :

Taking the mid value near about the middle of the column of mid values, let the assumed mean 'a' be 55. Then, we find the various deviations of the mid values from 55 in the following table and they are denoted by d_i.

x_i	$d_i = x_i - 55$	f_i	$f_i d_i$
5	–50	55	–2750
15	–40	57	–2280
25	–30	80	–2400
35	–20	75	–1500
45	–10	62	–620
55	0	47	0
65	10	25	250
75	20	7	140
85	30	2	60
95	40	0	0
		410=N	–9550+450 = –9100(= $\sum f_i d_i$)

Now, $f_i d_i = f_i(x_i - a) = f_i x_i - f_i a$

$$\Rightarrow \sum f_i d_i = \sum f_i x_i - \sum f_i a$$

$$\therefore \bar{d} = \frac{1}{N}\sum f_i d_i = \frac{1}{N}\sum f_i x_i - a\frac{\sum f_i}{N},$$ $\frac{1}{N}\sum f_i d_i$ is the mean deviation from 'a' denoted by $\bar{d}$

$$\Rightarrow \frac{1}{N}\sum f_i d_i = \bar{x} - a.\frac{N}{N}$$

giving $\bar{x} = a + \frac{1}{N}\sum f_i d_i$

$$\therefore \quad \bar{x} = 55 + \frac{1}{410}(-9100)$$

$$= 55 - 22.19$$

$$= 32.8$$ years (nearly).

(c) The step deviation or the change of origin and scale method

In the solution of the problem in the above example by the assumed mean method if you examine the column of the deviation i.e. d_i, you will observe that these deviations have the greatest common factor 10, in this case. In fact 10 is the width of each of the classes.

Thus, when the deviations have common factors, the HCF i.e. the greatest common factor say, h can be easily found and we construct a new table using $u_i = \frac{x_i - a}{h}$ as the new variate.

Thus taking $a = 55$ and $h = 10$ the new table is constructed as follows:

x_i	$u_i = \frac{x_i - a}{h}\left(= \frac{x_i - 55}{10}\right)$	f_i	$f_i u_i$
5	–5	55	–275
15	–4	57	–228
25	–3	80	–240
35	–2	75	–150
45	–1	62	– 62
55	0	47	0
65	1	25	25
75	2	7	14
85	3	2	6
95	4	0	0
		410	$\Sigma f_i u_i = -955 + 45$ $= -910$

Now, $u_i = \frac{x_i - a}{h}$

$\Rightarrow \quad hu_i = x_i - a$

$\Rightarrow \quad f_i hu_i = f_i x_i - af_i$

$\Rightarrow \quad \Sigma f_i hu_i = \Sigma f_i x_i - \Sigma af_i$

$\Rightarrow \quad h \Sigma f_i u_i = \Sigma f_i x_i - a \Sigma f_i$

$\Rightarrow \quad \frac{h}{N} \Sigma f_i u_i = \frac{1}{N} \Sigma f_i x_i - \frac{a}{N} \Sigma f_i$

$\Rightarrow \quad h\bar{u} = \bar{x} - a$

$\Rightarrow \quad \bar{x} = a + h\bar{u}$, $\bar{u}$ is the mean of the new variate u whose values are u_i (i = 1, 2,....,n)

Thus is this example , $\bar{x} = 55 + 10 \times \frac{1}{410} \times (-910)$

$= 55 - 22.19$

$= 32 \cdot 8$ years (nearly).

Note 1. This method of change of origin and scale makes the calculations simple. But this method is advantageous only when the classes are of equal width.

Note 2. In actual application of the formula we need not establish the relation between $\bar{x}$ and $\bar{u}$ etc.

Note 3. The other two measures of the mean namely the Geometric mean G and the Harmonic mean H are not discussed here.

13.5 Other measures of central tendency

As stated earlier, apart from the mean, two other measures of central tendency viz. the median and the mode are also discussed in statistics.

The Median

The Median is the value of the variate such that half of the total number of variates have their values less than or equal to it and the other half have their values greater than or equal to it.

For a grouped frequency distribution we first locate the class in which the median denoted by M lies.

For this we construct a column called the column of cumulative frequency.

We are using the same data of the example of section 13.4.

The data is reproduced with one more column of cumulative frequency.

Class	Frequency	Cumulative frequency
0 – 10	55	55
10 – 20	57	112
20 – 30	80	192
30 – 40	75	267
40 – 50	62	329
50 – 60	47	376
60 – 70	25	401
70 – 80	7	408
80 – 90	2	410
90 –100	0	410

Here, $N = 410$

$\therefore \quad \frac{N}{2} = 205$.

From the cumulative frequency column we observe that 192 persons have their ages below 30 years. And 267 persons have their ages below 40. Thus, the age such that 205 persons are younger and another 205 are older than it must be between 30 and 40.

Thus, 30 – 40 is the median class.

$\therefore$ Median age = M = 30 + a fraction of the class width 10.

We assume that the frequency 75 of the median class is uniformly distributed over the width i.e. 10 of the class, so that the remaining portion $\left(\frac{410}{2} - 192\right)$ must lie in a width of $\frac{10}{75}(205 - 192)$

$$\therefore \quad M = 30 + \frac{10}{75} \times 13$$

$$= 31.7 \text{ years (nearly).}$$

Thus, we use the formula

$$M = l + \frac{\frac{N}{2} - c}{f} \times h$$

where, l = lower limit of the median class

N= size of the sample or population

c = cumulative frequency of the class just before the median class

f = frequency of the median class

h = width of the median class

Note : The given data can be modified in the following ways

1.

Age in years	Number of persons
Below 100	410
″ 90	410
″ 80	408
″ 70	401
″ 60	376
″ 50	329
″ 40	267
″ 30	192
″ 20	112
″ 10	55

2.

Age in years	Number of persons
Above 0	410
,, 10	355
,, 20	298
,, 30	218
,, 40	143
,, 50	81
,, 60	34
,, 70	9
,, 80	2
,, 90	0

When the data is given in any one of the three forms, the median can be determined after due modification, if necessary.

We shall show how to determine the median from the more than and less than ogives of the data when they are drawn together.

13.6 The Mode

The mode of a data is the value of the variate for which there is maximum frequency.

For a grouped frequency distribution an examination of the column of frequency shows the class in which the mode lies.

In the example of section 13.4, we see that 20 – 30 is the modal class.

$\therefore$ Mode = 20 + a fraction of the class width 10.

We assume that the width 10 of the modal class is divided uniformly in the ratio $(f_m - f_1):(f_m - f_2)$ so that

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

where, f_m = frequency of the modal class

f_1 = ,, ,, ,, class just before the modal class

f_2 = ,, ,, ,, ,, just after ,, ,, ,,

l = lower limit of the modal class

h = width of the modal class.

$$\therefore \quad \text{Mode} = 20 + \frac{80-57}{2\times 80-57-75}\times 10$$

$$= 20 + \frac{23}{28}\times 10$$

$$= 20 + 8.2$$

= 28.2 years (nearly)

Note : In an ideal case, the mean, median and mode of a distribution coincide at the same point.

However, a rough test for consistency can be done by using Pearson's empirical formula viz.

Mean – Mode = 3 (Mean – Median) even though they do not coincide.

In our case, Mean – Mode $= 32.8 - 28.2$

$= 4.6$

and, 3 (Mean – Median) $= 3(32.8 - 31.7)$

$= 3.3$

Thus, the difference is not large.

13.7 Measures of Location or Partition values or Quantiles

For a grouped frequency distribution, certain values of the variate may give important information about the data. For example, we may be interested in the value of the variate such that it is more than one fourth of all the values of the variate in the data while is less than three fourths of all of them etc.

Such values are called the partition values or quantiles or measures of location because they are partitioning the whole data into parts.

Some of the important partition values are :

The First or Lower Quartile

This quartile denoted by Q_1 is the value of the variate such that one fourth of all the values of the variate in the data have their values less than or equal to it while three fourths have their values greater than or equal to it and Q_1 is given by

$$Q_1 = l + \frac{\frac{N}{4} - c}{f}\times h$$

where , $l =$ lower limit of the quartile class
$N =$ sum of all the frequencies
$c =$ cumulative frequency of the class just before the quartile class
$f =$ frequency of the quartile class
$h =$ width of the quartile class.

The Third or the Upper Quartile

This quartile denoted by Q_3 is the value of the variate such that three fourths of all the values of the variate are less than or equal to it while one fourth of them have their values greater than or equal to it. Its formula is

$$Q_3 = l + \frac{\frac{3}{4}N - c}{f} \times h$$

where, l = lower limit of the quartile class
N = sum of all the frequencies
c = cumulative frequency of the class just before the quartile class
f = frequency of the quartile class
h = width of the quartile class

In general, we can write

$$Q_i = l + \frac{\frac{i}{4}N - c}{f} \times h, \quad i = 1, 2, 3, \text{ with usual notations}$$

When $i = 2$, $$Q_2 = l + \frac{\frac{2}{4}N - c}{f} \times h$$

This is the same as the formula for the median. In fact the median is the second quartile.

Similarly, the Deciles denoted by D_i's and the Percentiles denoted by P_i's are also partition values defined as follows :

$$D_i = l + \frac{\frac{i}{10}N - c}{f} \times h, \quad i = 1, 2, \ldots\ldots\ldots\ldots, 9.$$

and $$P_i = l + \frac{\frac{i}{100}N - c}{f} \times h$$

$i = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots, 99.$

The notations have their usual meanings.

Example 2. For the data of section 13.4, the first quartile Q_1 and the third quartile Q_3 are calculated as follows :

Now, $\frac{N}{4} = \frac{410}{4} = 102.5$

Here, 10 – 20 is the Q_1 class

$$\therefore \quad Q_1 = 10 + \frac{102.5 - 55}{57} \times 10$$

$$= 10 + 8.33$$

$$= 18.3 \text{ (nearly)}$$

Again, $\frac{3}{4}N = \frac{3}{4} \times 410 = 307.5$

Here, 40 – 50 is the Q_3 class

$$\therefore \quad Q_3 = 40 + \frac{307.5 - 267}{62} \times 10$$

$$= 40 + 6.53$$

$$= 46.5 \text{ (nearly).}$$

In the same way, D_6, the 6th decile or the 60th percentile of the data is given by

$$D_6 = P_{60} = l + \frac{\frac{60}{100}N - c}{f} \times h \text{, with usual notations}$$

Now, $\frac{60}{100}N = \frac{6}{10} \times 410 = 246$

30 – 40 is thus the P_{60} class

$$\therefore \quad P_{60} = 30 + \frac{\frac{60}{100}N - 192}{75} \times 10$$

$$= 30 + \frac{246 - 192}{75} \times 10$$

$$= 30 + 7.2$$

$$= 37.2 \text{ years}$$

13.8 Applications of the measures of location

Although the measures of locations are simple as they look, they have many applications in taking important decisions objectively. Now a-days the results of public examinations, particularly competitive exams where a large number of students appear for limited number of seats, are declared with percentiles. These results enable the examining authority to know the relative position of a particular candidate with respect to other competitors.

Example 3. In an all India level examination of 100 marks, 157210 candidates competed for admission to a particular course. The grouped frequency distribution of the marks secured by the candidates are given in the following table.

Group of marks secured	Number of candidates
0 – 10	5606
10 – 20	8670
20 –30	10078
30 – 40	25686
40 – 50	25700
50 – 60	35006
60 – 70	24678
70 – 80	16462
80 – 90	4270
90 – 100	1054

The concerned authority found that there are seats only for the best 10% of the total number of the candidates.

Find the cut off marks. Also indicate the way how the relative position of every candidate can be shown at the time of declaration of the result. Illustrate the procedure by giving one example.

Solution : The given table is reproduced with one more column of cumulative frequency as follows:

Group of marks secured (classes)	No. of candidates (Frequency)	Cumulative frequency
0 - 10	5606	5606
10 - 20	8670	14276
20 – 30	10078	24354
30 – 40	25686	50040
40 – 50	25700	75740
50 – 60	35006	110746
60 – 70	24678	135424
70 – 80	16462	151886
80 – 90	4270	156156
90 – 100	1054	157210

Since the admission is to be given only to the uppermost 10%, we are to find the marks secured by the candidates such that 90% of the total number of candidates get marks below that mark and 10% get it and above it. Surely, it is P_{90}, i.e. the 90^{th} percentile.

Now, N = 157210 and $\frac{90}{100}\text{N} = \frac{90}{100}\times 157210$

$= 141489$

Here, 70–80 is the P_{90} class

∴ From the formula , $P_{90} = l + \dfrac{\frac{90}{100}N - c}{f}\times h$

we get,

$$P_{90} = 70 + \frac{141489 - 135424}{16462}\times 10$$

$$= 70 + \frac{60630}{16462}$$

$$= 74 \text{ (nearly)}.$$

Thus, those candidates who secure 74 marks and above can be given admission.

Now, each and every student gets a unique mark between 0 and 100. Again, we can also find all the percentiles from P_1 to P_{99}. P_0 and P_{100} are surely 0 and 100 respectively in this case.

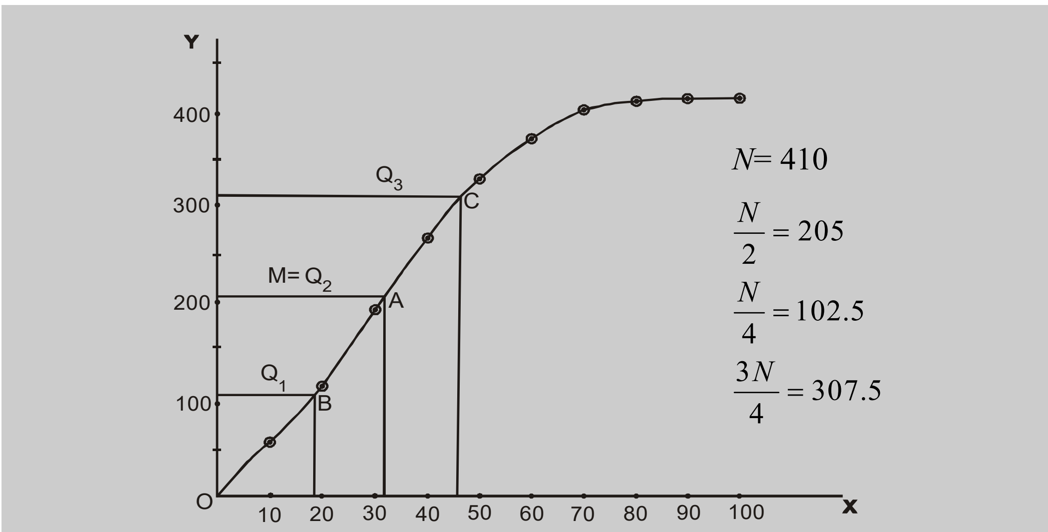
Thus a candidate getting any mark between 0 to 100 will have a percentile which indicates his/her relative position with respect to the total number of candidates.

For instance, in the above case, we see that P_{90}=74. Thus any candidate who gets 74 marks has percentile 90 showing that 90 % of all the candidates get marks below what he gets.

13.9 Graphical method of determination of partition values

For a grouped frequency distribution, the less than ogive is drawn in the following way. Points with the upper limit of each class as the abscissa and the corresponding less than cumulative frequency of the class as ordinate, are plotted. These points are joined by free hand.

For example, for the data of section 13.4, we plot the points (10, 55), (20,112), (30, 192), (40, 267), (50, 329), (60, 376),(70, 401),(80, 408),(90, 410) and (100, 410) and join them by free hand.



Along the y-axis, we locate points corresponding to ordinates

$$\frac{N}{2} = 205,\ \frac{N}{4} = 102.5 \text{ and } \frac{3N}{4} = 307.5$$

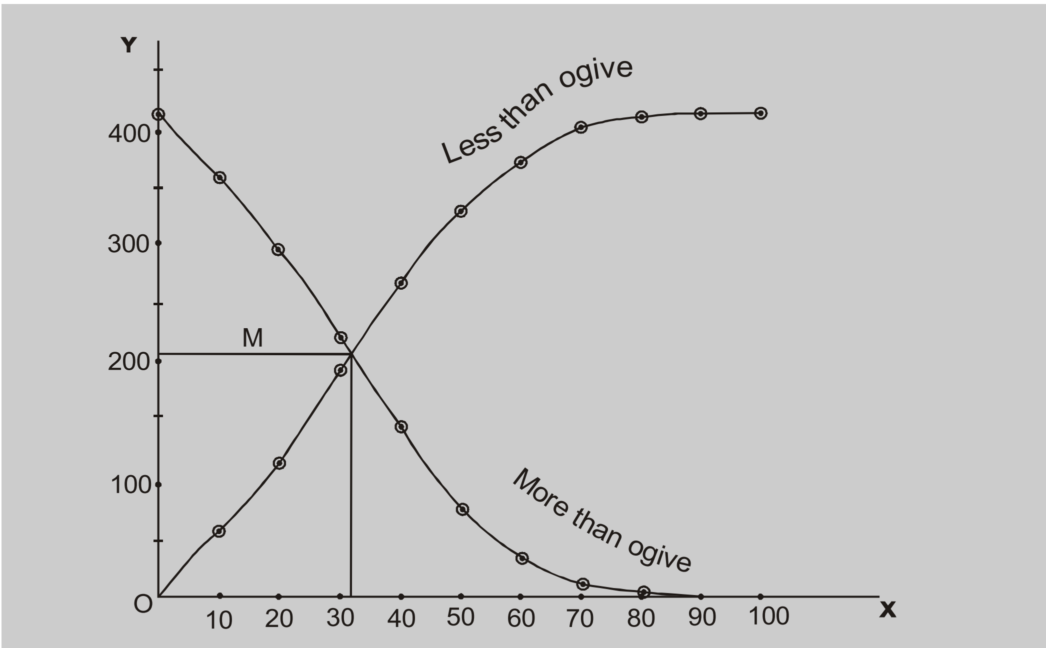
Through these points, lines are drawn parallel to the x- axis meeting the ogive at A, B and C respectively.

The abscissae of these points give Q_2, Q_1 and Q_3 respectively. And they are such that $Q_1<20$, $Q_2=M>30$ & $Q_3 > 45$ nearly .

If we draw both the less than and the more than ogive for the same data using the same axes of reference , the abscissa of the point of intersection of the two ogives gives the median M of the data as shown below.

To draw the more than ogive, we plot points with the lower limits of the classes as the abscissae and the more than cumulative frequency of the corresponding classes as the ordinates and join these points by free hand.

Thus, we plot the points (0,410), (10,355), (20,298), (30,218), (40,143),(50,81), (60,34), (70,9), (80,2), (90,0) and joining them by free hand we get the more than ogive.



The two ogives intersect at the point 'P' whose abscissa is more than 30. Thus M>30.

Note.1 Accurate values can be obtained by taking precise scale. We are showing only the principle.

Note.2 You must have seen that the Median and the second Quartile are both measures of central tendency and measures of location.

13.10 A general remark on the suitability of the different measures of central tendency

We have seen that there are three measures of central tendency viz, the mean, the median and the mode.

Again, there are three means viz, the AM, the GM and the HM.

It is therefore quite natural to seek the criteria for the determination of the most suitable measure of central tendency for a particular case. For such situation, we examine the case in the following way.

Even though we cannot give definite reasons for the adoption of a particular measure, the guiding principle that the measure of central tendency under study, should possess the representative character of the data, leads us to the choice of the right measure of central tendency.

The following few examples may illustrate the idea.

1. Suppose, we are interested in the yield of a field. Then, the yield of the past ten or twenty years and the average i.e. the AM of annual yields will give an idea of the productivity of the field.

   However, if you are interested in the information of success of the implementation of Sarva Shiksha Abhiyan (SSA), then you will surely be interested in the literacy rate of the children of the age group of 6 years to 14 years or in a round about way the answer to the question:

   " Is the modal age group of literacy percentage 6yrs –14 yrs ?"

   If yes, the scheme is bearing fruit. If no, the scheme is a failure.

2. In order to raise a certain sum of money for a cause, suppose we are to fix a uniform rate of subscription from each and every household of a locality where the income vary widely, then the knowledge of the median income will give a better idea of the rate.

3. Suppose, you are interested to open a business centre at a particular locality. Then your main concern is the number of the potential buyers and accordingly the income of people of that locality. Then surely, the modal class of the income groups of the people will be of interest to you.

   Thus, depending on the nature of the information that you are looking for , the appropriate measure of central tendency is to be fixed.

EXERCISE 13.1

1. The following are the numbers of children in a locality of 30 families.

   7, 4, 0, 4, 2, 1, 2, 5, 3, 1, 4, 6, 2, 1, 4, 3, 2, 0, 1, 2, 5, 4, 2, 3, 2, 2, 1, 2, 1, 3.

   Find the average number of children per family.

2. A shop dealing in electronic goods makes the following record of T.V. sets sold during a particular year-

Month	No. of T.V. sets sold
January	14
February	17
March	16
April	12
May	7
June	6
July	8
August	9
September	6
October	14
November	15
December	18

Find the average number of T.V. sets sold per month.

3. The following is the record of weight of children at the time of their birth as maintained in a maternity ward during a particular year--

Weight of children in kg.	No. of children
2.5	12
3.0	175
3.5	156
3.8	42
4.0	15
4.2	5

Find the average (mean) weight of a child at the time of birth.

4. Five coins were simultaneously tossed 1000 times, and at each toss the number of heads was observed. The number of tosses during which 0, 1, 2, 3, 4, 5 heads were obtained are shown in the table below. Find the mean number of heads per toss.

No of heads per toss	No. of tosses
0	38
1	144
2	342
3	287
4	164
5	25

5. A class master maintains the following record of absence of students in his class.

No. of students absent	No. of days
0	2
1	7
2	6
3	8
4	30
5	25
6	7
7	12
8	8
9	5
Total	**110**

Find the mean number of students absent per day.

6. The following is the frequency distribution of the number of teachers in Higher Secondary Schools in 1978 in India. Find the average number of teachers per Higher Secondary School in India for 1978.

No. of Teachers	No. of Hr. Sec. Schools
6 – 10	955
11 – 15	1067
16 – 20	1663
21 – 25	1492
26 – 30	1220
31 – 35	1129
36 – 40	745
41 – 45	637
46 – 50	442

Also, find the median number of teachers.

7. The expenditure for the consumption of water per month by 100 families is given below.

Expenditure on water (in Rs.)	No. of families
30 – 40	12
40 – 50	18
50 – 60	20
60 – 70	15
70 – 80	12
80 – 90	11
90 – 100	6
100 – 110	4
110 – 120	2
	100

Find the mean monthly expenditure of the families on water .

Also, determine the quartiles of the expenditure.

8. Find the median, lower and upper quartiles, 4th decile and 56th percentile of the following distribution of marks:

Marks	No. of students
0 – 4	10
4 – 8	12
8 – 12	18
12 – 14	7
14 – 18	5
18 – 20	8
20 – 25	4
25 and above	6

9. A school has 4 sections in class X having 45, 50, 42 and 30 students respectively. A test in Mathematics was conducted in all the sections and the mean marks of these sections are 62, 55, 58 and 49 respectively. Find the overall average mark of the class.

10. In the following distribution, the frequencies of two classes were missing. However, the mean of the data is given to be 50. Find the missing frequencies.

Class	Frequency
0 – 20	17
20 – 40	....
40 – 60	32
60 – 80	....
80 –100	19

Total 120

11. Find the median, the two quartiles and the mode of the following distribution.

Marks below	No. of Students
10	15
20	35
30	60
40	84
50	96
60	127
70	198
80	250

By drawing the less than ogive, indicate the positions of the quartiles.

12. Find the median, lower and upper quartiles , 8th decile, 56th percentile and the mode of the following distribution of 245 workers.

Daily savings (in ₹)	No. of workers
1 – 2.99	6
3 – 4.99	53
5 – 6.99	85
7 – 8.99	56
9 –10.99	21
11 –12.99	16
13 –14.99	4
15 – 16.99	4

[**Hint :** Convert into continuous classes]

ANSWER

(1) 2.63 (2) 11.83 (3) 3.31 Kg (4) 2.47

(5) 4.75 (6) Mean = 25.04, Median = 23.82

(7) Mean = ₹ 63.60, Q_1 = ₹ 47.22, Q_3 = ₹ 78.33

(8) Median = 10.89 , Q_1 = 6.5, Q_3 = 18.13, D_4 = 9.33, P_{56} = 11.82

(9) 56.56

(10)

Class	Frequency
20 - 40	28
60 - 80	24

(11) Median = 59.35, Q_1 = 31.04, Q_3 = 68.52,
Mode = 66.77

(12) Median = ₹ 6.49, Q_1 = ₹ 5.05, Q_3= ₹ 8.41,

D_8 = ₹ 8.85, P_{56} = ₹ 6.84, Mode = ₹ 6.04

SUMMARY

In this chapter, we have studied the following

1. Measures of central tendency: Mean, Median and Mode. We consider mainly the AM.

   AM is obtained by three methods viz, the direct method, the assumed mean method and the step deviation method. Median and Mode are also calculated by using the respective formula.

2. Measures of location or partition values namely, Quartiles, Deciles and Percentiles are also calculated.

3. Representation of grouped frequency distribution particularly the cumulative frequency by curves called ogives are shown.

   Calculation of particular values from ogives are also shown.

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CHAPTER

14

PROBABILITY

14.1 Introduction

We are familiar with the word probability. In fact, we have seen that it is the measure of chance and the probability that we are discussing is objective probability. We have also seen that the probability of the occurrence of an event is a real number lying between 0 and 1. In extreme cases, the end values may be assumed. In other words, for any event E, the probability P(E) of its occurrence is such that $0 \le P(E) \le 1$. You may note that the probability of an event may be an irrational number as you will come across later on (see Appendix II).

The definition of this measure of chance is given in three seemingly different forms and you are familiar with the one called the empirical or experimental definition. One of the other two is the classical or mathematical or a priori definition of probability due to Laplace and the last is the set theoritic or axiomatic or modern definition due to A. Kolmogorov.

We shall not discuss the last definition. This is discussed in higher classes with more sophisticated tools of modern Mathematics.

However, the value of the probability of an event E as calculated from these three seemingly different stand points converge to the same value as the number of trials becomes larger and larger.

In the following few sections, we shall discuss probability of the occurence of an event from the classical or mathematical approach.

But before the actual discussion, we shall consider some terms associated with the probability of an event.

14.2 Random or non-deterministic experiment

An experiment, whose result cannot be uniquely predicted even if the previous results of the same experiment conducted under similar conditions are all known is called a random or more precisely a non-deterministic experiment.

For example, tossing a coin 100 times may give 51 heads and 49 tails. With this prior information, we cannot say the outcome of the 101^{th} toss beforehand. In our daily social and personal lives, we come across a number of such examples and some of them are so important that the fate of some individuals or even a powerful government may depend on the outcome of such unpredictable phenomena.

Take the case of monsoon in India. How important is this phenomenon to the life of individuals and the government of the country! With all the records of the previous years, nobody can say precisely the shape of things to come in the coming years. Such is the nature of a non-deterministic experiment.

A non-deterministic experiment is also sometimes known as a trial. It is to be noted that one or more trials may constitute an experiment.

14.3 Sample space and event

The totality of all the possible outcomes of an experiment is called the sample space of the experiment. Any component of a sample space is an event.

For example, in tossing a coin once, the possible outcomes are a head or a tail to be denoted by H and T respectively. If we take these two outcomes together and write S to denote the collection of these two denoted by {H,T}, i.e. S={H,T}, then S is the sample space of the experiment of tossing a coin once.

In this case, to get H or to get T are events and they are denoted by {H} and {T} respectively. Note that H means a head while {H} means the event of getting a head. The cases of getting none of them and getting any one of H and T are also events. In fact, for the experiment of tossing a coin , there are 4 events. Similarly, the sample space S of tossing two coins once or one coin twice is given by S={HH, HT, TH, TT}.

You can verify that for this there are 16 events (How?).

Now what does {HH} represent? It is the event of getting head in both the tosses. Similarly, {HH, HT, TH} represents the event of getting at least one head in tossing two coins once or one coin twice.

14.4 Equally likely events

Events are said to be equally likely if there is no valid reason to say that one event has more chance to occur than the others.

For example, in tossing an unbias coin, one of the faces will turn up but there is no valid reason to say that a particular face will surely turn up or has more chance to turn up.

14.5 Mutually exclusive events

Events are said to be mutually exclusive if the happening of one forecloses the happening of all the others.

For example, in tossing a fair die once, one of the six faces will turn up. But when a face, say, the one denoted by [2] or [· ·] turns up, any one of the remaining five faces namely [1],[3],[4],[5] and [6] or [·],[···],[: :],[:·:],[:::] will not turn up.

14.6 Independent events

Events are said to be independent if the occurence of one has no effect on the occurence of the other or others.

For example : If a coin is tossed twice, the event of getting H or T in the second toss is independent of the outcome of the first toss.

14.7 Elementary events

An elementary event is one which cannot be further subdivided. This is analogus to an atom which is the smallest unit that is taking part in a chemical reaction.

Therefore , some writers use the term 'atom' of an event to represent an elementary event.

e.g. In the tossing of a coin once, the events {H} and {T} are elementary events.

On the other hand, in the experiment of tossing a coin twice, the event E of getting at least one head consists of the three mutually exclusive events viz {HH}, {HT} and {TH}. These constitute the event E. Thus, E is not an elementary event.

Further, any one of the three events is not an elementary event because, the event say {HT} is formed by two elementary events namely {H} in the 1st and {T} in the second toss. Similarly, {TT} is the event of getting T in both the tosses.

14.8 Exhaustive set of events

A set of events is said to be exhaustive if all the possible outcomes are included.

For example, in tossing a fair die once, the six events {1},{2},{3},...,{6} constitute a set of exhaustive events.

14.9 Favourable outcomes

For every experiment, out of the set of exhaustive outcomes, those entailing the occurence of a particular event are called the favourable ones for the event.

For example, consider the following situation.

In an urn, there are 2 white and 4 black balls, all similar in all respects except in colour. One ball is drawn at random.

If E is the event of drawing a white ball, then, out of the six exhaustive ways of drawing, only two are favourable to the event E.

14.10 Classical or Mathematical or a Priori definition of Probability due to Laplace

Out of n exhaustive, equally likely and mutually exclusive outcomes, if *m* are favourable to the event A, then the probability of the occurence of the event A denoted by P(A) is the ratio *m*:*n* and we write $P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to A}}{\text{Total number of possible outcomes}}$

From the definition, it is clear that $0 \le P(A) \le 1$.

When the probability of an event is 0, we say that the event will never occur. On the other hand, if the probability of an event is 1, then we say that the event is sure to happen.

Conversely, an impossible event has its probability 0 and a sure event has its probability 1.

We use the symbol $P\left(\overline{A}\right)$ to denote the probability of not happening of the event A.

In the definition, out of *n* exhaustive, equally likely and mutually exclusive outcomes, *(n–m)* are not favourable to the event A.

$$\therefore \qquad P\left(\overline{A}\right) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$$\text{or,} \qquad P(A) + P\left(\overline{A}\right) = 1.$$

A and $\overline{A}$ are called complementary events of each other. You are to note that A and $\overline{A}$ are mutually exclusive events and one of them is sure to happen i.e. A and $\overline{A}$ form an exhaustive system of events.

Similarly, if A, B, C are three mutually exclusive events forming an exhaustive system then, $P(A)+P(B)+P(C) = 1$.

Example. 1 In an urn, there are 2 white, 3 black and 5 yellow balls. One ball is drawn at random. Find the probability that it is (i) white (ii) black (iii) yellow.

$$\begin{bmatrix} W & -2 \\ B & -3 \\ Y & -5 \end{bmatrix}$$

Solution : There are altogether (2+3+5) i.e. 10 balls and choosing any one of them can be done in 10 ways. These 10 ways are all equally likely, mutually exclusive in the sense that when one is chosen, the remaining will not be chosen and further, the 10 ways form an exhaustive system because all the possible ways have been included.

(i) If W represents the event of choosing 1 white ball out of the 2, there are 2 ways which are favourable to W. Then by definition, $P(W)=\frac{2}{10}$.
Similarly, if B and Y respectively denote the events of drawing a black and a yellow ball then, $P(B)=\frac{3}{10}$

& $P(Y)=\frac{5}{10}$, for cases (ii) & (iii)

Now, $P(W)+P(B)+P(Y)=\frac{2}{10}+\frac{3}{10}+\frac{5}{10}=\frac{10}{10}=1$.

Note 1. If A and B are two events then, we can use the symbol $P(A\cup B)$ to denote the happening of one of them. If they form an exhaustive system, then, surely, $P(A\cup B)=1$. Further, if A and B are mutually exclusive, then $P(A\cup B)$ = P(A) + P(B).

Thus, P(A) + P(B) = 1, when A and B are both mutually exclusive and exhaustive events.

Note 2. If A and B are not mutually exclusive then, $P(A\cup B)\neq$ P(A)+P(B)

Note 3. $P(A\cup B)$ exists even if A and B are not mutually exclusive and also when they do not form an exhaustive system of events. Even then, $P(A\cup B)$ denotes the probability of happening of one of them. However, we shall not discuss the case here.

Example 2. A fair die is tossed twice. Write the sample space. How many sample points are there? Find the probability that the sum of points obtained in the two tosses is (i) equal to 10. (ii) greater than 10. (iii) less than 10.

Solution : The sample space S is given by

S = {(1,1),(1,2),(1,3),(1,4), (1,5),(1,6),
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(4,2),(5,3),(5,4),(5,5),(5,6),
(6,1),6,2),(6,3),(6,4),(6,5),(6,6)}

There are, in all 6×6 i.e. 36 samples or outcomes.

Let A, B and C denote the events of getting the sum of points equal to greater than and less than 10 respectively.

The outcomes favourable to A are (4,6),(5,5) and (6,4) which are 3 in number.

$$\therefore \quad P(A) = \frac{\text{Number of outcomes favourable to A}}{\text{Total number of possible outcomes}} = \frac{3}{36} = \frac{1}{2}$$

The outcomes favourable to B are (5,6),(6,5), and (6,6).

$$\therefore \quad P(B) = \frac{3}{36} = \frac{1}{12}$$

Since A, B, C are mutually exclusive events forming an exhaustive system, therefore P(A)+ P(B) + P(C)=1

i.e. $\frac{1}{12} + \frac{1}{12} + P(c) = 1$

$$\therefore \quad P(C) = 1 - \frac{1}{6} = \frac{5}{6}$$

Independent events

As stated earlier, two events A and B are said to be independent if the occurence of A has no effect on the occurence of B and vice-versa.

We use the symbol AB to denote the event of the combined occurence of A and B.

Further, if A and B are independent , then we have P(AB)=P(A)P(B)

Example 3. If a coin is tossed twice, what is the probability of getting head in both the cases?

Solution: We can look upon the problem from two stand points

One : If S is the sample space of the tossing of a coin twice, then, S={HH, HT, TH, TT}

Thus, out of the four exhaustive no. of outcomes only one is favourable to {HH}

$$\therefore P(HH) = \frac{1}{4}$$

Two : Probability of getting head in the 1st toss $= \frac{1}{2}$ i.e. $P(H) = \frac{1}{2}$

and Probability of getting head in the second toss $= \frac{1}{2}$ i.e. $P(H) = \frac{1}{2}$

Since, the outcome of the first toss has no effect on the outcome of the second toss, we get

$$P(HH) = P(H)P(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

14.13 A note on playing cards

Playing cards mean those cards popularly used in many indoor games like Bridge, Rummy etc.

A deck or pack of cards consists of 52 cards divided into two sets of 26 cards each. One set is red set and the other set is black set. In each colour set, there are two suits of 13 cards each. In a suit, there are 3 face cards called King, Queen and Jack. In each suit, there is one card called Ace and nine numeral cards starting from 2 and ending at 10. When one says four powers in a suit, it means Ace, King, Queen and Jack of the suit. Thus, there are two colours, four suits, 12 face cards consisting of four Kings, four Queens and four Jacks. Before any draw, cards are well suffled to ensure that each cards has equal chance of being drawn.

[A demonstration of playing cards and dice to the students is desirable so that they become familiar with the names and terms associated with these pieces of games of chance]

The four suits have their names, colours, symbols and number of cards as follows:

Suit name	**Colour**	**Symbol**	**No. of Cards**			
			Ace	**Face**	**Numeral**	**Total**
Heart	Red	♥	1	3	9	13
Diamond	Red	♦	do	do	do	do
Club	Black	♣	do	do	do	do
Spade	Black	♠	do	do	do	do

Example 4. A die is tossed 4 times. Find the probability that 6 appears at least once.

Solution : The probabilty that 6 appears in a single toss $= \frac{1}{6}$

$\therefore$ The probability that 6 does not appear in a toss $= 1 - \frac{1}{6} = \frac{5}{6}$

Now, the appearances in all the 4 tosses are all independent. Thus, the probability

that 6 does not appear in all the 4 tosses $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^4$

Let us consider two events namely A, the event in which all the four tosses do not give any 6 and B, the event in which there is at least one 6 in the 4 tosses.

Now, these two events are mutually exclusive and they form an exhaustive system of events. We are to find P(B).

But, P(A)+P(B)=1

$\Rightarrow$ P(B)=1$-$P(A)

$$= 1 - \left(\frac{5}{6}\right)^4$$

Example 5. From a well suffled pack of cards, one card is drawn at random. Find the probability that the drawn card is a power.

Solution: There are 52 cards is the pack and so one card can be drawn in 52 ways.

Futher, there are 4 suits in the pack and each suit has four powers (viz. Ace, King, Queen and Jack). So, there are 16 powers in the pack and out of these, one may be drawn in 16 ways.

Thus, out of the 52 ways, 16 are favourable to the event.

Thus, the reqd. probability $= \frac{16}{52} = \frac{4}{13}$.

Example 6. Two dice are thrown. Find the probability that the sum of their points is 11.

Solution : The favourable cases of giving a sum of 11 points are shown below:

Point on the 1st die	Point on the second die
5	6
6	5

That is to say, there are only 2 favourable cases of getting the sum of 11 points, because no other pair of points will give 11 as sum of points. But there are 6×6 ways in which the two dice can turn up giving different sums of points ranging from (1+1) i.e. 2 to (6+6) i.e. 12.

Thus, the reqd. probability $=\dfrac{2}{6\times6}=\dfrac{1}{18}$

Example 7. From a pack of cards, two cards are drawn at random (one after another without replacing). Find the probability that both are aces.

Solution : One card out of 52, can be drawn in 52 ways. And one ace out of 4, can be drawn in 4 ways.

$\therefore$ The probability that the first card drawn is an ace $=\dfrac{4}{52}=\dfrac{1}{13}$

Having drawn one ace in the first draw, we are left with 51 cards of which 3 are aces.

So, the probability that the second card drawn is also an ace $=\dfrac{3}{51}=\dfrac{1}{17}$

The required probability that both the cards drawn are aces

$=\dfrac{1}{13}\times\dfrac{1}{17}$ [$\because$ P(AB)=P(A) P(B) when A and B are independent]

$=\dfrac{1}{221}$.

14.14 Reconciliation between the empirical and the classical definition of probability.

In the empirical definition, we take the probability of an event A as the ratio

$$\frac{\text{The no. of trials in which A appeared}}{\text{Total no. of trials}}$$

$=\dfrac{m}{n}$ (say)

Supposing in tossing a coin 1000 times if H appeared 507 times then,

$P(H)=\dfrac{507}{1000}=0.507$. On the other hand, by the classical definition, $P(H)=\dfrac{1}{2}=0.5$.

In the tossing of a fair coin a large number of times, it has been observed that the ratio $\frac{m}{n}$ approaches to 0.5. The larger the number of trials i.e. n, the nearer is the ratio $\frac{m}{n}$ to 0.5.

Thus, like the case in any deterministic experiment, giving allowance to experimental errors, we take the theoritical or classical value as the probability of the event to which empirical values converge for large values of n.

EXERCISE 14.1

1. Give some events in our day to day life that are mutually exclusive.

2. If two events A and B are such that P(A) + P(B) =1, then, write P(B) in terms of $P(\overline{A})$.

3. A, B, C are three events which are equally likely, mutually exclusive but not forming an exhaustive system of events. Then, show that $P(A)=P(B)=P(C) \neq \frac{1}{3}$. Give one example of such a situation.

4. Distinguish between subjective and objective probabilities giving examples in each case.

5. When do you say that a die is fair ?

6. When a fair coin is tossed 3 times, how many outcomes can be there? Are the outcomes equally likely?

7. From an urn containing 4 white and 5 red balls, two balls are drawn at random. Find the probability that at least one is white.

8. From a pack of cards, two cards are drawn at random after a thorough suffle. Find the probability that both are kings.

9. Two fair dice are rolled. Find the probability that the sum of the points is 7. Also, find the sum of points which is the most probable by showing all the possible sum of points in a chart or table.

10. Given that p is the probability that a person aged x years will die in a year, find the probability that none of the four persons all aged x years will die in a year.

[This probability p can be obtained from a table called life table in the study of vital statistics]

11. In the above excercise if Mr. A is one of the four persons, find the probability that at least one of them will die in a year and Mr. A is the first person to die.

12. From a well suffled pack of cards, two cards are drawn at random. Find the probability that both the cards are diamonds.

13. A letter is chosen at random from the word **'ASSASSINATION'.** Find the probability that the letter is (i) a vowel (ii) a consonant.

14. In a 20-20 cricket match, a batsman hits a boundary 5 times out of 24 balls he faced. Find the probability that he did not hit a boundary in a ball he faced.

15. A die is thrown. Find the probability of the following events:

(a) A prime number will appear.

(b) A number less than 6 will appear.

(c) A number more than 6 will appear.

(d) A number less than or equal to 3 will appear.

16. There are four men and three ladies in a council. If two council members are selected at random for a committee, how likely is it that both are ladies?

ANSWER

1. **The result of a lottery, the result of game, the gender of a person, the result of a student's examination, the result of the competition for a gold medal among a number of contenders etc.**

2. $P(B) = P(\overline{A})$

5. **When the six faces have equal probabilities of turning up in every toss.**

6. **2^3 i.e. 8. Yes**

7. $\frac{13}{18}$ 8. $\frac{1}{221}$ 9. $\frac{1}{6}, 7$

10. $(1-p)^4$ 11. $\frac{1-(1-p)^4}{4}$ 12. $\frac{1}{17}$

13. (i) $\frac{6}{13}$ (ii) $\frac{7}{13}$ 14. $\frac{19}{24}$ 15. (a) $\frac{1}{2}$ (b) $\frac{5}{6}$ (c) 0 (d) $\frac{1}{2}$

16. $\frac{1}{7}$

SUMMARY

In this chapter we have studied the following items:

1. Classical definition of probability

2. $P(A) + P(\bar{A}) = 1$

Here, $\bar{A}$ is the event of non-occurence of the event A.

3. For mutually exclusive events A and B,

$P(A \cup B) = P(A) + P(B)$.

4. If A and B are mutually exclusive and exhaustive then

$$P(A \cup B) = P(A) + P(B)$$

and $P(A) + P(B) = 1$

5. For independent events A and B

$$P(AB) = P(A)P(B)$$

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APPENDIX 1 PROOF IN MATHEMATICS

A 1.1 Introduction

In class IX while discussing the import of proof in Mathematics, you have seen the following two facts viz.

(a) All true statements and deductions from them do not contradict any accepted axiom. On the other hand a statement leading to a contradiction to an axiom is taken as false.

(b) Deductive logic is used for giving a general proof of a statement. Even though inductive logic may lead to an indication of the pattern of the behaviour of the problem, it may not be adequate to give a general proof.

In this chapter, we shall discuss the type of methods that we are to adopt to give mathematical proof of a statement.

To give a mathematical proof of a proposition, there are two approaches. One is the direct method. For example, to prove the equality of two quantities or statements say, p and q, we may start from p and arrive at q or we may start from q and arrive at p or we may work from both p and q and arrive at the same result. Anyone of the above approaches is a direct method.

The other method is the indirect method.

In this method, we may use either the method of contradiction or the method of exhaustion.

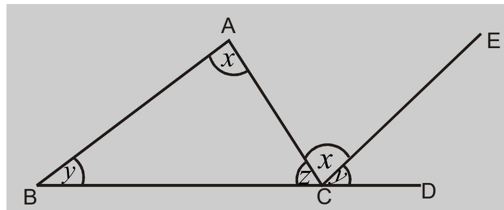
In the indirect method we make a hypothesis and proceed accordingly. If the conclusion does not contradict the hypothesis or does not violate an accepted axiom then, the hypothesis is taken to be true and draw the necessary conclusion. On the other hand if the hypothesis leads to a contradiction or violates an acepted axiom, then we reject the hypothesis and draw a different conclusion.

We shall illustrate these methods by way of examples.

Example 1. Show that the sum of the three angles of a triangle is 180°.

Solution :

Direct Method



ABC is a triangle and x, y, z are the measures of its three angles A, B and C respectively.

We are to show that x+y+z = 180°.

Through the point C, a line segment CE is drawn parallel to BA.

Then, by the property of the parallel lines BA and CE met by the transversal CA we get,

x + y +z = the straight angle BCD

= 180°

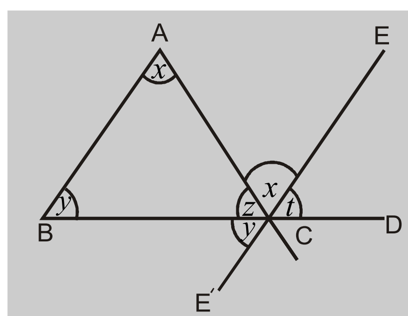
Now let us prove the same proposition by the **method of exhaustion**.

The sum of the three angles in degrees i.e. (x + y + z) is a real number and 180 is also a real number. Between any two real numbers, by the Trichotomy law, one of the following must hold and not more than one can hold. They are (i) x + y + z >180

(ii) x + y +z< 180

(iii) x + y + z = 180

If the first two cases are not true, then the third is sure to hold. Let ABC be a triangle with the angles x, y and z.



$E'CE$ is drawn through C and ∥ to BA.

BC is produced to D.

If possible, let x + y + z >180

$\Rightarrow$ x+y>180 − z

$\Rightarrow$ x + y > x + t $\because$ $\angle$BCD−z

=x+t   (As shown in the figure)

$\Rightarrow$ y > t

But y and t are vertically opposite angles. Thus, we are arriving at a contradiction to a known accepted theorem.

Thus, the hypothesis that x + y + z > 180 is wrong.

Again if we take x + y + z<180

then , $x + y < 180 - z$

$\Rightarrow x+y< x+t$

$\Rightarrow y< t$. This is also a contradiction to the same theorem.

Thus, (ii) is also wrong.

$\therefore$ the only possibility is $x + y + z = 180$.

This method is indirect and is called the method of exhaustion.

We have seen that the direct method of proving the proposition is very much more convenient.

But there are occasions where direct method becomes impossible or inconvenient as seen in the following example.

Example 2. Show that there is no greatest rational number less than $\sqrt{2}$.

Solution : We know that $\sqrt{2}$ is an irrational number considering the endless sequence of increasing reactional numbers obtained at different stages of extracting the square root of 2 by the long division method as shown below ,

$1.4< 1.41<1.414<1.4142 <\cdots\cdots\cdots<\sqrt{2}$

The rational numbers 1.4, 1.41, 1.414, 1.4142, etc are gradually becoming nearer and nearer to $\sqrt{2}$ though all of them are less than $\sqrt{2}$.

Therefore, one natural question is "Is there a rational number in this sequence which is the greatest of all rational numbers less than $\sqrt{2}$?"

To answer this question by direct method is impossible as the number of rational numbers less than $\sqrt{2}$ is infinite and it is impossible to find a general formula to represent all of them.

However, if we use the indirect method, we can prove that there is no such rational number which is the greatest among all the rational numbers less than $\sqrt{2}$.

Method of contradiction

If possible let us think that there is the rational number k which is the greatest of all the rational numbers less than $\sqrt{2}$

i.e. k is the greatest rational number less than $\sqrt{2}$

i.e. k is the greatest rational number satisfying

$k^2 < 2$...........(i)

Now, consider the rational number $k_1 = \frac{2k+2}{k+2}$

We see that $k_1 - k = \frac{2k+2}{k+2} - k = \frac{2k+2-k^2-2k}{k+2}$

$$= \frac{2-k^2}{k+2} > 0$$

$$\therefore \quad k_1 > k .$$

Again, $2 - k_1^2 = 2 - \left(\frac{2k+2}{k+2}\right)^2$

$$= \frac{2k^2+8k+8-4k^2-8k-4}{(k+2)^2}$$

$$= \frac{4-2k^2}{(k+2)^2} = \frac{2(2-k^2)}{(k+2)^2} > 0 .$$

Thus, $k_1^2 < 2$

i.e. $k_1 < \sqrt{2}$

Hence, $k < k_1 < \sqrt{2}$

Thus, there is another rational number k_1 which is greater than k but less than $\sqrt{2}$.

Thus, the hypothesis that there exists the greatest rational number k, less than $\sqrt{2}$ is false.

In other words, there is no greatest rational number less than $\sqrt{2}$.

A1.2 Systemetic Reasoning

The following example is given to illustrate the art of reasoning to arrive at a right conclusion.

Example 3.

We are reproducing a story leading to a problem and its solution (Yanglom and Yanglom, 1959).

The inhabitants of the south side of Canberra are known to be incapable of giving an untruthful answer to a question, but the reverse is the case for the inhabitants of the north side. A tourist lost in Canberra wishes to find out which side he is in by asking an inhabitant of the town, who may have come from either side, questions which will settle the problem, and which require answers 'Yes' or 'No' only.

Solution : Suppose the question of the tourist is " Do you belong to this part of the town?"

There are two possibilities regarding the identity of the inhabitant. He is either a northerner or a southerner.

Also there are two possibilities regarding the side of the town where he is standing. It may be the north side or the south side of the town.

Again, there are two answers that the tourist can get , Yes or No.

Let us examine the possiblities in the following tabular form.

Question : "Do you belong to this side of the town ?"

Identity of the inhabitant / Side of the town where the tourist is.	Northerner	Southerner	Conclusion, regarding the side of the town where, the tourist is
North side of the town	No (because northerner tells lies)	No (because southerner tells the truth)	He is in the north side if the answer is no, in any case
South side of the town	Yes (because he tells a lie)	Yes(because he tells the truth)	He is in the south side of the town if the answer is yes, in any case.

A 1.3 Statistical Reasoning

The following example illustrates the application of Statistical reasoning in giving opinion on the veracity of certain statements.

Example 4. President George W. Bush Jr. of the USA made a statement in the early part of May 2008 when the world as a whole was passing through a period of stress due to the sudden rise in the price of food stuff.

He made the statement that the sudden rise in the price of food stuff in the world market was due to the change of food habit of the people of the middle class families of developing countries like India.

He was blaming the growing demand of the people of India for the rise of the price.

Considering the following statistical data, examine the veracity of the statement.

Country	**India**	**USA**
Population consumed	1.19×10^9	301×10^6
food stuff during the period 2006-2007	193×10^6 tonnes	277×10^6 tonnes

Source : The Telegraph 7th May 2008 (Editorial).

Solution : The per capita consumption of India during the year under report

$$= \frac{193 \times 10^6 \times 10^3}{1.19 \times 10^9}$$

$$= \frac{19300}{119}$$

$$= 162 \text{ kg (nearly).}$$

The per capita consumption of the USA during the same year

$$= \frac{277 \times 10^6 \times 10^3}{301 \times 10^6}$$

$$= 922.6 \text{ kg (nearly).}$$

Thus, on the average, a citizen of the USA consumed nearly more than 5 times the average consumption of an average Indian.

Hence, president Bush's statement is not based on fact and is not tenable.

A 1.4 Converse of an implication

In our day-to-day life and also in the mathematical realm we come across converse of implications which may or may not be true.

If a statement p implies a statement q, then we write

$$p \Rightarrow q$$

The converse of this statement is $q \Rightarrow p$

The converse may or may not be true even though the implication is true.

Example 1. Find the converse of the implication:
If a whole number is even, then it is divisible by 2.

Solution : Let p : a whole number is even
q: the number is divisible by 2.

Now the implication is $p \Rightarrow q$ and we see that the implication is true.

The converse is $q \Rightarrow p$ which says that if a whole number is divisible by 2, then it is even. This converse is also true.

In such cases, we write $p \Leftrightarrow q$ which is a double way implication. We also use the term "p implies and is implied by q".

Example 2. Find the converse of the implication: Moths fly when it becomes dusk.

Solution : Let p : It has become dusk

& q : Moths fly

Now, the given implication is $p \Rightarrow q$ which is taken to be true.

Now, the converse of the implication is $q \Rightarrow p$

i.e. when the moths fly, it becomes dusk, which may not be true.

Thus, $q \Rightarrow p$ may not be true even if $p \Rightarrow q$ is true.

Example 3. Find the converse of the implication: For the quadratic equation $ax^2 + bx + c = 0$, if the roots are equal in magnitude but opposite in sign, then b=0.

Solution : Let p : the roots of the quadratic equation are equal in magnitude but opposite in sign.

q : b = 0

Then the given implication is $p \Rightarrow q$

Now, the converse of the implication is $q \Rightarrow p$. Which means that if b = 0, then the roots of the quadratic equation are equal in magnitude but opposite in sign.

In this case, $p \Leftrightarrow q$

Example 4. Find the converse of the implication:
If x = 4, then $x^2 = 16$.

Solution : Let p be x = 4 and q be $x^2 = 16$

Then, $p \Rightarrow q$ is the given implication.

The converse of the implication is $q \Rightarrow p$

i.e. when $x^2 = 16$, x = 4 which is not true.

Thus ,$q \Rightarrow p$ even though $p \Rightarrow q$.

A1.5 Negation of statement

The negation of a statement is the denial of an assertion made in the statement. For example , for the statement "Summer is hot in Delhi", the negation is "Summer is not hot in Delhi".

The negation of a statement p is denoted by ~p (tilde p).

Thus, negation of x = 5 is $x \neq 5$ or if p stands for x = 5, then ~p stands for $x \neq 5$.

Examine the following statement of a witty person who seemed to have withdrawn his earlier statement on facing an angry reaction from the audience he was addressing.

Once a famous person was addressing an eager audience who took pains to wait for him when he failed to turn up at the appointed time.

The very first sentence of his talk was " Half of you are fools". The audience who had already become restive reacted angrily and shouted at him to withdraw his statement.

The man noded and said "No, no, half of you are not fools".

The audience calmed down and listened to his speech.

In fact the man was not withdrawing his earlier statement. His statement "No, no, half of you are not fools" is not the negation of "Half of you are fools".

If the first statement of the man is p, then the second statement is also p stated in a different way and it is not ~p.

The actual withdrawal that the audience expected should be like "None of you is a fool".

Negation of a compound statement requires careful consideration

For example, the negation of " Polar regions are cold and dry" is "Polar regions are hot or wet". Similarly, the negation of "He is dead or disabled" is " He is alive and active".

We are to note that **(i)** negation of p is ~ p
(ii) ,, of p and q is ~ p or ~ q
(iii) ,, of p or q is ~ p and ~ q

Example 5. Write the negations of

(i) Delhi is the capital of India

(ii) 7>4 and 10+6 = 16

(iii) $\mathbf{x - y \geq 0}$ **or** $\mathbf{y - z \leq 0}$**.**

Solution: **(i)** Delhi is not the capital of India.
or It is not the case that Delhi is the capital of India.

(ii) $7 \not> 4$ or $10+6 \neq 16$

(iii) x – y< 0 and y – z > 0.

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APPENDIX 2 MATHEMATICAL MODELLING

A 2.1 Introduction

You have seen earlier that a model is a standard or an example for imitation or comparison and modelling is the work of making models.

A mathematical model is the presentation of a problem in mathematical language and finding its solutions under specified conditions with provisions for modifications to suit changing situations.

Mathematical models are also made to give a vivid idea on some complex entity or some very small or very big objects.

A globe is a model of the earth. There are models of the DNA structure of living organisms.

Now-a-days, mathematical models of the doses of medicines to be administered to patients are made for effective control of diseases.

The Government of India is thinking of making a mathematical model of the problem of flood control in the Brahmaputra valley of Assam.

In fact, a mathematical model is now taken as a fore runner of any solution to complex problems.

A 2.2 Modelling through various methods

Mathematical modelling can be done through

- **(a)** Simple linear equations and inequations
- **(b)** Geometry
- **(c)** Trigonometry
- **(d)** Differential equations
- **(e)** Graphs and diagrams
- **(f)** Probability and statistics etc.

We are giving two examples to illustrate.

Example 6. Size of a typical Atom

Considering the following four bodies namely, the earth, a drop of water, a toy ball and a hydrogen atom as perfect spheres of respective radii 6400km, 1mm, 24.68 cm and .37A°, justify Lord Kelvin's statement in respect of the size of an atom which more or less runs as follows:

"When a drop of water is magnified as large as the Earth, the same degree of magnification will make an atom as large as a toy ball"

Solution : Let V_1, V_2, V_3 and V_4 all in cubic centimetres denote respectively the volumes of the earth, the drop of water, the toy ball and a typical hydrogen atom.

Then,

$$V_1 = \frac{4}{3}\pi(6400 \times 1000 \times 100)^3$$

$$= K(64 \times 10^7)^3$$

$$= K64^3 \times 10^{21}, \; K = \frac{4}{3}\pi$$

$$V_2 = K\left(\frac{1}{10}\right)^3 = K10^{-3}$$

$$V_3 = K(24.68)^3$$

$$V_4 = K(.37 \times 10^{-8})^3 = K\left(\frac{37}{100}\right)^3 \times 10^{-24}$$

Now, $$\frac{V_1}{V_2} = 64^3 \times 10^{24}$$

and $$\frac{V_3}{V_4} = \left(\frac{24.68 \times 100}{37}\right)^3 \times 10^{24}$$

$$= (64)^3 \times 10^{24}$$

$\therefore \quad \dfrac{V_1}{V_2} = \dfrac{V_3}{V_4} = (64 \times 10^8)^3$ which is the measure of the magnification.

This shows that Kelvin's statement is perfectly justified.

Example 7. A function was held in a hall, the floor of which is a rectangular figure having its length one and half times its breadth. The floor has a carpeted path running parallel to its length just passing through the mid points of the opposing shorter sides.

The seats were arranged uniformly throughout the interior of the hall and no seat was vacant.

All of a sudden, a bomb went off and there was a chaos even though no major damage was done except some marks of the bomb pieces on the walls.

After the incident, security personnel found that the bomb was planted at a spot under the carpet equidistant from the boundary walls. Further, they found that the maximum distance reached by the bomb fragments was equal to the distance of the spot from the nearest corners.

Mr. A was reported to be present in the function . Find the probability that Mr. A was out of the danger zone.

Solution : In the figure, ABCD is the floor of the hall

We take, AB= CD = $2a$

and BC = AD = $3a$

P is the spot where the bomb was planted.

Then, PL = PM = PN = a

The fragments of the bomb were all inside the region $CD\overset{\frown}{RTS}C$ where

$$PR = PT = PS = \sqrt{2}a$$

Had there been no walls, the interior of the circle with centre P and radius $\sqrt{2}a$ must have been the danger zone.

Mr. A would be out of the danger zone if he took a seat in the region $AB\overset{\frown}{STR}A$.

$\therefore$ The reqd. prob.

$$= \frac{\text{Area of the region } AB\overset{\frown}{STR}A}{\text{Area of floor of the hall}}$$

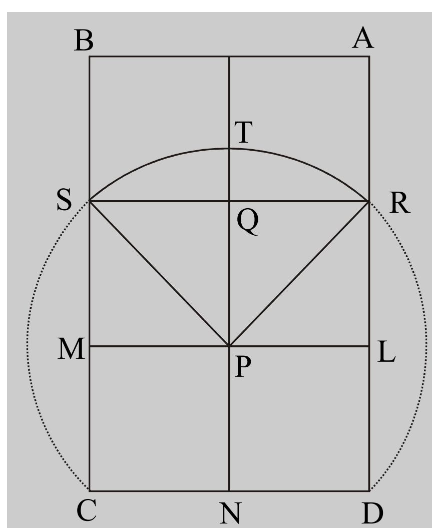
$$= \frac{A_1 (say)}{2a.\ 3a} = \frac{A_1}{6a^2}$$

Now, the area of the sector $P\overset{\frown}{RTS}P$

$$= \frac{1}{2}.\frac{\pi}{2}(\sqrt{2}a)^2$$

$$= \frac{\pi a^2}{2}$$

Area of the $\Delta PRS = \frac{2a^2}{2} = a^2$



∴ Area of the region

$$\text{RSTR} = \frac{\pi a^2}{2} - a^2$$

$$= \frac{a^2}{2}(\pi - 2)$$

$$\therefore \quad A_1 = 2a.a - \frac{a^2}{2}(\pi - 2)$$

$$= \frac{a^2}{2}(6 - \pi)$$

Thus, the reqd. prob. $= \dfrac{\frac{a^2}{2}(6-\pi)}{6a^2}$

$$= \frac{6-\pi}{12}.$$

Note. From the finding of the report of the security personnel, the spot Q can be another point where the bomb can be planted.

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