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CHAPTER-1

BINARY OPERATIONS

1.1 Introduction

Many interesting characteristics of the number system are associated with the four fundamental operations of addition, subtraction, multiplication and division. Let us investigate what such an operation does in the number system. Take the case of addition in the system of real numbers. Although we are quite familiar with the process of addition, still it is not easy to give a formal definition of addition of real numbers. You know that any two given real numbers can be added together. In other words, there is no pair of real numbers that cannot be added together. Also for any two real numbers, the sum (the result of adding together the numbers) is again a real number. So, given any two real numbers say, x and y, we get on adding, another real number $x+y$. Thus, addition can be looked upon as a rule of forming a real number $x+y$ corresponding to a given pair (x, y) of real numbers. The other three operations may also be treated similarly. Only in the case of division, the given pair of numbers (x, y) should be such that $y \neq 0$ (to avoid division by zero). Generalising the idea of these four fundamental operations, we shall develop in this chapter, the concept of binary operations on arbitrary sets and discuss their classification according to the properties they possess.

1.2 Binary Operation on a set

Definition 1.1 Let S be a non-empty set and 'o' be a mapping of the cartesian product S×S to S. Then 'o' is called a binary operation or binary composition or an internal composition on the set S.

Thus, a binary operation o on the set S, assigns to each ordered pair $(x, y) \in S \times S$ a uniquely determined element say, $z \in S$. We denote the o-image of (x, y) i.e. the element z by xoy and call it the composite (or product) of x and y under o.

As examples, we can define binary operations o and $*$ on the set Z of integers by

$$xoy = x+y$$

and $$x*y = xy; \quad x,\ y \in \mathrm{Z}$$

These binary operations are nothing but the usual addition and multiplication of integers.

According to the definition, division is not a binary operation on Z for division does not assign any integer to the ordered pair of integers (2, 3) or (3, 0). Again the impossibility of division by zero restricts division from being a binary operation on R also. However if we exclude zero from the set of real numbers, then division becomes a binary operation on the resulting set R^* of all non-zero real numbers, in as much as the quotient of a non-zero real number by a non-zero real number is a non-zero real number.

We can define several mappings from a given set to another. Likewise we can define several binary operations on a given set. For instance, on the infinite set R of real numbers, infinite number of binary operations may be defined.

An example of binary operation free from usual addition, multiplication etc. is that of set intersection or union.

Let P(S) be the set of all subsets of a given set S. Then the maps $(\mathrm{A},\mathrm{B}) \rightarrow \mathrm{A} \cap \mathrm{B}$ and $(\mathrm{A},\mathrm{B}) \rightarrow \mathrm{A} \cup \mathrm{B}$ where A, B are subsets of S, are binary operations on P(S), for the intersection or union of any two subsets of S is again a subset of S.

Definition 1.2 Let A and S be non-empty sets and $f: \mathrm{A} \times \mathrm{S} \rightarrow \mathrm{S}$ be a mapping. Then f is called an *external binary operation* on S over A.

Thus, an external binary operation f on S over A assigns to each ordered pair $(a,x) \in \mathrm{A} \times \mathrm{S}$ a uniquely determined element say, $y = f(a,x) \in \mathrm{S}$. Here, $f(a,x)$ is denoted multiplicatively as ax.

Multiplication of a vector by a scalar is an example of external binary operation on the set of vectors over the set of scalars.

Note : Henceforth “binary operation” will mean as defined in definition 1.1 and the word external will be specifically mentioned while considering the case of external binary operation.

Definition 1.3 A set equipped with one or more binary operations (external or internal) is known as an *algebraic structure.*

If o is a binary operation on a set S, then the pair (S, o) is an algebraic structure.

1.3 Associativity and Commutativity

Definition 1.4 A binary operation o on a set S is said to be *associative* if $(xoy)oz = xo(yoz)$ for every $x, y, z \in S$.

Definition 1.5 A binary operation o on a set S is said to be *commutative* if $xoy = yox$ for every $x, y \in S$.

The usual addition and multiplication on the set Z of integers are associative as well as commutative for

$$(a+b)+c = a+(b+c),\ (ab)c = a(bc) \text{ for all } a, b, c \in Z,$$

and $a+b=b+a,\ ab=ba$ for all $a, b \in Z$.

Similarly, the usual addition and multiplication on Q (or R) are also associative as well as commutative. Subtraction on Z (or Q or R) is a binary operation which is neither associative nor commutative.

In fact, if $a, b, c \in Z$, then

$$a-(b-c) \neq (a-b)-c \text{ whenever } c \neq 0$$

and $$a-b \neq b-a \text{ whenever } a \neq b.$$

Example 1. Let $*$ be the binary operation defined on the set Q of rational numbers by $x*y = x+y-xy;\ x,y \in Q$. Show that $*$ is associative as well as commutative.

Solution : Let $x, y, z \in Q$. Then

$$\begin{aligned}(x*y)*z &= (x+y-xy)*z \\ &= a*z \quad \text{where } a=x+y-xy \\ &= a+z-az \\ &= (x+y-xy)+z-(x+y-xy)z \\ &= x+y-xy+z-zx-yz+xyz \\ &= x+y+z-xy-yz-zx+xyz\end{aligned}$$

and $$\begin{aligned}x*(y*z) &= x*(y+z-yz) \\ &= x+(y+z-yz)-x(y+z-yz) \\ &= x+y+z-yz-xy-zx+xyz \\ &= x+y+z-xy-yz-zx+xyz\end{aligned}$$

Hence $(x*y)*z = x*(y*z)$ for all $x, y, z \in Q$ and accordingly $*$ is associative.

Again, $x*y = x+y-xy$

$= y+x-yx$

$= y*x$ for all $x, y \in Q$

Hence $*$ is commutative as well.

Example 2. Show that the binary operation o defined on Z by

$xoy = x+2y$; $x,y \in Z$ is neither associative nor commutative.

Solution : For o to be associative we should have

$(xoy)oz = xo(yoz)$, for every $x,y,z \in Z$.

Now, LHS $= (x+2y)oz$

$= x+2y+2z$

And RHS $= xo(y+2z)$

$= x+2(y+2z)$

$= x+2y+4z$

Taking the particular case where $x=y=z=1$, we have

LHS = 1+2+2=5 and RHS = 1+2+4=7

Thus, $1o(1o1) \neq (1o1)o1$ and hence o is not associative.

In order that o is commutative, we should have

$xoy=yox$, for all $x,y \in Z$.

But $xoy = x+2y$ and $yox = y+2x$ so that when $x=0, y=1$,

we have $xoy = 2 \neq 1 = yox$. Hence o is not commutative.

1.3.1 Generalised product under an Associative Binary Operation

Let o be an associative binary operation on a set S. Then we can define inductively, the composite or product of any n elements $x_1, x_2, \ldots\ldots, x_n \in S$ under o as follows :

$x_1 o x_2 o x_3 = (x_1 o x_2) o x_3$

$x_1 o x_2 o x_3 o x_4 = (x_1 o x_2 o x_3) o x_4$

....

$x_1 o x_2 o \ldots\ldots\ldots o x_n = (x_1 o x_2 o \ldots\ldots o x_{n-1}) o x_n$

By making repeated use of associativity, one can see that in the product

$x_1 o x_2 o .... o x_n$ the factors may be grouped in any manner without altering the value of the product so long as the order of the elements is unchanged. Besides being associative, if o is commutative also, then the order of the factors may also be changed randomly without altering the value of the product.

1.3.2 Power of an Element relative to an Associative Binary Operation

Let o be an associative binary operation on a set S. Then for any $x \in$ S, and for any $n \in$ N, the nth power of x denoted by x^n is defined by

$$x^n = xoxo......ox \qquad (n \text{ factors each equal to } x)$$

It can be easily proved that

$$x^m o x^n = x^{m+n} \text{ for all } m, n \in \text{N}.$$

In case the associative binary operation is denoted additively, the nth multiple (additive power) of x denoted by nx is defined by $nx = x + x + ... + x$ (n terms).

In this case also we can easily prove that

$$mx + nx = (m+n)x, \qquad \text{for all } m, n \in \text{N}.$$

1.4 Distributivity

Definition 1.6 Let $*$ and o be two binary operations on a set S. Then we say that $*$ is distributive over o, if

$$x*(yoz) = (x*y)o(x*z) \text{ .... (i)}$$

and $$(yoz)*x = (y*x)o(z*x) \text{ .... (ii)}$$

for all $x, y, z \in$ S.

Thus, distributivity is a relation that exists between two binary operations. The conditions (i) and (ii) are known as left distributive law and right distributive law respectively and the two together are referred to as distributive law. If $*$ is commutative, the two conditions are identical.

In the set of real numbers, multiplication is distributive over addition and also over subtraction, but addition is not distributive over multiplication. In the power set P(S) of a set S, the binary operation of union distributes over intersection and vice-versa i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

and $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \text{ for all } A, B, C \in P(S).$$

Example 3. Two binary operations, addition and multiplication are defined on the set N×N as follows :

$$(a,\ b)+(c,\ d)=(a+c,\ b+d)$$

and $(a,\ b)\times(c,\ d)=(ac+bd,\ ad+bc)$

Prove that the multiplication distributes over the addition.

Solution : We are to prove that for all $a,b,c,d,e,f \in \mathrm{N}$,

$$(a,b)\times[(c,d)+(e,f)]=(a,b)\times(c,d)+(a,b)\times(e,f)$$

Now, LHS $= (a,b)\times(c+e,\ d+f)$

$=\big(a(c+e)+b(d+f),\ a(d+f)+b(c+e)\big)$

$=(ac+ae+bd+bf,\ ad+af+bc+be)$

and RHS $=(ac+bd,\ ad+bc)+(ae+bf,\ af+be)$

$=(ac+bd+ae+bf,\ ad+bc+af+be)$

$=(ac+ae+bd+bf,\ ad+af+bc+be)$

$\therefore$ LHS=RHS

Hence the result.

1.5 Subsets closed under a Binary Operation

Definition 1.7 Let o be a binary operation on a set S and H be a subset of S. Then H is said to be closed under o, if for every pair (a,b) of elements $a,b \in \mathrm{H}$, the composite aob is also an element of H i.e. if $(a,b) \in \mathrm{H}\times\mathrm{H} \Rightarrow aob \in \mathrm{H}$.

For example, consider the binary operation of addition on the set R of all real numbers. As a subset of R, the set Q of all rational numbers is closed under addition for the sum of any two rational numbers is again a rational number. However, the set Q^c of all irrational numbers is not closed under addition for the sum of two irrational numbers needs not be an irrational number. For instance, $2+\sqrt{3}$ and $2-\sqrt{3}$ are irrational numbers whereas their sum 4 is not an irrational number.

Note : For the algebraic structure (S,o), if H is a subset of S closed under o, then o is a binary operation on H also i.e. (H,o) is also an algebraic structure.

Example 4. Show that the set H={1,o,–1} is closed under multiplication but not under addition.

Solution : Here, H×H={(1,1), (1, 0), (1,–1), (0, 1), (0,0), (0, –1), (–1, 1), (–1, 0), (–1,–1)}

For each ordered pair $(a,b) \in \mathrm{H}\times\mathrm{H}$, we see that the product ab is 1 or 0 or -1. This means that $ab \in \mathrm{H}$ whenever $(a,b) \in \mathrm{H}\times\mathrm{H}$. Hence H is closed under multiplication.

For H to be closed under addition, we should have $a+b \in \mathrm{H}$ whenever $(a,b) \in \mathrm{H} \times \mathrm{H}$. However, $(1,1) \in \mathrm{H} \times \mathrm{H}$ and $1+1=2 \notin \mathrm{H}$.

Hence, H is not closed under addition.

Example 5. Let S = {0,1,2,3,4,5} and '$+_6$' be defined by

$$a +_6 b = c, \qquad a,b \in \mathrm{S}$$

where c is the remainder when $a+b$ is divided by 6. Prove that $+_6$ is a binary operation on S and that the subset H={0,2,4} of S is closed under $+_6$.

Solution : When any integer is divided by 6, the remainder will be one of the six numbers 0, 1, 2, 3, 4, 5. So, for any two integers a and b, if $a+b$ is divided by 6, the remainder will be a member of the given set S. Consequently $a +_6 b = c \in \mathrm{S}$ whenever $(a,b) \in \mathrm{S} \times \mathrm{S}$ and therefore $+_6$ is a binary operation on S.

Again, H×H= {(0,0), (0,2), (0,4), (2,0), (2,2), (2,4), (4,0), (4,2), (4,4)}

And

$0+_6 0=0$	$2+_6 0=2$	$4+_6 0=4$
$0+_6 2=2$	$2+_6 2=4$	$4+_6 2=0$
$0+_6 4=4$	$2+_6 4=0$	$4+_6 4=2$

Thus, $a+_6 b \in \mathrm{H}$ for any $(a, b) \in \mathrm{H}\times\mathrm{H}$. Hence H is closed under $+_6$.

Remark : The binary operation '$+_6$' is known as *"addition modulo 6"*.

Let (S,o) be an algebraic structure and H be a finite subset of S. To examine whether H is closed under o or not, a short cut method may be adopted by forming what is called the *composition table*. The method is illustrated in the following example.

Example 6. A binary operation $\times_5$ (*multiplication modulo 5*) is defined on the set W of whole numbers by

$$a \times_5 b = c; \quad a,b \in \mathrm{W}$$

where c is the remainder when ab is divided by 5.

Show that H={1, 2, 3, 4} is closed under this binary operation.

Solution : We can exhibit all possible multiplications in the form of the composition table shown below :

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

[Here, the composite of an entry in the first column (left of the vertical line) and an entry in the first row (above the horizontal line) is entered at the intersection of the corresponding row and column. For instance, the composite of 2 in the first column and 3 in the first row is 1, the second row (in which 2 occurs) third column (in which 3 occurs) entry inside the lines.]

Since each entry in the above table is a member of H, therefore H is closed under the binary composition.

1.6 Identity Element

An algebraic structure (S,o) is said to be with *identity element* if there exists $e \in S$ such that $xoe = eox = x$, for every $x \in S$.

For the algebraic structure (Z,+), 0(zero) is the identity element since $a+0=0+a=a$ for every $a \in Z$. And for (Q,.), 1 is the identity element since $x.1=1.x=x$, for every $x \in Q$.

Again consider the algebraic structure (N,+) where N is the set of all natural numbers. There is no identity element for this structure as $0 \notin N$. Thus, identity element may or may not exist in a given algebraic structure.

Theorem 1.1 The identity element for an algebraic structure, if it exists, is unique.

Proof : Let (S,o) be an algebraic structure. Let if possible, e_1 and e_2 be identity elements in S. Then

$e_1 o e_2 = e_1$, since e_2 is an identity element,

and $e_1 o e_2 = e_2$, since e_1 is an identity element.

But $e_1 o e_2$ is uniquely determined as the composite of e_1 and e_2 and so $e_1 = e_2$. Hence the theorem.

1.7 Inverse of an Element

Let (S,o) be an algebraic structure with identity element e and let x be an element of S. An element $y \in S$ if it exists, is said to be an inverse of x if $xoy = yox = e$.

Usually, inverse of x is denoted by x^{-1} and is specified by $xox^{-1} = x^{-1}ox = e.$ And any two elements x and y in S are inverse of each other if and only if $xoy = yox = e.$ It follows that the identity element e is the inverse of itself as $eoe=e.$

In (Z,+), every element has an inverse. In fact for $a \in Z$, $-a$ is the inverse since $a+(-a)=(-a)+a=0,$ the identity element. And in (R,.), every non-zero element x has an inverse $\frac{1}{x} \in R.$

Theorem 1.2 If (S,o) is an algebraic structure with identity, in which the binary operation o is associative, then the inverse of an element of S if it exists, is unique.

Proof : Suppose y and y' are inverses of the same element $x \in S.$

Then, $xoy = yox = e$

and $xoy' = y'ox=e$

where $e \in S$ is the identity element.

Now, $(yox)oy' = eoy'=y'$ .... (i)

and $yo(xoy') = yoe= y$ .... (ii)

Since o is associative,

$(yox)oy' = yo(xoy')$

$\therefore$ $y'=y$ (from (i) and (ii)).

This proves the theorem.

In an algebraic structure with identity, an element is said to be *invertible* if its inverse exists.

Theorem 1.3 Let (S,o) be an algebraic structure with identity, in which o is associative. If x and y are two invertible elements of S, then xoy is also invertible and $(xoy)^{-1} = y^{-1}ox^{-1}$.

Proof : Let e be the identity element in S.

Then, $xox^{-1} = x^{-1}ox = e$

and $yoy^{-1} = y^{-1}oy=e$

Now, $(xoy)\ o\ (y^{-1}ox^{-1}) = xo\ (yoy^{-1})ox^{-1}$ (by associativity)

$= (xoe)ox^{-1}$

$= xox^{-1}$

$= e$

and $(y^{-1}ox^{-1})o(xoy) = y^{-1}o(x^{-1}ox)oy$

$= y^{-1}o(eoy)$

$= y^{-1}oy$

$= e$

Thus, $(xoy)o(y^{-1}ox^{-1}) = (y^{-1}ox^{-1})o(xoy) = e$

Hence xoy and $y^{-1}ox^{-1}$ are inverse of each other.

$\therefore \quad (xoy)^{-1} = y^{-1}ox^{-1}$

This proves the theorem.

Example 7. A binary operation o is defined on Q by $xoy = \frac{xy}{5};\ x, y \in Q$.

Find the identity element and the inverse of $\frac{2}{3}$ if they exist.

Solution : Let e denote the identity element. Then for any $x \in Q$

$$eox = xoe = x$$

$$\Rightarrow \quad \frac{xe}{5} = x$$

$$\Rightarrow \quad e = 5 \in Q$$

Thus, the identity element is 5.

Again, let a be the inverse of $\frac{2}{3}$. Then

$$ao\frac{2}{3} = 5 \qquad \text{(the identity element)}$$

$$\Rightarrow \quad \frac{a \times \frac{2}{3}}{5} = 5$$

$$\Rightarrow \quad \frac{2a}{3} = 25$$

$$\Rightarrow \quad a = \frac{75}{2} \in Q$$

Thus, the inverse of $\frac{2}{3}$ is $\frac{75}{2}$.

Example 8. If S is a non-empty set, find the identity element (if it exists) for the algebraic structure $(P(S), \cup)$. Also examine whether B^{-1} exists or not for any subset B of S.

Solution : Let E denote the identity element. Then for any $A \in P(S)$

i.e. for any subset A of S, $E \cup A = A$.

But this relation holds for arbitrary subset A if and only if E is the empty set. So, $E = \phi$ is the identity element in P(S).

Again, $B^{-1} \cup B = \phi$

$\Rightarrow$ $B^{-1} = \phi$ and $B = \phi$

Thus, B^{-1} exists if and only if $B = \phi$.

EXERCISE 1.1

1. If E is the set of all even natural numbers and F, the set of all odd natural numbers, answer the following :
   (a) Is addition a binary operation on F ?
   (b) Is multiplication a binary operation on F ? If yes, find whether identity element exists or not.
   (c) Is addition a binary operation on E ? If yes, find whether identity element exists or not.
   (d) Is multiplication a binary operation on E ? If yes, find whether identity element exists or not.
2. State whether each of the following definitions of $*$ gives a binary operation on N or not. Give justification of your answer :
   (i) $a * b = a - b$
   (ii) $a * b = |a - b|$
   (iii) $a * b = a^2b$
   (iv) $a * b = b$
   (v) $a * b = a+ab$
   (vi) $a * b = a^b$
   (vii) $a * b = ab-1$
   (viii) $a * b = ab+1$.

3. Prove that the following binary operations on N are commutative but not associative :

   (i) $a * b = 2a + 2b \quad (a, b \in N)$

   (ii) $a * b = 2^{ab}$

   (iii) $a * b = (a-b)^2$

   (iv) $a * b = ab+1$.

4. Show that the binary operation on N defined by $a*b=b$ is associative but not commutative.

5. Show that each of the following binary operation $*$ on Q is neither associative nor commutative :

   (i) $x * y = x - y + 1 \quad (x, y \in Q)$

   (ii) $x * y = 2x + 3y$

   (iii) $x * y = x + xy$

   (iv) $x * y = xy^2$.

6. Prove that the binary operation o on Z defined by ao$b = a+b-5$ is associative as well as commutative.

7. Prove that the binary operation $*$ defined on Z by $a*b = 3a+5b$ is neither associative nor commutative. Also prove that the usual multiplication on Z distributes over $*$.

8. Let binary operations o and $*$ on R be defined by

   xo$y = 2x + 2y$ and $x * y = x$.

   Show that o is commutative but not associative and $*$ is associative but not commutative. Also show that o distributes over $*$.

9. Prove that the binary operation o on N defined by aob= maximum of a and b is associative and commutative. Find the identity element and invertible elements of (N,o).

10. Investigate the set of integers, the set of rational numbers and the set of irrational numbers for closure under the following binary operations :

    (i) addition (ii) subtraction (iii) multiplication (iv) division.

11. Prove that there is no non-empty finite subset of N closed under addition.

12. Prove that the only non-empty finite subset of N closed under multiplication is $\{1\}$.

13. Find whether the identity element exists or not for each of the following algebraic structures :

    (i) (N,+) (ii) (N, .) (iii) (Z,+) (iv) (Z, .)

    (v) (Q,+) (vi) (Q, .) (vii) $(P(S),\cap)$ (viii) $(P(S),\cup)$

    (S is any set and P(S) is the power set of S).

14. Let S={1, 2, 3, 4, 5, 6, 7}. Find the identity element of the algebraic structure $(P(S),\cap)$. Also find the inverse of A={2,4,6}, if it exists.

15. Consider the binary operation $*$ on Q defined by

    $$x*y = x+y-xy.$$

    Find the identity element of (Q, $*$). Also find x^{-1} for $x \in Q$. For what value of x does the inverse not exist ?

16. Form the composition table for the set S={1, 2, 3, 4, 5, 6} with respect to the binary operation of multiplication modulo 7. Deduce that S is closed under the operation. From the table, find the identity element and the inverse of each element of S. Also calculate 2^6 in S.

17. Form the composition table for the set S={0, 1, 2, 3, 4, 5} with respect to the binary operation of addition modulo 6. From the table, find the identity element and the inverse of each element of S.

18. Let a binary operation $*$ on N be defined by

    $a*b$= HCF of a and b.

    Show by means of a composition table that the set H={1, 2, 3, 4, 5, 6} is closed under $*$.

19. A binary operation o on N, is defined by

    aob = LCM of a and b.

    Form a composition table for the set H={1, 2, 3, 4, 5} with respect to o. State whether H is closed under o or not.

20. Prove that the set $S=\{3n : n \in Z\}$ is closed under usual addition and multiplication. Examine the algebraic structures (S,+) and (S,.) for existence of identity and invertible elements.

ANSWER

1. (a) No, (b) Yes, Identity element is 1 $(\in F)$.
   (c) Yes, Identity does not exist. (d) Yes, Identity does not exist.
2. (i) No, for $1,2 \in N$ but $1*2=1-2=-1 \notin N$.
   (ii) No, for $2 \in N$ but $2*2=(2-2=0 \notin N$.
   (iii) Yes, for $a*b=a^2b \in N$ whenever $a,b \in N$.
   (iv) Yes, for $a*b=b \in N$ whenever $a,b \in N$.
   (v) Yes, $a+ab \in N$ whenever $a,b \in N$.
   (vi) Yes, $a^b \in N$ whenever $a,b \in N$.
   (vii) No, $1*1=1-1=0 \notin N$.
   (viii) Yes, $ab+1 \in N$ whenever $a,b \in N$.
9. Identity =1, 1 is the only invertible element.
10. Z, Q are closed under addition, subtraction and maltiplication but not under division. Q^c is not closed under any of the compositions.
13. (i) Does not exist (ii) Exists (1) (iii) Exists (0) (iv) Exists (1)
    (v) Exists (0) (vi) Exists (1) (vii) Exists (S) (viii) Exists (ϕ).
14. Identity = S, A^{-1} does not exist.
15. Identity = 0, $x^{-1} = \dfrac{x}{x-1}$, x^{-1} does not exist when x=1.
16.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

The identity element is 1. The inverses of 1, 2, 3, 4, 5, 6 are 1, 4, 5, 2, 3, 6 respectively. $2^6=1 \in S$.

17.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

The identity element is 0. The inverses of 0,1,2,3,4,5 are 0,5,4,3,2,1 respectively.

19.

o	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

H is not closed under o for there are entries in the composition table which are not members of H.

20. For (S,+), the identity is 0 and inverse of any element $3n$ is $3(-n)$ i.e. $-3n$.

For (S,.), the identity does not exist and hence inverse of any element does not exist.

CHAPTER 2

SEQUENCES, A.P., G.P. AND H.P.

2.1 Introduction

We have discussed something about sequences and arithmetic progression (A.P.) in the Mathematics (general course) for class-X. In this chapter, we shall discuss again something more about these terms. We shall also discuss about geometric progression (G.P.), harmonic progression (H.P.) and find the sum of some important finite series.

2.2 Sequence

Recall that a sequence is an ordered set of real numbers a_1, a_2, a_3, ..., a_n, ... which is formed according to some specific rule so that corresponding to any definite positive integer n, there is a definite number a_n. The numbers a_1, a_2, a_3, ..., a_n ,... are called terms or elements or members of the sequence.

Consider the set of numbers $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, ..., which is obviously an ordered set. Now let us see the relation between any particular element of the set and its position in the set. For this set, we can write

$$a_1 = \frac{1}{2} = \frac{1}{1+1}$$

$$a_2 = \frac{2}{3} = \frac{2}{2+1}$$

$$a_3 = \frac{3}{4} = \frac{3}{3+1}$$

....

Following the pattern, we see that

$$a_n = \frac{n}{n+1}$$

Thus, we see that the above set of numbers are formed according to the specific rule a_n (nth element of the set) $=\frac{n}{n+1}$. So, it is a sequence.

A sequence is said to be finite if the number of its elements is finite, otherwise it is said to be infinite. A finite sequence $a_1, a_2, a_3, ..., a_k$, is denoted by $\{a_n\}_{n=1}^{k}$ and an infinite sequence $a_1, a_2, a_3, ..., a_n, ...$ is denoted by $\{a_n\}_{n=1}^{\infty}$ or simply by $\{a_n\}$, where a_n is the nth term of the sequence. By assigning to n, the values 1, 2, 3, ... successively in the formula for a_n, we can determine the elements of the sequence $\{a_n\}$. In general, if the nth term of a particular sequence is known, then by assigning different natural numbers to n, all the terms of the sequence, and hence the sequence may be determined. Therefore, the nth term in any sequence is called the *general term* of the sequence.

Some examples of sequence are given below :

(a) Finite sequences

(i) $1, 2, 3, ..., 50$ i.e. $\{n\}_{n=1}^{50}$

(ii) $1^2, 2^2, 3^2, ..., 10^2$ i.e. $\{n^2\}_{n=1}^{10}$

(iii) $2, 4, 6, ..., 150$ i.e. $\{2n\}_{n=1}^{75}$

(iv) $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ..., \frac{1}{100}$ i.e. $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{100}$

(b) Infinite sequences

(i) $1, 3, 5, ..., 2n-1 ...$ i.e. $\{2n-1\}$

(ii) $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ..., \frac{1}{2^n}, ...$ i.e. $\left\{\frac{1}{2^n}\right\}$

(iii) $1, \frac{1}{4}, \frac{1}{9}, ..., \frac{1}{n^2} ...$ i.e. $\left\{\frac{1}{n^2}\right\}$

(iv) $1, -1, 1, -1, ...,$ i.e. $\left\{(-1)^{n-1}\right\}$

Example 1. Find the general term (i.e. the nth term) of each of the following sequences :

(i) $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, ...$

(ii) $1, 3, 1, 3, 1, 3, ...$

Solution : (i) We have $a_1 = \frac{1}{2} = \frac{1}{1^2+1}$

$$a_2 = \frac{2}{5} = \frac{2}{2^2+1}$$

$$a_3 = \frac{3}{10} = \frac{3}{3^2+1}$$

$$a_4 = \frac{4}{17} = \frac{4}{4^2+1}$$

....

Following the pattern, the general term of the sequence is given by

$$a_n = \frac{n}{n^2+1}$$

(ii) We have $a_1 = 1 = 2-1 = 2+(-1)^1$

$$a_2 = 3 = 2+1 = 2+(-1)^2$$

$$a_3 = 1 = 2-1 = 2+(-1)^3$$

$$a_4 = 3 = 2+1 = 2+(-1)^4$$

and so on.

Hence, the general term of the sequence is given by

$$a_n = 2+(-1)^n$$

or $$a_n = \begin{cases} 1, \text{ if } n \text{ is odd} \\ 3, \text{ if } n \text{ is even.} \end{cases}$$

Example 2. Find the first five terms of each of the following sequences :

(i) $\left\{\frac{n^2}{n+1}\right\}$ (ii) $\left\{1+(-2)^n\right\}$

Solution : (i) We have $a_n = \frac{n^2}{n+1}$

$$\therefore \quad a_1 = \frac{1^2}{1+1} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{2+1} = \frac{4}{3}$$

$$a_3 = \frac{3^2}{3+1} = \frac{9}{4}$$

$$a_4 = \frac{4^2}{4+1} = \frac{16}{5}$$

$$a_5 = \frac{5^2}{5+1} = \frac{25}{6}$$

Hence, the first five terms of the given sequence are $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}$ and $\frac{25}{6}$.

(ii) We have $a_n = 1+(-2)^n$

$$\therefore \quad a_1 = 1+(-2)^1 = 1-2 = -1$$

$$a_2 = 1+(-2)^2 = 1+4 = 5$$

$$a_3 = 1+(-2)^3 = 1-8 = -7$$

$$a_4 = 1+(-2)^4 = 1+16 = 17$$

$$a_5 = 1+(-2)^5 = 1-32 = -31$$

Hence, the first five terms of the given sequence are –1, 5, –7, 17 and –31.

Example 3. Determine the sequence $\{n(n+3)\}_{n=1}^{20}$.

Solution : We have $a_n = n(n+3)$

$$\therefore \quad a_1 = 1\times(1+3) = 4$$

$$a_2 = 2\times(2+3) = 10$$

$$a_3 = 3\times(3+3) = 18$$

$$a_4 = 4\times(4+3) = 28$$

....

....

$$a_{20} = 20\times(20+3) = 460$$

Hence, given sequence is 4, 10, 18, 28, ..., 460.

EXERCISE 2.1

1. Find the first five terms of each of the following sequences :

(i) $\left\{1+(-1)^n\right\}$ (ii) $\left\{(-1)^{n-1}\right\}$ (iii) $\left\{\frac{3n-1}{n+2}\right\}$ (iv) $\left\{\frac{2n-1}{n}\right\}$

(v) $\left\{2^n+3\right\}$ (vi) $\left\{\frac{n^2+1}{3n-1}\right\}$ (vii) $\left\{\frac{1}{(2n-1)^2}\right\}$ (viii) $\left\{\frac{(-1)^n}{n!}\right\}$

2. Find the first four terms of each of the following sequences whose general term is :

(i) $\frac{n-1}{n}$ (ii) $\frac{n+1}{n+2}$ (iii) $\frac{1}{3^{n-1}}$ (iv) $\frac{\lfloor n}{n+1}$

(v) $n(n+1)$ (vi) $1+\frac{1}{2}+\frac{1}{3}+.........+\frac{1}{n}$ (vii) $\sqrt{n-1}-\sqrt{n}$ (viii) $1-\frac{(-1)^n}{2}$

3. Find the general term (i.e. the *n*th term) of each of the following sequences :

(i) 0, 3, 8, 15, 24, ... (ii) $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \ldots$

(iii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ (iv) 1, 0, 1, 0, 1, 0, ...

(v) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (vi) $\frac{3}{4}, \frac{5}{16}, \frac{7}{36}, \frac{9}{64}, \ldots$

(vii) $0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \ldots$ (viii) $\frac{1!}{2}, \frac{2!}{5}, \frac{3!}{8}, \frac{4!}{11} \ldots$

(ix) $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots$ (x) $1, 1+\sqrt{2}, \sqrt{2}+\sqrt{3}, \sqrt{3}+\sqrt{4}, \ldots$

4. Find the *n*th term of the sequence $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5} \ldots$ and hence obtain the 9th term.

5. Determine the following sequences :

(i) $\left\{\frac{1}{n^2+2}\right\}_{n=1}^{10}$ (ii) $\{3n-1\}_{n=1}^{15}$ (iii) $\{n(n+2)\}_{n=1}^{50}$ (iv) $\left\{\frac{1}{3^{n-1}}\right\}_{n=1}^{100}$

ANSWER

1. (i) 0, 2, 0, 2, 0 (ii) 1, –1, 1, –1, 1 (iii) $\frac{2}{3}, \frac{5}{4}, \frac{8}{5}, \frac{11}{6}, 2$

(iv) $1, 1, \frac{4}{3}, 2, \frac{16}{5},$ (v) 5, 7, 11, 19, 35 (vi) $1, 1, \frac{5}{4}, \frac{17}{11}, \frac{13}{7}$

(vii) $1, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}, \frac{1}{81}$ (viii) $\frac{1}{1!}, \frac{1}{2!}, -\frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}$

2. (i) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ (ii) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ (iii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ (iv) $\frac{1!}{2}, \frac{2!}{3}, \frac{3!}{4}, \frac{4!}{5}$

(v) 2, 6, 12, 20 (vi) $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}$ (vii) $-1, 1-\sqrt{2}, \sqrt{2}-\sqrt{3}, \sqrt{3}-\sqrt{4}$ (viii) $\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}$

3. (i) $n^2 - 1$ (ii) $\frac{2n-1}{n}$ (iii) $\frac{1}{2^{n-1}}$ (iv) $\frac{1+(-1)^{n-1}}{2}$ (v) $\frac{2n-1}{2n}$

(vi) $\frac{2n+1}{4n^2}$ (vii) $(-1)^n + \frac{1}{n}$ (viii) $\frac{\underline{|n}}{3n-1}$ (ix) $\frac{1}{n(n-1)}$ (x) $\sqrt{n-1} + \sqrt{n}$

4. $\frac{(-1)^{n+1}}{n+1}, \frac{1}{10}$

5. (i) $\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, ..., \frac{1}{102}$ (ii) 2, 5, 8, ..., 44

(ii) 3, 8, 15, ..., 2600 (iv) $1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3^2}, ..., \frac{1}{3^{99}}$.

2.3 Arithmetic Progression (A.P.)

Let us consider the sequence 1, 3, 5, 7, ...

We observe that

$$a_1 = 1$$
$$a_2 = 3 = 1 + 2 = a_1 + 2$$
$$a_3 = 5 = 3 + 2 = a_2 + 2$$
$$a_4 = 7 = 5 + 2 = a_3 + 2$$

Thus, we see that the first term is 1 and each of the other terms is obtained by adding a constant number (viz. 2) to the term preceding it.

We also observe that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 =$............ This means that the difference of any two consecutive terms, taken in the same order is constant. Such a sequence is called *an arithmetic progression*.

Definition : A sequence $\{a_n\}$ is called an arithmetic progression (A.P.) if there exists a number d such that $a_{n+1} - a_n = d \; \forall n \in \text{N}$. The number d is called the common difference (C.D.) of the A.P.

An A.P. is completely determined, if we know the first term and the common difference. In fact, if a is the first term and d is the common difference of an A.P., then the A.P. is $a, a+d, a+2d, a+3d,$ ....

The following are examples of A.P. :

(i) 1, 4, 7, 10, 13, ...; C.D. = 3

(ii) 2, –1, –4, –7, –10, ...; C.D. = –3

(iii) –10, –6, –2, 2, 6, ...; C.D. = 4

2.4 The n^{th} Term of an A.P.

Let a be the first term and d be the common difference of an A.P. Then the A.P. is $a, a+d, a+2d, a+3d, ...$

Denoting the successive terms by $a_1, a_2, a_3, a_4, ...,$ we have

$$a_1 = a = a + (1-1)d$$
$$a_2 = a + d = a + (2-1)d$$
$$a_3 = a + 2d = a + (3-1)d$$
$$a_4 = a + 3d = a + (4-1)d$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Following the pattern, we have

$$a_n = a + (n-1)d$$

Thus, for an A.P. whose first term is a and common difference is d, the nth term (or the general term) a_n is given by

$$a_n = a + (n-1)d.$$

Example 4. If the nth term of a sequence be $2n-3$, show that the sequence is an A.P. Hence find its first term and the common difference.

Solution : If a_n denote the nth term of the given sequence, we have

$$a_n = 2n-3$$

$\therefore$ $a_{n+1} = 2(n+1) - 3 = 2n - 1$

Now, $a_{n+1} - a_n = (2n-1) - (2n-3)$

$= 2$, a constant.

Thus, we see that, $a_{n+1} - a_n = 2 \forall n \in \mathrm{N}$.

Hence the given sequence i.e. $\{2n-3\}$ is an A.P. with common difference 2.

Also, the first term of the A.P. $= a_1 = 2.1 - 3 = -1$.

Example 5. If the first term of an A.P. is 2 and common difference is 3, find its 10th term and 15th term.

Solution : Here, first term, $a=2$

and common difference, $d=3$

$\therefore$ 10th term $= a_{10} = a + (10-1)d$

$= 2 + 9 \times 3 = 29$

and 15th term $= a_{15} = a + (15-1)d$

$= 2 + 14 \times 3 = 44$.

Example 6. Find the 20th term of the A.P. 3, 7, 11, 15, ...

Solution : Here, first term, $a=a_1=3$

and common difference, $d = a_2 - a_1 = 7 - 3 = 4$

$\therefore \quad a_{20} = a + (20-1)d$

$= 3+19\times4$

$= 79.$

Example 7. Examine if 70 is a term of the A.P. 5, 8, 11, 14, 17, ...

Solution : For the given A.P., we have

first term, $a=5$

and common difference, $d=8-5=3$

If 70 be the nth term of the A.P., then

$70=a+(n-1)d$

$\Rightarrow \quad 70=5+(n-1)\times3$

$\Rightarrow \quad 70=3n+2$

$\Rightarrow \quad 3n=68$

$\Rightarrow \quad n=\frac{68}{3}=22\frac{2}{3}$

But this is absurd, in as much as the natural number n cannot be fractional.

Hence, 70 is not a term of the given A.P.

Example 8. Which term of the A.P. –3, 1, 5, 9, ... is 65 ?

Solution : For the given A.P., we have

$a=-3$ and $d=1-(-3)=4$

Let 65 be the nth term of the A.P.

Then $65=a+(n-1)d$

$\Rightarrow \quad 65=-3+(n-1)\times4$

$\Rightarrow \quad 4n=72$

$\Rightarrow \quad n=18$

Hence, 65 is the 18th term of the given A.P.

Example 9. The 12th and 15th terms of an A.P. are 68 and 86 respectively. Find its 18th term.

Solution : Let a be the first term and d be the common difference of the A.P.

Then by question, we have

$a_{12}=68$

$\Rightarrow \quad a+11d=68$ .... (1)

and $\quad a_{15}=86$

$\Rightarrow \quad a+14d=86$ .... (2)

Solving (1) and (2), we get

$a=2$ and $d=6$

$\therefore \quad a_{18}=a+17d=2+17\times6=104.$

Example 10. The pth and qth term of an A.P. are respectively q and p. Prove that the $(p+q)^{th}$ term is 0.

Solution : Let a be the first term and d be the common difference of the A.P.

By question, we have

$a_p=q$

$\Rightarrow \quad a+(p-1)d=q$ .... (1)

and $\quad a_q=p$

$\Rightarrow \quad a+(q-1)d=p$ .... (2)

Subtracting (2) from (1), we get

$(p-q)d=q-p$

$$\Rightarrow \quad d=\frac{q-p}{p-q}=-1$$

Then from (1), we have

$a+(p-1)\times(-1)=q$

$\Rightarrow \quad a=p+q-1$

$\therefore \quad (p+q)^{th}$ term $=a_{p+q}=a+(p+q-1)d$

$=(p+q-1)+(p+q-1)\times(-1)$

$=0.$

Example 11. If a^2, b^2, c^2 are in A.P., show that $\frac{1}{b+c}, \frac{1}{b+c}, \frac{1}{a+b}$ are also in A.P.

Solution : Since a^2, b^2, c^2 are in A.P., we have

$$b^2 - a^2 = c^2 - b^2 \quad \text{................} \quad \text{(i)}$$

Now, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in A.P.

if $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

i.e. if $\frac{b-a}{b+c} = \frac{c-b}{a+b}$

i.e. if $(b-a)\ (b+a) = (c-b)\ (c+b)$

i.e. if $b^2 - a^2 = c^2 - b^2$, which is true by (i)

Hence $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Example 12. If the pth, qth and rth terms of an A.P. are respectively x, y and z, prove that

$$(q-r)x + (r-p)y + (p-q)z = 0.$$

Solution : Let a be the first term and d be the common difference of the A.P.

Then by question, we have

$$x = a + (p-1)d \quad \text{................} \quad (1)$$

$$y = a + (q-1)d \quad \text{................} \quad (2)$$

$$z = a + (r-1)d \quad \text{................} \quad (3)$$

Multiplying (1), (2), (3) by $(q-r)$, $(r-p)$, $(p-q)$ respectively and adding, we get

$$(q-r)x + (r-p)y + (p-q)z$$

$$= (q-r)[a + (p-1)d] + (r-p)[a + (q-1)d] + (p-q)[a + (r-1)d]$$

$$= [(q-r) + (r-p) + (p-q)]a + [(q-r)(p-1) + (r-p)(q-1) + (p-q)(r-1)]d$$

$$= 0 \times a + 0 \times d = 0 \cdot$$

2.5 Arithmetic Mean

When three quantities are in A.P., the middle one is called the arithmetic mean (A.M.) between the other two. Thus, if a, x, b are in A.P., then x is the A.M. between a and b, and we have

$$x-a=b-x$$

$$\Rightarrow \quad 2x=a+b$$

$$\Rightarrow \quad x=\frac{1}{2}(a+b)$$

Hence A.M. between a and b is $\frac{1}{2}$ $(a+b)$.

Again, if $a, x_1, x_2, ..., x_n, b$ be in A.P., then $x_1, x_2, ..., x_n,$ are called the n arithmetic means between a and b.

To insert a given number of arithmetic means between two given quantities :

Let a and b be two given quantities, and let n be the number of arithmetic means to be inserted between a and b.

Thus we have altogether $(n+2)$ terms of an A.P. of which the first term is a and the $(n+2)^{th}$ term (i.e. the last term) is b.

If d be the common difference, then

$$b=a+\{(n+2)-1\}d$$

$$\Rightarrow \quad b-a=(n+1)d$$

$$\Rightarrow \quad d=\frac{b-a}{n+1}$$

Hence the n arithmetic means between a and b are $a+d, a+2d, a+3d, ... a+nd$

i.e. $$a+\frac{b-a}{n+1}, a+\frac{2(b-a)}{n+1}, ..., a+\frac{n(b-a)}{n+1}.$$

Example 13. Insert 4 arithmetic means between 2 and 32.

Solution : Let x_1, x_2, x_3, x_4 be the 4 arithmetic means between 2 and 32. Then 2, x_1, x_2, x_3, x_4, 32 are in A.P.

Here, 2 is the first term and 32 is the 6th term.

If d be the common difference, then

$$32 = 2 + (6-1)\ d$$

$$\Rightarrow \quad 30=5d$$

$$\Rightarrow \quad d=6$$

$\therefore \quad x_1 = 2 + d = 2 + 6 = 8$

$x_2 = 2 + 2d = 2 + 2 \times 6 = 14$

$x_3 = 2 + 3d = 2 + 3 \times 6 = 20$

and $\quad x_4 = 2 + 4d = 2 + 4 \times 6 = 26$

Hence the required means are 8, 14, 20 and 26.

2.6 Sum of First *n* Terms of an A.P.

Let a be the first term and d be the common difference of an A.P.

If l be the nth term of the A.P., then

$$l = a + (n-1)d \text{ .... } (1)$$

Let S be the sum of the first n terms of the A.P.

Then, $\quad S = a + (a + d) + (a + 2d) + ... + l$ ....(2)

On writing the terms in the reverse order, we have

$$S = l + (l - d) + (l - 2d) + ... + a \text{ ....}(3)$$

Adding the corresponding terms in (2) and (3), we get

$$2S = (a + l) + (a + l) + (a + l) + ... + (a + l)$$

$= n(a+l)$, since there are n terms

$$\therefore \quad S = \frac{n}{2}(a + l) \text{ ....}(4)$$

$$= \frac{n}{2}[a + a + (n-1)d] \qquad \text{[using (1)]}$$

$$= \frac{n}{2}[2a + (n-1)d] \text{ .... } (5)$$

(4) or (5) may be used as the standard result for the sum of first n terms of an A.P.

Example 14. Find the sum of the first 30 terms of the A.P. 1, 4, 7, 10, ...

Solution : Here, $a=1$, $d=4-1=3$, $n=30$

$$\therefore \text{ Reqd. sum } = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{30}{2}[2 \times 1 + (30-1) \times 3]$$

$$= 15(2 + 87)$$

$$= 15 \times 89 = 1335.$$

Example 15. Find the sum of the following series :

$24+22+20+ \ldots +6$

Solution : The terms of the given series are in A.P. of which, first term, a=24 and common difference, $d = 22 - 24 = -2$.

Let n be the number of terms in the series.

Then, $a_n = 6$

$\Rightarrow a+(n-1)d = 6$

$\Rightarrow 24+(n-1)(-2) = 6$

$\Rightarrow 18=2(n-1)$

$\Rightarrow n-1=9$

$\Rightarrow n=10$

$\therefore$ Reqd. sum $= \frac{n}{2}(a+l)$

$= \frac{10}{2}(24+6)$ $[\because\ l = a_n = 6]$

$= 5\times30$

$=150.$

Example 16. How many terms of the A.P. 24, 20, 16, 12, ... must be taken so that the sum may be 72. Explain the double answer.

Solution : Here, a=24 and d=20–24=–4.

Let n terms of the A.P. give a sum 72.

$\therefore \frac{n}{2}[2a+(n-1)d] = 72$

$\Rightarrow \frac{n}{2}[2\times24+(n-1)\times(-4)] = 72$

$\Rightarrow 2n(12-n+1) = 72$

$\Rightarrow n(13-n) = 36$

$\Rightarrow n^2 - 13n + 36 = 0$

$\Rightarrow (n-4)(n-9) = 0$

$\Rightarrow n=4, 9$

Both these value of n satisfy the conditions of the question, for, if we write down the first 9 terms, we get 24, 20, 16, 12, 8, 4, 0, –4, –8. It is seen that the sum of the last five consecutive terms is zero. Hence, the sum of the first 9 terms is the same as the sum of the first 4 terms of the A.P.

Example 17. The 5th and 12th terms of an A.P. are 14 and 35 respectively. Find the sum of the first 20 terms of the A.P.

Solution : Let a be the first term and d be the common difference of the A.P. Then,

$a_5 = 14$

$\Rightarrow \quad a + 4d = 14$ .... (1)

and $a_{12} = 35$

$\Rightarrow \quad a + 11d = 35$ ....(2)

Solving (1) and (2), we get

$a = 2$ and $d = 3$

$$\therefore \quad \text{Reqd. sum} = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{20}{2}[2 \times 2 + (20-1) \times 3]$$

$$= 10 \times 61 = 610 \cdot$$

Example 18. If x, y, z are respectively the sum of the first p, q, r terms of an A.P., prove that $\frac{x}{p}(q-r) + \frac{y}{q}(r-p) + \frac{z}{r}(p-q) = 0$.

Solution : Let a be the first term and d be the common difference of the A.P. Then,

$$x = \frac{p}{2}[2a + (p-1)d] \Rightarrow \frac{x}{p} = \frac{1}{2}[2a + (p-1)d] \quad .... (1)$$

$$y = \frac{a}{2}[2a + (q-1)d] \Rightarrow \frac{y}{q} = \frac{1}{2}[2a + (q-1)d] \quad .... (2)$$

$$z = \frac{r}{2}[2a + (r-1)d] \Rightarrow \frac{z}{r} = \frac{1}{2}[2a + (r-1)d] \quad .... (2)$$

Multiplying (1), (2), (3) by $(q–r)$, $(r–p)$, $(p–q)$ respectively and adding, we get

$$\frac{x}{p}(q-r) + \frac{y}{q}(r-p) + \frac{z}{r}(p-q)$$

$$= \frac{1}{2}[2a + (p-1)d](q-r) + \frac{1}{2}[2a + (q-1)d](r-p)$$
$$+ \frac{1}{2}[2a + (r-1)d](p-q)$$
$$= a[(q-r) + (r-p) + (p-q)] + \frac{d}{2}[(p-1)(q-r)$$
$$+ (q-q)(r-p) + (r-1)(p-q)]$$
$$= a \times 0 + \frac{d}{2} \times 0$$
$$= 0.$$

EXERCISE 2.2

1. Find the 15th and 50th terms of the A.P. 1, 3, 5, 7, ...
2. Find the 21st term of the A.P. 7, 4, 1, –2, –5, 8, ...
3. (i) Which term of the A.P. 1, 4, 7, 10, ... is 55 ?

   (ii) Which term of the A.P. $3, \frac{11}{3}, \frac{13}{3}, 5,$ .... is 9 ?
4. Is 216 a term of the A.P. 3, 8, 13, 18, ... ? If not, find the term nearest to it.
5. The first term and the common difference of an A.P. are respectively 39 and –7. Find the 10th term.
6. The first term and 12th term of an A.P. are respectively 5 and 49. Find the common difference.
7. How many numbers divisible by 15 are there between 20 and 400 ?
8. If the nth term of a sequence is $3n + 4$, show that the sequence is an A.P. Hence find the first term and the common difference.
9. Find the 25th term and the common difference of the A.P. whose nth term is $4n+1$.
10. The 8th and 15th terms of an A.P. are 4 and –24 respectively. Find its 12th term.
11. The 13th and 22nd terms of an A.P. are respectively 6 and 9 ; which term is 8 ?
12. The pth and qth terms of an A.P. are respectively q and p. Find the nth term.
13. A sequence $\{a_n\}$ is given by

    $$a_n = n^2 - 1, n \in \mathrm{N}.$$

    Show that it is not an A.P.

14. If a, b, c are in A.P., show that

(i) $b+c, c+a, a+b$ are also in A.P.

(ii) $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

(iii) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

15. If x, y, z be respectively pth, qth, rth terms of an A.P., show that

$$p(y-z)+q(z-x)+r(x-y)=0.$$

16. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., show that a^2, b^2, c^2 are also in A.P.

17. If the nth term of 3, 5, 7, 9, ... is the same as that of $9, 10\frac{1}{2}, 12, 13\frac{1}{2},$ ...., find n.

18. The sum of three numbers in A.P. is 21 and the sum of their squares is 179. Find the numbers.

19. The sum of three numbers in A.P. is 24 and the product of the two extremes is 55. Find the numbers.

20. The sum of four numbers in A.P. is 48 and the product of the two extremes is 108. Find the numbers.

21. Find the arithmetic mean between

(i) 10 and 20 (ii) –5 and 5 (iii) –5 and 9.

22. Insert (i) 2 arithmetic means between 2 and 11.

(ii) 3 arithmetic means between 6 and 22.

(iii) 4 arithmetic means between 5 and 20.

(iv) n arithmetic means between 1 and n^2.

(v) 3 arithmetic means between $2n+1$ and $2n-1$.

23. There are n arithmetic means between 4 and 64. If the ratio of the fourth mean to the eighth is 7:13, find n.

24. If $a+b+c \neq 0$ and $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

25. Find the sum of the first

(i) 20 terms of the A.P. 1, 5, 9, 13, ...

(ii) 25 terms of the A.P. 9, 12, 15, 18, ...

(iii) 30 terms of the A.P. $1\frac{2}{3}, 2, 2\frac{1}{3}, 2,\frac{2}{3},$............

(iv) 40 terms of the A.P. 10, 8, 6, 4, ...

(v) n terms of the A.P. $3n$, $3n$–1, $3n$–2, ...

(vi) n terms of the A.P. $\frac{1}{1+\sqrt{a}}, \frac{1}{1-a}, \frac{1}{1-\sqrt{a}},$ ....

26. Find the sum of the following series :

(i) 5 + 8 + 11 + ... + 47

(ii) 4 + 7 + 10 + ... + 49

(iii) 4 + 8 + 12 + ... + 80

(iv) $(\sqrt{2}+1)+\sqrt{2}+(\sqrt{2}-1)........+(\sqrt{2}-14)$

(v) $(x-y)^2+(x^2+y^2)+(x+y)^2+........+(x^2+y^2+18xy)$.

27. (i) How many terms of the A.P. 5, 9, 13, 17, ... must be taken so that the sum be 1224 ?

(ii) How many terms of the A.P. 3, 8, 13, 18, ... must be taken so that the sum may be 1010 ?

28. How many terms of the A.P. 22, 18, 14, 10 .... must be taken so that the sum may be 64. Explain the double answer.

29. The 5th and 11th terms of an A.P. are 41 and 20 respectively. Find the sum of the first 12 terms.

30. The 12th term of an A.P. is –13 and the sum of the first four terms is 24. Find the sum of the first 10 terms.

31. The first term of an A.P. is 7 and the sum of the first 15 terms is 420. Find the common difference of the A.P.

32. The sum of the first 15 terms and that of the first 22 terms of an A.P. are 495 and 1034 respectively. Find the sum of the first 18 terms.

33. The sum of the first 21 terms of an A.P. is 28 and that of the first 28 terms is 21. Show that one term of the A.P. is zero, and find the sum of the preceding terms.

34. The sum of the first 10 terms of an A.P. is 30 and the sum of the next 10 terms is –170. Find the sum of the next 10 terms following these.

35. Find the sum of the integers between 21 and 99 divisible by 6.

36. Find the A.P. when the sum to $n$terms is (i)n^2 (ii) $2n^2+5n$.

37. If the first term of an A.P. be a, its common difference be $2a$ and the sum of the first n terms be S, prove that $n=\sqrt{\frac{S}{a}}$.

38. The sum of the first n terms & that of the first m terms of an A.P. are m and n respectively. Show that the sum of the first $(m+n)$ terms is $-(m+n)$.

39. If the sum of m terms of an A.P. be equal to the sum of n terms, prove that the sum of $(m+n)$ terms is zero.

40. If the pth term of an A.P. is $\frac{1}{q}$ and the qth term is $\frac{1}{p}$, prove that the sum of the first pq terms is $\frac{1}{2}(pq+1)$.

ANSWER

1. 29, 99 **2.** –53 **3.** (i) 19th terms (ii) 10th term **4.** No, 218

5. –24 **6.** 4 **7.** 25 **8.** 7, 3 **9.** 101, 4 **10.** –12

11. 19th term **12.** $p+q-n$ **17.** 13 **18.** 3, 7, 11 **19.** 5, 8, 11

20. 6, 10, 14, 18 **21.** (i) 15 (ii) 0 (iii) 2

22. (i) 5, 8 (ii) 10, 14, 18 (iii) 8, 11, 14, 17

(iv) $n+a$, $2n+3$, $3n+4$, ...$3n^2+n+1$ (v) $2n+\frac{1}{2}, 2n-\frac{1}{2}$.

23. (i) 9

25. (i) 780 (ii) 1125 (iii) 195 (iv) –1160

(v) $\frac{n(5n+1)}{2}$ (vi) $\frac{n}{2(1-a)}[n-3)\sqrt{a}+2]$

26. (i) 390 (ii) 424 (iii) 840

(iv) $8\left(2\sqrt{2}-\sqrt{13}\right)$ (v) $11(x^2+y^2+8xy)$

27. (i) 24 (ii) 20

28. 4 and 8 **29.** 429 **30.** 0 **31.** 3 **32.** 702

33. $28\frac{1}{2}$ **34.** –370 **35.** 780 **36.** (i) 1, 3, 5, ... (ii) 7, 11, 15, ...

2.7 Geometric Progression (G.P)

Let us consider the sequence 2, 6, 18, 54, ...

We observe that

$a_1=2$

$a_2= 6=3\times2 =3a_1$

$a_3= 18 = 3\times6=3a_2$

$a_4= 54 = 3\times18 =3a_3$

....

Thus, we see that the first term is 2 and each of the other terms is obtained by multiplying the term preceding it by a fixed number (viz. 3).

We also observe that $\frac{a_2}{a_1}=\frac{a_3}{a_2}=\frac{a_4}{a_3}=\ldots\ldots\ldots$ i.e. the ratio of any term (except the first) to the term preceding it is the same. Such a sequence is called a geometric progression.

Definition : A sequence $\{a_n\}$ is called a geometric progression (G.P) if there exists a non-zero number r such that $\frac{a_{n+1}}{a_n}=r\,\forall n\in\mathrm{N}$. The number r is called the common ratio (C.R.) of the G.P.

A G.P. is completely determined, if we know the first term and the common ratio. In fact, if a is the first term and r is the common ratio of a G.P., then the G.P. is a, ar, ar^2, ar^3, ...

The following are examples of G.P. :

(i) 1, 2, 4, 8, ... ; C.R. = 2

(ii) $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3},$ .... ; C.R.= $\frac{1}{3}$

(iii) 3, –6, 12, –24, 48, ...; C.R.= –2.

2.8 The n^{th} Term of a G.P.

Let a be the first term and r be the common ratio of a G.P. Then the G.P. is a, ar, ar^2, ar^3, ... Denoting the successive terms by a_1, a_2, a_3, a_4, ..., we have

$a_1 = a = ar^{1-1}$

$a_2 = ar = ar^{2-1}$

$a_3 = ar^2 = ar^{3-1}$

$a_4 = ar^3 = ar^{4-1}$

....

Following the pattern, we have

$a_n = ar^{n-1}$.

Thus, for a G.P. whose first term is a and common ratio is r, the n^{th} term (or the general term) a_n is given by

$$\boxed{a_n = ar^{n-1}}$$

Example 19. Find the specified term of the following G.P. :

(i) 10^{th} term of 1, 2, 4, 8, ...

(ii) 9^{th} term of 2, –1, $\frac{1}{2}$, $-\frac{1}{4}$, ...

Solution : (i) Here, $a = a_1 = 1$ and $r = \frac{a_2}{a_1} = \frac{2}{1} = 2$.

$$\therefore\ 10^{th} \text{ term } = a_{10} = ar^{10-1}$$
$$= 1 \times 2^9 = 512$$

(ii) Here, $a = 2$ and $r = \frac{-1}{2}$.

$$\therefore\ a_9 = ar^{9-1} = 2 \times \left(-\frac{1}{2}\right)^8$$
$$= 2 \times \frac{1}{256} = \frac{1}{128}$$

Example 20. The 6^{th} and 11^{th} terms of a G.P. are 96 and 3072 respectively. Find the 15^{th} term of the G.P.

Solution : Let a be the first term and r be the common ratio. Then

$$a_6 = 96$$
$$\Rightarrow\ ar^{6-1} = 96$$

$\Rightarrow\ ar^5 = 96$ .... (1)

and $a_{11} = 3072$

$\Rightarrow\ ar^{11-1} = 3072$

$\Rightarrow\ ar^{10} = 3072$ .... (2)

Dividing (2) and (1), we get

$$r^5 = \frac{3072}{96}$$

$\Rightarrow\ r^5 = 32$

$\Rightarrow\ r^5 = 2^5$

$\Rightarrow\ r = 2$

Then from (1), we have

$a \times 2^5 = 96$

$\Rightarrow\ 32a = 96$

$\Rightarrow\ a = 3$

$\therefore\quad a_{15} = ar^{15-1} = 3 \times 2^{14} = 49152$

Example 21. Three numbers, whose sum is 15, are in A.P.; when 1, 4, 19 are added to them respectively, the results are in G.P. Find the numbers.

Solution : Let the three numbers in A.P. be $a-d$, a, $a+d$.

Then $(a-d) + a + (a+d) = 15$

$\Rightarrow\ 3a = 15$

$\Rightarrow\ a = 5$

Also it is given that $a - d + 1$, $a + 4$, $a + d + 19$ are in G.P. i.e. $6 - d$, 9, $24 + d$ (taking $a = 5$) are in G.P.

$$\therefore\ \frac{9}{6-d} = \frac{24+d}{9}$$

$\Rightarrow\ 81 = (6-d)(24+d)$

$\Rightarrow\ 81 = 144 - 18d - d^2$

$\Rightarrow\ d^2 + 18d - 63 = 0$

$\Rightarrow\ (d-3)(d+21) = 0$

$\Rightarrow\ d = 3,\ -21$

Taking $d = 3$, three numbers are $5 - 3, 5, 5 + 3$ i.e. 2, 5, 8.

Again, taking $d = -21$, the numbers are $5 - (-21), 5, 5 + (-21)$ i.e. 26, 5, – 16.

Hence the three numbers are either 2, 5, 8 or 26, 5, – 16.

Example 22. Divide 26 into three parts which are in G.P., such that their product is 216.

Solution : Let the three parts (which are in G.P.) be $\frac{a}{r}$, a, ar. Then

$$\frac{a}{r}.a.ar = 216$$

$$\Rightarrow a^3 = 6^3$$

$$\Rightarrow a = 6$$

Also, $$\frac{a}{r} + a + ar = 26$$

$$\Rightarrow \frac{6}{r} + 6 + 6r = 26$$

$$\Rightarrow \frac{6}{r} + 6r = 20$$

$$\Rightarrow \frac{3}{r} + 3r = 10$$

$$\Rightarrow \frac{3 + 3r^2}{r} = 10$$

$$\Rightarrow 3 + 3r^2 = 10r$$

$$\Rightarrow 3r^2 - 10r + 3 = 0$$

$$\Rightarrow (r - 3)(3r - 1) = 0$$

$$\therefore \quad r = 3, \frac{1}{3}$$

Hence the three parts are $\frac{6}{3}$, 6, 6×3 i.e. 2, 6, 18.

(Observe that when we take $r = \frac{1}{3}$, we get the same set of numbers 18, 6, 2.)

Example 23. If a, b, c, d are in G.P., Prove that $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are in G.P.

Solution : Let r be the common ratio of the G.P. Then

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$$

$$\therefore \quad b = ar, \quad c = br = ar^2, \quad d = cr = ar^3.$$

Now, $$\frac{b^2 + c^2}{a^2 + b^2} = \frac{(ar)^2 + (ar^2)^2}{a^2 + (ar)^2} = \frac{a^2r^2(1 + r^2)}{a^2(1 + r^2)} = r^2$$

and $\dfrac{c^2+d^2}{b^2+c^2} = \dfrac{(ar^2)^2+(ar^3)^2}{(ar)^2+(ar^2)^2} = \dfrac{a^2r^4(1+r^2)}{a^2r^2(1+r^2)} = r^2$

Thus, $\dfrac{b^2+c^2}{a^2+b^2} = \dfrac{c^2+d^2}{b^2+c^2} \ (= r^2)$

Hence, $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$ are in G.P.

Example 24. If x, y, z be respectively the p^{th}, q^{th}, r^{th} terms of a G.P., prove that $x^{q-r}\, y^{r-p}\, z^{p-q} = 1$.

Solution : Let a be the first term and k be the common ratio of the G.P. Then

$$x = ak^{p-1},\ y = ak^{p-1},\ z = ak^{r-1.}$$

$$\therefore\ x^{q-r}y^{r-p}z^{p-q} = \left(ak^{p-1}\right)^{q-r}.\left(ak^{q-1}\right)^{r-p}.\left(ak^{r-1}\right)^{p-q}$$

$$= a^{q-r}k^{(p-1)(q-r)}.a^{r-p}k^{(q-1)(r-p)}.e^{p-q}k^{(r-1)(p-q)}$$

$$= a^{(q-r)+(r-p)+(p-q)}.k^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$= a^0.k^0$$

$$= 1$$

2.9 Geometric Mean (G.M.)

When three quantities are in G.P., the middle one is called the geometric mean (G.M.) between the other two. Thus, if a, x, b are in G.P., then x is the geometric mean between a and b, and we have

$$\frac{x}{a} = \frac{b}{x}$$

$$\Rightarrow x^2 = ab$$

$$\Rightarrow x = \sqrt{ab}\,.$$

Again, if a, x_1, x_2,...................., x_n, b, are in G.P., then x_1, x_2,........, x_n are called the n geometric means between a and b.

To insert a given number of geometric means between two given quantities :

Let a and b be two given quantities, and let n be the number of geometric means to be inserted between a and b.

Then we have altogether $(n + 2)$ terms of a G.P. of which the first term is a and the last term i.e. the $(n + 2)^{th}$ term is b.

If r be the common ratio, then

$$b = ar^{(n+2)-1}$$

$$\Rightarrow r^{n+1} = \frac{b}{a}$$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence the n geometric means between a and b are ar, ar^2, ar^3, ...,ar^n i.e.
$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$, $a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$, $a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$,................,$a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$.

Example 25. Insert 3 geometric means between 4 and 64.

Solution : Let x_1, x_2, x_3 be the 3 geometric means between 4 and 64. Then 4, x_1, x_2, x_3, 64 are in G.P.

Here, 4 is the first term and 64 is the 5th term.

If r be the common ratio, then

$$64 = 4r^{5-1}$$

$$\Rightarrow r^4 = 16$$

$$\Rightarrow r^4 = 2^4$$

$$\Rightarrow r = 2$$

$\therefore$ $x_1 = 4r = 4\times2 = 8$

$x_2 = 4r^2 = 4\times2^2 = 16$

and $x_3 = 4r^3 = 4\times2^3 = 32$

Hence the required means are 8, 16 and 32.

2.10 Sum of First *n* Terms of a G.P.

Let a be the first term and r be the common ratio of a G.P.

Then the nth term of the G.P. is ar^{n-1}.

Let S denote the sum of the first n terms of the G.P.

Then,

$$S = a + ar + ar^2 + ...+ ar^{n-1}$$

and $Sr = ar + ar^2 + ...+ ar^{n-1} + ar^n$

By subtraction, $S - Sr = a - ar^n$

$$\Rightarrow S(1-r) = a(1-r^n)$$

$$\Rightarrow S = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \qquad (r \neq 1)$$

If $r > 1$, we write $S = \frac{a(r^n-1)}{r-1}$;

and if $r < 1$, we write $S = \frac{a(1-r^n)}{1-r}$.

Note : The above formula fails when $r = 1$. However, in this case $S = a + a + a + ...$ to n terms $= na$.

Example 26. Find the sum of the first 10 terms of the G.P. 1, 3, 9, 27, ...

Solution : Here, $a = 1$, $r = \frac{3}{1} = 3$, $n = 10$.

$$\therefore \text{ Reqd. sum } = \frac{a(r^n-1)}{r-1} = \frac{1(3^{10}-1)}{3-1}$$

$$= \frac{59049-1}{2} = \frac{59048}{2} = 29524$$

Example 27. If a, b, c are in A.P. and x, y, z are in G.P., prove that $x^{b-c} . y^{c-a} . z^{a-b} = 1$.

Solution : Since a, b, c are in A.P., therefore

$$b - a = c - b$$

$$\Rightarrow a - b = b - c$$

$$\Rightarrow a + c = 2b$$

Again, since x, y, z are in G.P., therefore

$$\frac{y}{x} = \frac{z}{y}$$

$$\Rightarrow xz = y^2$$

Then, $x^{b-c} . y^{c-a} . z^{a-b} = x^{a-b} . y^{c-a} . z^{a-b}$ $[\because b - c = a - b]$

$= (xz)^{a-b} . y^{c-a}$

$= (y^2)^{a-b} . y^{c-a}$ $[\because xz = y^2]$

$= y^{2a-2b} . y^{c-a}$

$= y^{2a-2b+c-a}$

$= y^{a+c-2b}$

$= y^{2b-2b}$ $[\because a + c = 2b]$

$= y^0 = 1$

Example 28. How many terms of the G.P. 2, 3, $4\frac{1}{2}$,............ must be taken to give a sum equal to $26\frac{3}{8}$?

Solution : Here, $a = 2$ and $r = \frac{3}{2}$.

Let n terms of the G.P. give a sum $26\frac{3}{8}$.

$$\therefore \quad \frac{a(r^n - 1)}{r - 1} = 26\frac{3}{8}$$

$$\Rightarrow \quad \frac{2\left[\left(\frac{3}{2}\right)^n - 1\right]}{\frac{3}{2} - 1} = \frac{211}{8}$$

$$\Rightarrow \quad 4\left[\left(\frac{3}{2}\right)^n - 1\right] = \frac{211}{8}$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^n - 1 = \frac{211}{32}$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^n = \frac{211}{32} + 1$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^n = \frac{243}{32}$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^n = \left(\frac{3}{2}\right)^5$$

$$\Rightarrow \quad n = 5$$

Hence the required number of terms is 5.

Example 29. If S be the sum, P, the product and R, the sum of the reciprocals of n terms of a G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$

Solution : Let a be the first term and r be the common ratio of the G.P. Then

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

$$P = a \,.\, ar \,.\, ar^2 \,.\, ar^3 \ldots\ldots\ldots\ldots\ldots\ldots ar^{n-1}$$

$$= a^n r^{1+2+3+\ldots\ldots\ldots+n(n-1)}$$

$$= a^n r^{\frac{n(n-1)}{2}} \quad \left[\because 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n-1}{2}\{2.1 + (n-2).1\} = \frac{n(n-1)}{2}\right]$$

$$\text{and R} = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots\dots\dots\dots\dots + \frac{1}{ar^{n-1}}$$

$$= \frac{\frac{1}{a}\left\{1 - \left(\frac{1}{r}\right)^n\right\}}{1 - \frac{1}{r}}$$

$$= \frac{\frac{1}{a}\left(\frac{r^n - 1}{r^n}\right)}{\frac{r-1}{r}} = \frac{r^n - 1}{ar^{n-1}(r-1)}$$

$$\text{Now, } P^2 = \left[a^n r^{\frac{n(n-1)}{2}}\right]^2$$

$$= a^{2n} r^{n(n-1)}$$

$$\text{and} \left(\frac{S}{R}\right)^n = \left[\frac{a(r^n - 1)}{r-1} \times \frac{ar^{n-1}(r-1)}{r^n - 1}\right]^n$$

$$= (a^2 r^{n-1})$$

$$= a^{2n} r^{n(n-1)}$$

$$\text{Hence, } P^2 = \left(\frac{S}{R}\right)^n$$

EXERCISE 2.3

1. Find the specified term of the following G.P. :

   (i) 14th term of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \dots\dots\dots\dots$

   (ii) 7th term of $81, -27, 9, -3, \dots\dots\dots\dots$

   (iii) 10th term of $\frac{1}{\sqrt{2}}, \sqrt{2}, 2\sqrt{2}, \dots\dots\dots\dots$

   (iv) 8th term of $p^2, pq, q^2, \dots\dots\dots\dots$

2. Find the value of k so that the following may be in G.P. :

   (i) $k+1,\ 2k+2,\ 5k-2$ (ii) $3k+1,\ 6k-4,\ 3k-2$

   (iii) $k-1,\ 3k-3,\ 8k-2$

3. The fourth and seventh terms of a G.P. are 54 and 1458 respectively. Find the 10th term.

4. (i) Which term of the G.P. 9, 3, 1, ... is $\frac{1}{243}$?

   (ii) Which term of the G.P. 32, -16, 8, -4, ... is $\frac{1}{32}$?

5. If the sum of three numbers in G.P. is 104 and their product is 13824, find the numbers.

6. Divide 42 into three parts which are in G.P. such that their product is 512.

7. Divide 31 into three parts which are in G.P. such that the sum of their squares is 651.

8. Three numbers whose sum is 18, are in A.P.. When 2, 4, 11 are added to them respectively, the resulting numbers are in G.P.. Find the numbers.

9. The product of three numbers in G.P. is 729 and the sum of their products in pairs is 819. Find the numbers.

10. If a, b, c, are in G.P., show that (i) $a^2 + b^2,\ ab + bc,\ b^2 + c^2$ are in G.P.

    (ii) $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P.

11. If a, b, c, d are in G.P., prove that

    (i) $a + b,\ b + c,\ c + d$ are in G.P.

    (ii) $a^2 - b^2,\ b^2 - c^2,\ c^2 - d^2$ are in G.P.

    (iii) $(a - b)^2, (b - c)^2, (c - d)^2$ are in G.P.

    (iv) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

12. If p^{th}, q^{th}, r^{th} terms of a G.P. are also in G.P., show that p, q, r are in A.P.

13. If a, b, c, d are in G.P., show that

    (i) $(b + c)(b + d) = (c + a)(c + d)$

    (ii) $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

    (iii) $(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

14. If 1, 1, 3, 9 be added respectively to the four terms of an A.P., a G.P. results. Find the four terms of the A.P..

15. If a, b, c be the p^{th}, q^{th}, r^{th} terms both of an A.P. and of a G.P., show that $a^{b-c}\, b^{c-a}\, c^{a-b} = 1$.

16. If a, b, c are in G.P. and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, show that x, y, z are in A.P.

17. Find the geometric mean between

(i) 3 and 27 (ii) $\sqrt{2}$ and $8\sqrt{2}$ (iii) $\frac{1}{5}$ and 125

18. Insert (i) 2 geometric means between 2 and $\frac{1}{4}$.

(ii) 2 geometric means between $-\frac{1}{3}$ and $\frac{9}{8}$.

(iii) 3 geometric means between 3 and 48.

(iv) 3 geometric means between – 2 and $-\frac{1}{8}$.

(v) 5 geometric means between 8 and $\frac{1}{8}$.

(vi) 3 geometric means between a and $\frac{1}{a}$.

19. The arithmetic mean between two numbers is 15 and their geometric mean is 9. Find the numbers.

20. If a be the arithmetic mean between b and c, and p, q be the geometric means between them, show that $p^3 + q^3 = 2abc$.

21. If a, b, c be in G.P. and x, y be the arithmetic means between a, b and b, c respectively, show that

(i) $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$ (ii) $\frac{a}{x}+\frac{c}{y}=2$.

22. Prove that the product of n geometric means between a and b is $(ab)^{\frac{n}{2}}$.

23. Find the sum of the first

(i) 10 terms of the G.P. 1, 2, 4, 8, ...

(ii) 8 terms of the G.P. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, ...

(iii) 12 terms of the G.P. 8, 4, 2, 1, ...

(iv) 7 terms of the G.P. 1, – 3, 9, – 27, ...

(v) 9 terms of the G.P. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$, ...

(vi) n terms of the G.P. $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}$, ...

(vii) n terms of the G.P. 3, – 6, 12, – 24, ...

24. (i) How many terms of the G.P. 1, 3, 9, 27, ... must be taken so that their sum is equal to 3280 ?

(ii) How many terms of the G.P. $1\frac{1}{3}$, 2, 3, ... must be taken so that their sum is equal to $\frac{211}{12}$?

25. Find the least value of n, for which $1 + 3 + 3^2 + ...+ 3^n > 1000$.

26. The 5th term of a G.P. is 48 and the 12th term is 6144. Find the sum of the first 10 terms.

27. In a G.P., the first term is 5, the last term is 320 and the sum is 635. Find the 4th term.

28. The sum of the first 6 terms of a G.P. is 9 times the sum of the first 3 terms. If the 7th term be 384, find the sum of the first 10 terms.

29. The sum of the first 10 terms of a G.P. is 33 times the sum of the first 5 terms. Find the common ratio.

30. The first and the last terms of a G.P. are respectively 3 and 768 and the sum is 1533. Find the number of terms and the common ratio.

31. If S_1, S_2, S_3 be the sums of the first n terms, $2n$ terms, $3n$ terms respectively of a G.P., prove that

(i) $S_1^2 + S_2^2 = S_1(S_2 + S_3)$

(ii) $S_1(S_3 - S_2) = (S_1 + S_2)^2$

ANSWER

1. (i) 1024 (ii) $\frac{1}{9}$ (iii) $526\sqrt{2}$ (iv) $\frac{q^7}{p^5}$

2. (i) 6 (ii) 1 (iii) 7

3. 39366 4. (i) 8th term (ii) 11th term

5. 8, 24, 72 6. 2, 8, 32 7. 1, 5, 25

8. 3, 6, 9 or 18, 6, – 6 9. 1, 9, 81 14. 1, 3, 5, 7

17. (i) 9 (ii) 4 (iii) 5

18. (i) 1, $\frac{1}{2}$ (ii) $\frac{1}{2}, -\frac{4}{3}$ (iii) 6, 12, 24 (iv) $-1, -\frac{1}{2}, -\frac{1}{4}$ or $1, -\frac{1}{2}, \frac{1}{4}$

(v) 4, 2, 1, $\frac{1}{2}, \frac{1}{4}$ (vi) $\sqrt{a}$, 1, $\frac{1}{\sqrt{a}}$

19. 3, 27

23. (i) 1023 (ii) $\frac{3280}{2187}$ (iii) $\frac{4095}{2048}$ (iv) 547 (v) $\frac{171}{256}$ (vi) $\frac{5}{4}\left(1-\frac{1}{5^n}\right)$

(vii) $1 - (-2)^n$

24. (i) 8 (ii) 5	25. 6	26. 26.3069
27. 40	28. 6138	29. 2
30. 8 and 2		

2.11 Harmonic Progression (H.P.)

Let us consider the sequence . We observe that in this sequence, the reciprocals of the terms form the sequence 2, 5, 8, 11, ..., which is an A.P. Such a sequence is called a *harmonic progression*. Thus, the reciprocals of the terms of a harmonic progression form an A.P.

Definition : A sequence $\{a_n\}$ is called a harmonic progression (H.P.) if the sequence d such that .

Obviously, the terms of an H.P. may be determined in a manner similar to that of an A.P. As every H.P corresponds to an A.P, problems relating to H.P are solved with reference to the corresponding A.P.

The following are examples of H.P. :

(i) (1, 2, 3, 4, ... are in A.P.)

(ii) (4, 1, –2, –5, ... are in A.P.)

(iii) (a, $a + d$, $a + 2d$, $a + 3d$, ... are in A.P.)

2.12 Harmonic Mean (H.M.)

When three quantities are in H.P., the middle one is called the *harmonic mean* (H.M.) between the other two. Thus, if H be the harmonic mean between a and b, then a, H, b are in H.P. and consequently are in A.P.

$\Rightarrow \quad \frac{2}{H} = \frac{a+b}{ab}$

$\Rightarrow \quad H = \frac{2ab}{a+b}$

Hence, the harmonic mean between a and b is $\frac{2ab}{a+b}$.

2.13 Relation Between A.M., G.M. and H.M. of Two Unequal Quantities

To show that (i) A.M., G.M. and H.M. are in G.P.

(ii) A.M. > G.M. > H.M.

Let A, G and H be respectively the arithmetic mean, geometric mean and harmonic mean between two unequal positive real numbers a and b.

Then, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$.

(i) Now, $A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b}$

$= ab$

$= G^2$

$\Rightarrow \quad \frac{G}{A} = \frac{H}{G}$

$\Rightarrow$ A, G, H are in G.P.

(ii) Again, $A - G = \frac{a+b}{2} - \sqrt{ab}$

$= \frac{1}{2}\left(a + b - 2\sqrt{ab}\right)$

$= \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ [$\because$ a and b are positive and unequal]

$\Rightarrow \quad A - G > 0$

$\Rightarrow \quad A > G$ .... (1)

Also, $A \times H = G^2$

$\Rightarrow \quad \frac{G}{H} = \frac{A}{G} > 1$ [$\because$ $A > G$]

$\Rightarrow \quad \frac{G}{H} > 1$

$\Rightarrow \quad G > H$ .... (2)

Combining (1) and (2), we get

$A > G > H$

Note. It may be noted that the above inequality reduces to an equality (i.e. A = G = H) when $a = b$.

Example 30. Find the 6th term of the H.P. 4, 2, $\frac{4}{3}$, ...

Solution : Let a be the first term and d be the common difference of the A.P. corresponding to the given H.P. Then,

$$a = \frac{1}{4} \text{ and } d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$\therefore$ 6th term of the corresponding AP

$$= a + 5d$$

$$= \frac{1}{4} + 5 \times \frac{1}{4} = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$$

$\therefore$ 6th term of the given H.P. = $\frac{2}{3}$

Example 31. Find the 15th term of the H.P. whose 2nd term is 2 and 31st term is $\frac{4}{31}$.

Solution : Let a be the first term and d be the common difference of the A.P. corresponding to the given H.P.

Then by question, we have

$$a + d = \frac{1}{2} \quad \text{..} (1)$$

and $$a + 30d = \frac{31}{4} \quad \text{..} (2)$$

Solving (1) and (2), we get

$$a = \frac{1}{4} \text{ and } d = \frac{1}{4}$$

$\therefore$ 15th term of the A.P. $= a + 14d$

$$= \frac{1}{4} + 14 \times \frac{1}{4} = \frac{15}{4}$$

Hence, 15th term of the H.P. = $\frac{4}{15}$.

Example 32. If a, b, c are in H.P., show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H.P.

Solution : a, b, c are in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \dfrac{a+b+c}{a}, \dfrac{a+b+c}{b}, \dfrac{a+b+c}{c}$ are in A.P.

$\Rightarrow 1+\dfrac{b+c}{a}, 1+\dfrac{c+a}{b}, 1+\dfrac{a+b}{c}$ are in A.P.

$\Rightarrow \dfrac{b+c}{a}, \dfrac{c+a}{b}, \dfrac{a+b}{c}$ are in A.P.

$\Rightarrow \dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P.

Remark : In the above example, we have used the following facts :

(i) If each term of an A.P. is multiplied by a non-zero constant, the resulting sequence is still an A.P..

(ii) If a constant is added to each term of an A.P., the resulting sequence is still an A.P..

Example 33. Insert three harmonic means between 1 and $\dfrac{1}{9}$.

Solution : Let x_1, x_2, x_3 be the three harmonic means between 1 and $\dfrac{1}{9}$. Then $1, x_1, x_2, x_3, \dfrac{1}{9}$ are in H.P. Therefore $1, \dfrac{1}{x_1}, \dfrac{1}{x_2}, \dfrac{1}{x_3}, 9$ are in A.P. Obviously, 1 is the first term and 9 is the 5^{th} term of the A.P. If d be the common difference of the A.P., then

$$9 = 1 + 4d$$

$$\Rightarrow 4d = 8$$

$$\Rightarrow d = 2$$

$\therefore \quad \dfrac{1}{x_1} = 1 + 2 = 3, \quad \dfrac{1}{x_2} = 3 + 2 = 5$ and $\dfrac{1}{x_3} = 5 + 2 = 7.$

$\therefore \quad x_1 = \dfrac{1}{3}, x_2 = \dfrac{1}{5}$ and $x_3 = \dfrac{1}{7}$

Hence the required means are $\dfrac{1}{3}, \dfrac{1}{5}$ and $\dfrac{1}{7}$

Example 34. If $a^x = b^y = c^z = k$ and a, b, c are in G.P., show that x, y, z are in H.P.

Solution : We have $a^x = b^y = c^z = k$ (say)

$$\Rightarrow a = k^{\frac{1}{x}}, b = k^{\frac{1}{y}}, c = k^{\frac{1}{z}}$$

Since a, b, c are in G.P., therefore

$$\frac{b}{a} = \frac{c}{b}$$

$$\Rightarrow \frac{k^{\frac{1}{y}}}{k^{\frac{1}{x}}} = \frac{k^{\frac{1}{z}}}{k^{\frac{1}{y}}}$$

$$\Rightarrow k^{\frac{1}{y}-\frac{1}{x}} = k^{\frac{1}{z}-\frac{1}{y}}$$

$$\Rightarrow \frac{1}{y}-\frac{1}{x} = \frac{1}{z}-\frac{1}{y}$$

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\Rightarrow x, y, z$ are in H.P.

EXERCISE 2.4

1. Find the specified term of each of the following H.P. :
   (i) 10^{th} term of $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}$,.................
   (ii) 5^{th} term of $\frac{3}{4}, 1, \frac{3}{2}$,......................
   (iii) 6^{th} term of $3, 1\frac{1}{2}, 1$, ...
   (iv) 20^{th} term of $1, 1\frac{3}{5}, 4$, ...
   (v) n^{th} term of $11\frac{2}{3}, 8\frac{3}{4}, 7, 5\frac{5}{6}$,..............
2. Find the H.P. whose
   (i) 1^{st} term is $3\frac{1}{8}$ and 4^{th} term is $1\frac{7}{13}$.
   (ii) 4^{th} term is $\frac{1}{12}$ and 14^{th} term is $\frac{1}{42}$.
   (iii) 7^{th} term is $\frac{2}{5}$ and 17^{th} term is $\frac{2}{25}$.
3. Find the 19^{th} term of the H.P. whose 5^{th} and 10^{th} terms are $-\frac{36}{151}$ and $-\frac{4}{35}$ respectively.
4. Insert (i) two harmonic means between $\frac{1}{3}$ and $\frac{1}{81}$
   (ii) three harmonic means between $2\frac{2}{5}$ and 12
   (iii) four harmonic means between 1 and 6
   (iv) three harmonic means between a and b.

5. If the p^{th} term of an H.P. be q and the q^{th} term be p, prove that

   (i) $(p+q)^{th}$ term is $\dfrac{pq}{p+q}$

   (ii) n^{th} term is $\dfrac{pq}{n}$

   (iii) $(pq)^{th}$ term is 1.

6. If the p^{th}, q^{th} and r^{th} terms of an H.P. be a, b, c respectively, show that $(q-r)\,bc + (r-p)\,ca + (p-q)\,ab = 0$.

7. If a^2, b^2, c^2 are in A.P., prove that $b+c$, $c+a$, $a+b$ are in H.P.

8. If a, b, c be in A.P. and p, q, r be in H.P., show that $\dfrac{a+c}{bq} = \dfrac{p+r}{pr}$.

9. If $\dfrac{a+b}{2}$, b, $\dfrac{b+c}{2}$ be in H.P., show that a, b, c are in G.P.

10. If a, b, c are in H.P., show that $\dfrac{1}{a}+\dfrac{1}{b+c}$, $\dfrac{1}{b}+\dfrac{1}{c+a}$, $\dfrac{1}{c}+\dfrac{1}{a+b}$ are also in H.P.

11. If a, b, c be in A.P., b, c, d in G.P. and c, d, e in H.P., prove that a, c, e are in G.P.

12. If a, b, c be in G.P., show that $\log_a x$, $\log_b x$, $\log_c x$ are in H.P.

13. If a, b, c are in A.P. and b, c, d are in H.P., prove that $ad = bc$.

14. If x_1, x_2, x_3, ..., x_n are in H.P., show that $x_1x_2 + x_2x_3 +$ .... $+ x_{n-1}x_n =$ $(n-1)x_1x_n$.

15. If a, b, c, d are in H.P., prove that $a + d > b + c$.

16. The G.M. and H.M. between two numbers are 9 and $\dfrac{27}{5}$ respectively. Find the numbers.

17. If A.M. and G.M. of two positive numbers are 12 and 6 respectively, find their H.M.

18. If $\dfrac{b+c-a}{a}$, $\dfrac{c+a-b}{b}$, $\dfrac{a+b-c}{c}$ be in A.P., show that a, b, c are in H.P.

19. Which term of the H.P. 1, $\dfrac{8}{5}$, 4, ... is $-\dfrac{1}{8}$?

20. If A be the A.M. and H be the H.M. between a and b, prove that

$$\frac{a-\mathrm{A}}{a-\mathrm{H}} \times \frac{b-\mathrm{A}}{b-\mathrm{H}} = \frac{\mathrm{A}}{\mathrm{H}}.$$

ANSWER

1. (i) $\frac{1}{28}$ (ii) 0 (iii) $\frac{1}{2}$ (iv) $-\frac{8}{49}$ (v) $\frac{35}{n+2}$

2. (i) $3\frac{1}{8}, 2\frac{14}{43}, 1\frac{23}{27}, 1\frac{7}{13}, \ldots\ldots\ldots\ldots$

   (ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots\ldots\ldots\ldots$

   (iii) $-\frac{2}{7}, -\frac{2}{5}, -\frac{2}{3}, \ldots\ldots\ldots\ldots$

3. $-\frac{1}{19}$

4. (i) $\frac{1}{29}, \frac{1}{55}$ (ii) 3, 4, 6 (iii) $\frac{6}{5}, \frac{3}{2}$, 2, 3 (iv) $\frac{4ab}{a+3b}, \frac{2ab}{a+b}, \frac{4ab}{3a+b}$

16. 3 and 27 17. 1 19. 25^{th} term.

2.14 Sum of Some Important Finite Series

(i) Sum of the first n natural numbers.

Let $S = 1 + 2 + 3 + \ldots\ldots\ldots\ldots + n$.

This being a series in A.P. with first term = 1 and n^{th} term = n, we have

$$S = \frac{n}{2}(1+n) \quad \text{[Using the formula } S = \frac{n}{2}(a+l)]$$

$$= \frac{n(n+1)}{2}.$$

$$\therefore \quad \mathbf{1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}}.$$

(ii) Sum of the squares of the first n natural numbers.

Let $S = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2$.

We have, $r^3 - (r-1)^3 = 3r^2 - 3r + 1$.

Putting $r = 1, 2, 3, \ldots\ldots\ldots\ldots, n$ successively, we get

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$

$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$

$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$n^3 - (n-1)^3 = 3.n^2 - 3.n + 1$$

By adding, we get

$$n^3 = 3\,(1^2 + 2^2 + 3^2 + \text{..........} + n^2) - 3\,(1 + 2 + 3 + \text{..........} + n) + n$$

$$\Rightarrow n^3 = 3S - 3.\frac{n(n+1)}{2} + n$$

$$\Rightarrow 3S = n^3 + \frac{3n(n+1)}{2} - n$$

$$= \frac{n}{2}\{2n^2 + 3\,(n+1) - 2\}$$

$$= \frac{n}{2}(2n^2 + 3n + 1)$$

$$= \frac{n\,(n+1)\,(2n+1)}{2}$$

$$\therefore\ S = \frac{n\,(n+1)\,(2n+1)}{6}$$

$\therefore$ $\mathbf{1^2 + 2^2 + 3^2 + \text{..................} + n^2 = \frac{n\,(n+1)(2n+1)}{6}}$.

(iii) Sum of the cubes of the first n natural numbers.

Let $S = 1^3 + 2^3 + 3^3 + \text{......................} + n^3$.

We have, $(r + 1)^2 - (r - 1)^2 = 4r$

$\Rightarrow r^2.(r + 1)^2 - (r - 1)^2.r^2 = 4r^3$ (multiplying both sides by r^2)

Putting $r = 1, 2, 3, \text{...................}, n$ successively, we get

$$1^2.2^2 - 0^2.1^2 = 4.1^3$$

$$2^2.3^2 - 1^2.2^2 = 4.2^3$$

$$3^2.4^2 - 2^2.3^2 = 4.3^3$$

....

$$n^2\,(n + 1)^2 - (n - 1)^2 n^2 = 4.n^3$$

By adding, we get

$$n^2\,(n + 1)^2 = 4(1^3 + 2^3 + 3^3 + \text{....................} + n^3) = 4S$$

$$\Rightarrow S = \frac{n^2(n+1)^2}{4}$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2$$

$\therefore$ $\mathbf{1^3 + 2^3 + 3^3 + \text{...................} + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2}$.

2.15 The "$\sum$" Notation

The greek letter $\sum$ (sigma) is often used to denote the sum of a number of similar terms. For example, the sum 1 + 2 + 3 + ... + n may be denoted by $\sum_{r=1}^{n} r$ or simply by $\sum n$.

Thus, $\sum n = 1+2+3+ .... + n = \frac{n\,(n+1)}{2}$

$$\sum n^2 = 1^2+2^2+3^2+ .... + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

$$\sum n^3 = 1^3+2^3+3^3+ .... + n^3 = \left\{\frac{n\,(n+1)}{2}\right\}^2 .$$

Also, it is easy to see that $\sum (an+bn^2) = a\sum n + b\sum n^2$.

Example 35. Find the sum to n terms of the following series :

(i) 2 + 4 + 6 + ...

(ii) 1 + 3 + 5 + ...

Solution : (i) Here, t_n = n^{th} term of the series = $2n$

$\therefore$ Reqd. sum $= \sum t_n = \sum 2n = 2\sum n$

$= 2.\frac{n(n+1)}{2} = n(n+1)$

(ii) Here, t_n = n^{th} term of the series = $2n - 1$

$\therefore$ Reqd. sum $= \sum t_n = \sum (2n-1)$

$= 2\sum n - \sum 1$

$= 2.\frac{n\,(n+1)}{2} - n$ [$\because$ $\sum 1$ = 1+1+1+.....to n terms = n]

$= n\,(n+1) - n$

$= n^2$

Alternatively,

Reqd. sum $= \frac{n}{2}[2.1+(n-1).2]$ [$\because$ the given series being in A.P. with first term 1 and C.D. 2]

$= \frac{n}{2}(2+2n-2)$

$= n^2$

Example 36. Sum the series 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms.

Solution : Clearly, the n^{th} term of the series is $n(n + 1)(n + 2)$.

$\therefore$ The reqd. sum $= \sum n(n+1)(n+2)$

$$= \sum n(n^2+3n+2)$$

$$= \sum (n^3+3n^2+2n)$$

$$= \sum n^3 + 3\sum n^2 + 2\sum n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 3.\frac{n(n+1)(2n+1)}{6} + 2.\frac{3(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{4}[n(n+1)+2(2n+1)+4)$$

$$= \frac{n(n+1)}{4}.(n^2+5n+6)$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

Example 37. Sum the series 1 + (1+2) + (1+2+3) + ...to n terms.

Solution : Here, n^{th} term = 1 + 2 + 3 + ... + n = $\frac{n(n+1)}{2}$

$\therefore$ Reqd. sum $= \sum \frac{n(n+1)}{2}$

$$= \frac{1}{2}\sum (n^2+n)$$

$$= \frac{1}{2}\left[\sum n^2 + \sum n\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}.(2n+1+3)$$

$$= \frac{n(n+1)}{12}.2(n+2)$$

$$= \frac{1}{6}n(n+1)(n+2)$$

EXERCISE 2.5

Find the sum of the following series to *n* terms :

1. $1+4+7+10+$ ....
2. $1^2+3^2+5^3+$ ....
3. $1.2+2.3+3.4+$ ....
4. $1.3+3.5+5.7+$ ....
5. $1.2^2+2.3^2+3.4^2+$ ....
6. $1.3^2+2.42+3.5^2+$ ....
7. $1.2^2+3.5^2+5.8^2+$ ....
8. $1+(1+3)+(1+3+5)+$ ....
9. $1^3+3^3+5^3+$ ....
10. $1.1+2.3+3.5+$ ....
11. $1.3+2.5+3.7+$.........................
12. $1.2.4+2.3.7+3.4.10+$ ....
13. $1+\frac{1+2}{2}+\frac{1+2+3}{3}+$....................
14. $2.4+4.6+6.8+$ ....
15. $1.3.5+3.5.7+5.7.9+$ ....
16. $1.4.7+2.5.8+3.6.9+$ ....
17. $1.5.9+2.6.10+3.7.111+$ ....
18. $1.2.4.+2.3.5+3.4.6+$ ....
19. $1.3.4+2.4.5+3.5.6+$ ....
20. $1.3.5+2.4.6+3.5.7+$ ....
21. $1^2+\frac{1^2+2^2}{2}+\frac{1^2+2^2+3^2}{3}+$...................
22. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+$ ....
23. $\frac{1^3}{1}+\frac{1^3+2^3}{1+2}+\frac{1^3+2^3+3^3}{1+2+3}+$.............
24. $\frac{1^2}{2}+\frac{1^2+2^2}{3}+\frac{1^2+2^2+3^2}{4}+$.................
25. $2+(2+5)+(2+5+8)+$ ....

ANSWER

1. $\frac{1}{2}n(3n-1)$
2. $\frac{1}{3}n(4n^2-1)$
3. $\frac{1}{3}n(n+1)(n+2)$
4. $\frac{1}{3}n(4n^2+6n-1)$
5. $\frac{1}{12}n(n+1)(n+2)(3n+5)$
6. $\frac{1}{12}n(n+1)(3n^2+19n+32)$
7. $\frac{1}{2}n(9n^3+4n^2-4n+1)$
8. $\frac{1}{6}n(n+1)(2n+1)$
9. $n^2(2n^2-1)$
10. $\frac{1}{6}n(n+1)(4n-1)$
11. $\frac{1}{6}n(n+1)(4n+5)$
12. $\frac{1}{12}n(n+1)(9n^2+25n+4)$
13. $\frac{1}{4}n(n+3)$
14. $\frac{4}{3}n(n+1)(n+2)$
15. $n(2n^3+8n^2+7n-2)$
16. $\frac{1}{4}n(n+1)(n+6)(n+7)$
17. $\frac{1}{4}n(n+1)(n+8)(n+9)$
18. $\frac{1}{12}n(n+1)(n+2)(3n+13)$
19. $\frac{1}{12}n(n+1)(3n^2+23n+46)$
20. $\frac{1}{4}n(n+1)(n+4)(n+5)$
21. $\frac{1}{36}n(4n^2+15n+17)$
22. $\frac{1}{12}n(n+1)^2(n+2)$
23. $\frac{1}{6}n(n+1)(n+2)$
24. $\frac{1}{36}n(n+1)(4n+5)$
25. $\frac{1}{2}n(n+1)^2$

CHAPTER 3
MATHEMATICAL INDUCTION

3.1 Introduction

The word *'induction'* means inferring a general statement from the validity of particular cases. Mathematical induction is a method which is frequently used to establish the truth of a mathematical proposition or statement involving natural numbers and it provides a logical proof to generalise a mathematical result concerning natural numbers.

A mathematical proposition involving natural numbers is generally denoted by P(n), ($n \in$N). If we substitute $n = 4$ in the statement P(n), the particular statement so obtained is denoted by P(4). For example, if P(n) is the statement "$n(n+1)$ is even", then P(4) is the statement "4(4+1) is even" i.e., "20 is even".

The proof of a mathematical proposition by the method of mathematical induction is based on a principle known as the *Principle of Mathematical Induction* or simply *Principle of Induction*, which is given in the next section.

3.2 Principle of Mathematical Induction

It states that if P(n) be a mathematical proposition such that

(i) P(1) is true, and

(ii) P($k + 1$) is true whenever P(k) is true, where k is an arbitrary value of n (i.e. P(k) is true $\Rightarrow$ P(k+1) is true,)

then P(n) is true $\forall n \in$N.

Thus, the method of mathematical induction requires the following three basic steps in proving a mathematical proposition or theorem.

(i) Verification : Verify the validity of P(n) for $n = 1$ (least value of n) i.e. verify that P(1) is true.

(ii) Inductive property : Assume P(n) is true for $n = k$ (i.e. for some value k of n), and then deduce that P($k + 1$) is also true.

(iii) Conclusion : P(n) is true $\forall n \in N$.

Example 1. Let P(n) be the statement

"$n^2 + 3n$ is divisible by 4".

Is (i) P(1) true ?

(ii) P(2) false ?

Solution : When $n = 1$, $n^2 + 3n = 1^2 + 3 \times 1 = 4$, which is divisible by 4.

When $n = 2$, $n^2 + 3n = 2^2 + 3 \times 2 = 10$, which is not divisible by 4.

$\therefore$ P(1) is true, and P(2) is false.

Example 2. If P(n) is the statement

"$n^2 > 10$", $n \in N$,

prove that whenever P(k) is true, P(k+1) is also true.

Solution : Given P(k) is true i.e. $k^2 > 10$, we are to show that

P(k+1) is true i.e. $(k+1)^2 > 10$.

Since $k + 1 > k$, and also since $k^2 > 10$,

$\therefore \quad (k + 1)^2 > k^2 > 10 \Rightarrow (k + 1)^2 > 10$

Hence, P($k + 1$) is true.

Example 3. Prove by mathematical induction that

$$1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2} \quad \forall n \in N.$$

Solution : Let P(n) be the proposition

$$1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}.$$

(i) ***Verification :*** When $n = 1$,

L.S. = 1 and R.S. = $\frac{1(1+1)}{2} = \frac{1.2}{2} = 1.$

$\therefore$ L.S. = R.S.

Thus P(1) is true.

(ii) ***Inductive property :*** Let P(k) be true, k being some value of n.

Then

$$1 + 2 + 3 + \dots\dots\dots + k = \frac{k(k+1)}{2}$$

Adding $(k+1)$ to both sides, we get

$$1 + 2 + 3 + \dots\dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)\{(k+1)+1\}}{2}$$

$\therefore$ P(k + 1) is true

Thus, P(k + 1) is true whenever P(k) is true.

(iii) ***Conclusion :*** Hence, by the principle of mathematical induction, the proposition P(n) is true $\forall n \in$ N.

i.e. $1 + 2 + 3 + \dots\dots\dots\dots + n = \frac{n(n+1)}{2} \forall n \in \text{N}.$

Example 4. Prove by mathematical induction that

$$1^2 + 2^2 + 3^2 + \dots\dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \in \text{N}.$$

Solution : Let P(n) be the proposition

$$1^2 + 2^2 + 3^2 + \dots\dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(i) For $n = 1$, L.S. = $1^2 = 1$ and R.S. = $\frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1.$

$\therefore$ L.S. = R.S. Thus P(1) is true.

(ii) Assume that P(k) is true for some value k of n. Then

$$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k + 1)^2$ to both sides, we get

$$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)\{k(2k+1)+6(k+1)\}}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(2k^2+4k+3k+6)}{6}$$

$$= \frac{(k+1)\{2k(k+2)+3(k+2)\}}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}$$

$\therefore$ P(k + 1) is true.

Thus, P(k) is true $\Rightarrow$ P(k + 1) is true.

(iii) By the principle of mathematical induction, P(n) is true $\forall n \in$N

i.e. $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \forall n \in \text{N}$.

Example 5. Prove by mathematical induction that

$$1.2 + 2.3 + 3.4 + ... + n(n + 1) = \frac{n(n+1)(n+2)}{3}, n \in \text{N}.$$

Solution : Let P(n) be the proposition

$$1.2 + 2.3 + 3.4 + ... + n(n + 1) = \frac{n(n+1)(n+2)}{3}.$$

When $n = 1$, L.S. = 1.2 = 2 and R.S. = $\frac{1(1+1)(1+2)}{3} = \frac{1.2.3}{3} = 2$.

$\therefore$ L.S. = R.S.

Thus, P(1) is true.

Let us assume that P(k) is true for some value k of n. Then,

$$1.2 + 2.3 + 3.4 + \text{............} + k\,(k+1) = \frac{k(k+1)(k+2)}{3}$$

Adding $(k+1)\,(k+1+1)$ i.e. $(k+1)(k+2)$ to both sides, we get

$$1.2 + 2.3 + 3.4 + \text{..........} + k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \frac{(k+1)\{(k+1)+1\}\{(k+1)+2\}}{3}$$

$\therefore$ P(k+ 1) is true.

Thus, P(k) is true $\Rightarrow$ P($k+1$) is also true.

Hence, by the principle of mathematical induction, P(n) is true $\forall n \in$ N

i.e. $1.2 + 2.3 + 3.4 + \text{..........} + n(n+1) = \dfrac{n(n+1)(n+2)}{3} \forall n \in \text{N}$.

Example 6. Prove by mathematical induction that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{..............} + \frac{1}{n(n+1)} = \frac{n}{n+1}, \forall n \in \text{N}.$$

Solution: Let P(n) be the proposition

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{..............} + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

When $n = 1$, L.S. = $\dfrac{1}{1.2} = \dfrac{1}{2}$ and R.S. = $\dfrac{1}{1+1} = \dfrac{1}{2}$.

$\therefore$ L.S. =R.S.

Thus, P(1) is true.

Let us assume that P(k) is true for some value k of n

i.e. $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \text{.........} + \dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$.

Adding $\dfrac{1}{(k+1)(k+1+1)}$ i.e. $\dfrac{1}{(k+1)(k+2)}$ to both sides, we get

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \text{........} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{(k+1)+1}$$

$\therefore$ P(k + 1) is true.

Thus, P(k + 1) is true whenever P(k) is true.

$\therefore$ P(n) is true for all $n \in$N

i.e. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1} \forall n \in \text{N}$.

Example 7. Prove by the principle of mathematical induction that

$5^{2n}-1$ is divisible by 8 $\forall n \in$N.

Solution : Let P(n) be the statement '$5^{2n}-1$ is divisible by 8'.

For $n = 1$, $5^{2n}-1 = 5^2-1 = 24$ which is divisible by 8.

$\therefore$ P(1) is true.

Assume that P(k) is true for some value k of n

i.e. $5^{2k}-1$ is divisible by 8.

Writing k + 1 in place of k, we have

$5^{2(k+1)}-1 = 5^{2k+2}-1$

$= 5^{2k}.5^2-1$

$= 25.5^{2k}-25+24$

$= 25(5^{2k}-1)+24$, which is obviously divisible by 8 as $5^{2k}-1$ is divisible by 8.

Thus, P(k + 1) is true whenever P(k) is true.

$\therefore$ P(n) is true $\forall$ $n \in$N

i.e. $5^{2n}-1$ is divisible by 8 $\forall$ $n \in$N.

EXERCISE 3.1

1. If P(n) is the statement '$n^2 + 2$ is a multiple of 3', show that P(2) is true and P(3) is false.

2. If P(n) is the statement '$5^{2n} + 3n - 1$ is divisible by 9', state P(2).
   Is (i) P(1) true ?
   (ii) P(3) false ?

3. If P(n) is the statement '$n^2 + n > 15$' and if P(k) is true, prove that P(k +1) is true.

4. Prove by mathematical induction that $\forall\ n \in$ N,

   (i) $2 + 4 + 6 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 2n = n(n + 1)$.

   (ii) $1 + 4 + 7 + \ldots\ldots\ldots\ldots\ldots\ldots + (3n - 2) = \dfrac{n(3n-1)}{2}$.

   (iii) $1 + 6 + 11 + \ldots\ldots\ldots\ldots\ldots + (5n - 4) = \dfrac{1}{2}n(5n-3)$.

   (iv) $1 + 5 + 12 + 22 + \ldots\ldots\ldots + \dfrac{n(3n-1)}{2} = \dfrac{n^2(n+1)}{3}$.

   (v) $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots\ldots + n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$.

   (vi) $1.3 + 3.5 + 5.7 + \ldots\ldots\ldots\ldots\ldots + (2n - 1)(2n + 1) = \dfrac{n(4n^2+6n-1)}{3}$.

   (vii) $1+\dfrac{1}{2}+\dfrac{1}{2^2}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\dfrac{1}{2^{n-1}}=\dfrac{2^n-1}{2^{n-1}}$.

   (viii) $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\ldots\ldots\ldots\ldots\ldots\ldots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$.

   (ix) $1+\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\ldots\ldots\ldots\ldots\ldots\ldots+\dfrac{1}{3^{n-1}}=\dfrac{3^n-1}{2.3^{n-1}}$.

   (x) $2 + 2^2 + 3^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 2^n = 2(2^n - 1)$.

   (xi) $1.2 + 2.2^2 + 3.2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + n.2^n = (\mathrm{n} - 1)2^{n+1} + 2$.

   (xii) $1.2.3+2.3.4+3.4.5+\ldots\ldots\ldots\ldots\ldots+n(n+1)(n+2)=\dfrac{1}{4}n(n+1)(n+2)(n+3)$.

   (xiii) $1^2+3^2+5^2+\ldots\ldots\ldots\ldots\ldots+(2n-1)^2=\dfrac{1}{3}n(2n-1)(2n+1)$.

   (xiv) $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+\ldots\ldots\ldots\ldots\ldots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1}$.

(xv) $2^n > n$.

(xvi) $3^{2n} - 1$ is divisible by 4.

(xvii) $9^n - 1$ is divisible by 8.

(xviii) $a^n - b^n$ is divisible by $a - b$, $(a \neq b)$.

(xix) $x^n - 1$ is divisible by $x - 1$, $(x \neq 1)$.

(xx) $a^{2n} - b^{2n}$ is divisible by $a + b$, $(a \neq - b)$.

(xxi) $a^{2n+1} + b^{2n+1}$ is divisible by $a + b$, $(a \neq - b)$.

(xxii) $\left(1 + \frac{1}{n}\right)^n \leq n + 1$

ANSWER

2. '630 is divisible by 9'

   (i) Yes (ii) No

CHAPTER 4

BINOMIAL THEOREM

4.1 Introduction

We know that algebraic expressions like $x - zy$, $3x + \frac{1}{x}$, $a - 4x$, $a - \frac{1}{x}$ etc. which have got two terms, are called binomial expressions. We have also learnt how to multiply a binomial by another binomial or a binomial by itself.

Let us consider the binomial $(a + x)$ where $a, x \in R$.

By actual multiplication, we have

$$(a + x)^1 = a + x$$

$$(a + x)^2 = a^2 + 2ax + x^2$$

$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 \text{ and so on.}$$

In the above relations, each of the expressions on the right hand side is called binomial expansion of the binomial $(a + x)$ for the corresponding index i.e. 1, 2 and 3 respectively.

Here, we observe that the expansion of higher powers of $(a + x)$ like $(a + x)^4$, $(a + x)^5$, $(a + x)^6$ etc. become more and more inconvenient. Therefore, we look for a general formula which will help us in finding the expansion of higher powers of a binomial.

In this chapter, we shall discuss a theorem, known as the Binomial Theorem, which gives us the general rule for the expansion of $(a + x)^n$ where n is a positive integer. The more general case when n is any integer or a fraction, is dealt with in higher classes.

4.2 Binomial Theorem (for positive integral index)

Theorem : If a and x be any two real numbers and n be any positive integer, then

$$(a+x)^n = {}^nc_0a^n + {}^nc_1a^{n-1}x + {}^nc_2a^{n-2}x^2 + ... + {}^nc_ra^{n-r}x^r + ... + {}^nc_nx^n$$

Proof : We shall prove this theorem by the method of mathematical induction.

Let P(n) denote the proposition :

$$(a+x)^n = {}^nc_0a^n + {}^nc_1a^{n-1}x + {}^nc_2a^{n-2}x^2 + ... + {}^nc_ra^{n-r}x^r + ... + {}^nc_nx^n$$

When $n = 1$, we have

$$(a+x)^1 = a+x = {}^1c_0a^1 + {}^1c_1x^1 \qquad \left[\because {}^1c_0 = 1,\ {}^1c_1 = 1\right]$$

$\therefore$ P(1) is true.

Now, let us suppose that P(k) is true for some positive integer k, so that

$$(a+x)^k = {}^kc_0a^k + {}^kc_1a^{k-1}x + {}^kc_2a^{k-2}x^2 + ... + {}^kc_ra^{k-r}x^r + ... + {}^kc_kx^k$$

Multiplying both sides by $(a+x)$, we have

$$(a+x)^{k+1}$$

$$= (a+x)\left({}^kc_0a^k + {}^kc_1a^{k-1}x + {}^kc_2a^{k-2}x^2 + ... + {}^kc_ra^{k-r}x^r + ... + {}^kc_kx^k\right)$$

$$= \left({}^kc_0a^{k+1} + {}^kc_1a^kx + {}^kc_2a^{k-1}x^2 + ... + {}^kc_ra^{k-r+1}x^r + ... + {}^kc_kax^k\right) +$$
$$\left({}^kc_0a^kx + {}^kc_1a^{k-1}x^2 + {}^kc_2a^{k-2}x^3 + ... + {}^kc_ra^{k-r}x^{r+1} + ... + {}^kc_kx^{k+1}\right)$$

$$= {}^kc_0a^{k+1} + \left({}^kc_1 + {}^kc_0\right)a^kx + \left({}^kc_2 + {}^kc_1\right)a^{k-1}x^2 + ... + \left({}^kc_r + {}^kc_{r-1}\right)a^{k-r+1}x^r$$
$$+ ..... + {}^kc_kx^{k+1}$$

But, ${}^kc_0 = 1 = {}^{k+1}c_0;\ {}^kc_k = 1 = {}^{k+1}c_{k+1}$ and ${}^kc_r + {}^kc_{r-1} = {}^{k+1}c_r$

$\therefore \quad (a+x)^{k+1}$

$$= {}^{k+1}c_0a^{k+1} + {}^{k+1}c_1a^kx + {}^{k+1}c_2a^{k-1}x^2 + ... + {}^{k+1}c_ra^{k+1-r}x^r + ... + {}^{k+1}c_{k+1}x^{k+1}$$

$\therefore$ P($k+1$) is true

Thus, P(k) is true $\Rightarrow$ P($k+1$) is true

Therefore, by the principle of mathematical induction, P(n) is true $\forall n \in$ N i.e.

$$(a+x)^n = {}^nc_0a^n + {}^nc_1a^{n-1}x + {}^nc_2a^{n-2}x^2 + ... + {}^nc_ra^{n-r}x^r + ... + {}^nc_nx^n$$

Remarks : The expression on the right hand side of the above formula, is called the binomial expansion of $(a+x)^n$ for the positive integral index n and the coefficients ${}^nc_o, {}^nc_1, {}^nc_2, ..., {}^nc_n$ are called binomial coefficients.

Note : In the expansion of $(a + x)^n$, $n \in N$;

(i) The total number of terms is $n + 1$ (one more than the index n).

(ii) The sum of the indices of a and x in each term is equal to the index n.

(iii) The index of a in the first term is the same as that of the binomial $(a+x)$ and thereafter goes on decreasing by 1 in each subsequent term and it becomes 0 in the last term. On the other hand, the index of x in the first term is 0 and it goes on increasing by 1 and finally becomes equal to the index of the binomial.

(iv) The binomial coefficients in the terms equidistant from the beginning and the end are equal. This follows from the fact that ${}^nc_r = {}^nc_{n-r}$

4.3 Pascal's Triangle

The binomial coefficients ${}^nc_0, {}^nc_1, {}^nc_2, ..., {}^nc_n$ in the expansion of $(a + x)^n$ follow a pattern for different values of n.

When $n = 0$, ${}^0c_0 = 1$

When $n = 1$, ${}^1c_0 = 1, {}^1c_1 = 1$

When $n = 2$, ${}^2c_0 = 1, {}^2c_1 = 2, {}^2c_2 = 1$

When $n = 3$, ${}^3c_0 = 1, {}^3c_1 = 3, {}^3c_2 = 3, {}^3c_3 = 1$

When $n = 4$, ${}^4c_0 = 1, {}^4c_1 = 4, {}^4c_2 = 6, {}^4c_3 = 4, {}^4c_4 = 1$

When $n = 5$, ${}^5c_0 = 1, {}^5c_1 = 5, {}^5c_2 = 10, {}^5c_3 = 10, {}^5c_4 = 5, {}^5c_5 = 1$ and so on.

These coefficients can be arranged in the form of a triangle as follows :

1st row $(n = 0)$

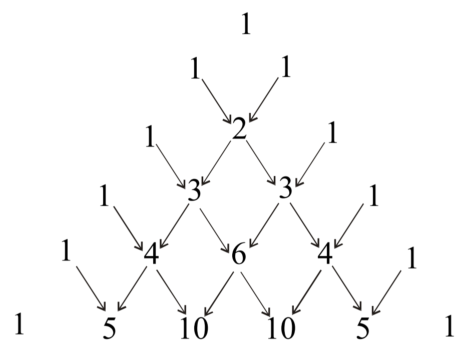
2nd row $(n = 1)$

3rd row $(n = 2)$

4th row $(n = 3)$

5th row $(n = 4)$

6th row $(n = 5)$ and so on.



Pascal's triangle

In the above triangle, it is observed that each entry (except the first and the last which are always 1) is the sum of the nearest two entries in the row immediately above it. Using this fact, we can determine the rows corresponding to $n = 6$, $n = 7$ etc.

This triangle gives us a handy rule for finding the coefficients of a Binomial expansion, especially when the value of n is small. The triangle is due to the celebrated mathematician B. Pascal and is known as the Pascal's triangle.

4.4 Some Simple Deductions

In the binomial expansion,

$$(a+x)^n = {}^nc_0 a^n + {}^nc_1 a^{n-1}x + {}^nc_2 a^{n-2}x^2 + \ldots.. + {}^nc_r a^{n-r}x^r + \ldots\ldots + {}^nc_n x^n$$

(i) If x is replaced by $-x$, we get

$$(a-x)^n = {}^nc_0 a^n - {}^nc_1 a^{n-1}x + {}^nc_2 a^{n-2}x^2 - \ldots\ldots.. + (-1)^r\ {}^nc_r a^{n-r}x^r + \ldots\ldots. + (-1)^n\ {}^nc_n x^n$$

(ii) If $a = 1$, we get

$$(1+x)^n = 1 + {}^nc_1 x + {}^nc_2 x^2 + \ldots\ldots.. + {}^nc_r x^r + \ldots\ldots. + x^n \qquad (\because\ {}^nc_0 = 1 = {}^nc_n)$$

(iii) If $a = 1$ and x is replaced by $-x$, we get

$$(1-x)^n = 1 - {}^nc_1 x + {}^nc_2 x^2 - \ldots\ldots\ldots. + (-1)^r\,{}^nc_r x^r + \ldots\ldots\ldots. + (-1)^n x^n$$

4.5 General Term in the Expansion of $(a + x)^n$

In the expansion of $(a + x)^n$, if we denote the first term by T_1, the second term by T_2, the third term by T_3 and so on, then

$$T_1 = T_{0+1} = {}^nc_0 a^n = {}^nc_0 a^n x^o$$

$$T_2 = T_{1+1} = {}^nc_1 a^{n-1} x^1$$

$$T_3 = T_{2+1} = {}^nc_2 a^{n-2} x^2$$

$$T_4 = T_{3+1} = {}^nc_3 a^{n-3} x^3 \quad \text{and so on.}$$

In general,

$$T_{r+1} = {}^nc_r a^{n-r} x^r$$

By putting $r = 0, 1, 2, 3, \ldots\ldots\ldots.., n$ in the $(r + 1)^{th}$ term T_{r+1}, we get all the terms of the expansion. So, this $(r + 1)^{th}$ term is called the general term in the expansion of $(a + x)^n$.

Thus, in the expansion of $(a + x)^n$,

the general term = $\mathbf{T_{r+1} = {}^nc_r a^{n-r} x^r}$

Note : In the expansion of $(a - x)^n$, **the general term =** $\mathbf{T_{r+1} = (-1)^r \, {}^nc_r a^{n-r} x^r}$

4.6 Middle Term(s) in the Expansion of $(a + x)^n$

We know that the number of terms in the binomial expansion of $(a + x)^n$ is $n + 1$. So, if n is even, the number of terms $(n + 1)$ is odd, so that there will be only one middle term. If however, n is odd, the number of terms$(n + 1)$ is even and hence, there will be two middle terms.

Case I : n is even.

Here, the number of terms is $(n + 1)$ which is odd. There is only one middle term which is obviously the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term.

Hence, $T_{\frac{n}{2}+1}$ is the middle term.

Case II : n is odd.

Here, the number of terms $(n + 1)$ being even, there are two middle terms, which are the $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term.

Thus, $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$ are the two middle terms.

Example 1. Expand $(3 + 2x)^5$ by using Binomial Theorem.

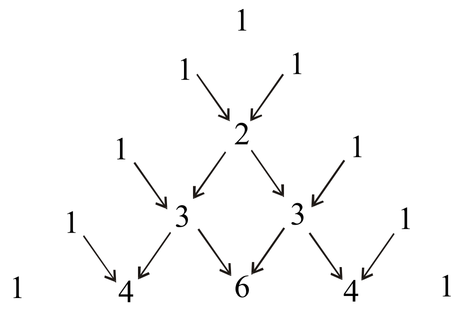
Solution : $(3+2x)^5 = {}^5c_0 3^5 + {}^5c_1 3^4(2x) + {}^5c_2 3^3(2x)^2 + {}^5c_3 3^2(2x)^3 + {}^5c_4 3(2x)^4 + {}^5c_5(2x)^5$

$= 1.243 + 5.81.2x + 10.27.4x^2 + 10.9.8x^3 + 5.3.16x^4 + 1.32x^5$

$= 243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5$

Example 2. Expand $(2x + y)^4$ by using Pascal's triangle.

Solution : The coefficients in the expansion of $(2x + y)^4$ are given by the 5th row of the following Pascal's triangle :



$\therefore \quad (2x+y)^4 = 1(2x)^4 + 4(2x)^3 y + 6(2x)^2 y^2 + 4(2x)y^3 + 1y^4$

$= 1.16x^4 + 4.8x^3 y + 6.4x^2 y^2 + 4.2xy^3 + 1.y^4$

$= 16x^4 + 32x^3 y + 24x^2 y^2 + 8xy^3 + y^4$

Example 3. Write the first four terms in the expansion of $(x+2y)^{10}$.

Solution : The first four terms in the expansion of $(x+2y)^{10}$ are

${}^{10}c_0 x^{10},\ {}^{10}c_1 x^9(2y),\ {}^{10}c_2 x^8 (2y)^2$ and ${}^{10}c_3 x^7 (2y)^3$

i.e. $1.x^{10},\ 10.x^9.2y,\ 45.x^8.4y^2$ and $120.x^7.8y^3$

i.e. $x^{10},\ 20x^9 y,\ 180x^8 y^2$ and $960x^7 y^3$

$\left(\text{Here } {}^{10}c_0 = 1,\ {}^{10}c_1 = 10,\ {}^{10}c_2 = \frac{9\times 10}{1\times 2} = 45 \text{ and } {}^{10}c_3 = \frac{8\times 9\times 10}{1\times 2\times 3} = 120\right)$

Example 4. Find the 6th term in the expansion of $\left(x-\frac{1}{x}\right)^8$

Solution : In the expansion of $\left(x-\frac{1}{x}\right)^8$, we have

$T_{r+1} = (-1)^r\, {}^8c_r x^{8-r}\left(\frac{1}{x}\right)^r$

$\therefore \quad T_6 = T_{5+1} = (-1)^5\ {}^8c_5 x^3\left(\frac{1}{x}\right)^5$

$= -1.56.x^3.\frac{1}{x^5}$

$= \frac{-56}{x^2}$

Example 5. Find the coefficient of x^6 in the expansion of $(x+3)^8$.

Solution : Let T_{r+1} be the term containing x^6 in the expansion.

Now, $T_{r+1} = {}^8c_r x^{8-r}(3)^r$

Since T_{r+1} contains x^6, we have

$8 - r = 6$

$\Rightarrow \ r = 2$

$\therefore \quad T_{2+1}$ i.e. T_3 contains x^6 and

hence the coefficient of $x^6 = {}^8c_2.3^2$

$= 28.9 = 152$

Example 6. Find the term independent of x in the expansion of $\left(2x-\frac{1}{3x^2}\right)^9$.

Solution : Let T_{r+1} be the term independent of x in the expansion of $\left(2x-\frac{1}{3x^2}\right)^9$

Now, $T_{r+1} = (-1)^r\ {}^9c_r(2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$

$= (-1)^r\ {}^9c_r(2)^{9-r}\left(\frac{1}{3}\right)^r(x)^{9-3r}$

Since T_{r+1} is independent of x, the index of x in this term is 0.

$\therefore\quad 9-3r=0$

$\Rightarrow\quad r=3$

$\therefore\quad T_{3+1}$ i.e. T_4 is the term independent of x

and $T_4 = (-1)^3\ {}^9c_3(2)^6\left(\frac{1}{3}\right)^3 = -1.\frac{7\times8\times9}{1\times2\times3}.64.\frac{1}{27}$

$= -84.64.\frac{1}{27}$

$= -\frac{1792}{27}$

Example 7. Find the middle term in the expansion of $\left(x+\frac{1}{x}\right)^8$.

Solution : Here, the index 8 is even.

$\therefore$ There is only one middle term and it is the $\left(\frac{8}{2}+1\right)^{\text{th}}$ term i.e. T_5.

Now, $T_5 = T_{4+1} = {}^8c_4x^{8-4}\left(\frac{1}{x}\right)^4$

$= \frac{5\times6\times7\times8}{1\times2\times3\times4}.x^4\frac{1}{x^4}$

$= 70$

Example 8. Find the middle terms in the expansion of $(2x-y)^7$.

Solution : Here, the index 7 is odd.

$\therefore$ There are two middle terms. They are the $\left(\frac{7+1}{2}\right)^{\text{th}}$ and $\left(\frac{7+3}{2}\right)^{\text{th}}$ terms i.e. T_4 and T_5.

Now, $T_4 = T_{3+1} = (-1)^3\ {}^7c_3 (2x)^{7-3} y^3$

$= -\frac{5\times6\times7}{1\times2\times3}.16x^4y^3$

$= -560\,x^4y^3$

and, $T_5 = T_{4+1} = (-1)^4\ {}^7c_4 (2x)^{7-4} y^4$

$= {}^7c_4.8x^3y^4$

$= \frac{5\times6\times7}{1\times2\times3}.8x^3y^4$

$= 280\,x^3y^4.$

4.7 Properties of Binomial Coefficients

(i) The sum of all the binomial coefficients is 2^n

i.e. ${}^nc_0 + {}^nc_1 + {}^nc_2 + ...+ {}^nc_n = 2^n$

(ii) The sum of the binomial coefficients of odd terms is equal to that of even terms, each being equal to 2^{n-1}.

i.e. ${}^nc_0 + {}^nc_2 + {}^nc_4 + ... = {}^nc_1 + {}^nc_3 + {}^nc_5 + ... = 2^{n-1}$

Proof : (i) we have

$(1+x)^n = {}^nc_0 + {}^nc_1x + {}^nc_2x^2 + ...+ {}^nc_nx^n$

Putting $x = 1$, we have

$$(1+1)^n = {}^nc_0 + {}^nc_1 + {}^nc_2 + ...+ {}^nc_n$$

$$\Rightarrow\quad {}^nc_0 + {}^nc_1 + {}^nc_2 + ...+ {}^nc_n = 2^n$$

(ii) we have

$(1-x)^n = {}^nc_0 - {}^nc_1x + {}^nc_2x^2 - ....+(-1)^n\ {}^nc_nx^n$

Putting $x = 1$, we have

$$(1-1)^n = {}^nc_0 - {}^nc_1 + {}^nc_2 - {}^nc_3 + ...$$

$$\Rightarrow\quad \left({}^nc_0 + {}^nc_2 + {}^nc_4 + ...\right) - \left({}^nc_1 + {}^nc_3 + {}^nc_5 + ...\right) = 0$$

$$\Rightarrow\quad {}^nc_0 + {}^nc_2 + {}^nc_4 + ... = {}^nc_1 + {}^nc_3 + {}^nc_5 + ... = k\ (\text{say})$$

Then, ${}^nc_0 + {}^nc_1 + {}^nc_2 + {}^nc_3 + ...+ {}^nc_n = 2k$

$$\Rightarrow \quad 2^n = 2k$$

$$\Rightarrow \quad k = \frac{2^n}{2} = 2^{n-1}$$

$$\therefore \quad {}^n c_0 + {}^n c_2 + {}^n c_4 + \ldots\ldots\ldots\ldots = {}^n c_1 + {}^n c_3 + {}^n c_5 + \ldots\ldots\ldots\ldots = 2^{n-1}$$

Example 9. Show that

(i) $^n c_1 + 2.{}^n c_2 + 3.{}^n c_3 + \ldots\ldots\ldots\ldots\ldots\ldots + n.{}^n c_n = n.2^{n-1}$

(ii) $^n c_0 + 2.\,{}^n c_1 + 3.\,{}^n c_2 + \ldots\ldots\ldots\ldots\ldots + (n+1).\,{}^n c_n = (n+2).2^{n-1}$

Solution : (i) We have

$$r.{}^n c_r = r.\frac{\lfloor n}{\lfloor r \, \lfloor n-r}$$

$$= \frac{r.n\lfloor n-1}{r.\lfloor r-1 \, \lfloor (n-1)-(r-1)}$$

$$= n.\frac{\lfloor n-1}{\lfloor r-1 \, \lfloor (n-1)-(r-1)}$$

$$= n.{}^{n-1}c_{r-1}$$

$$\therefore \quad {}^n c_1 = n.{}^{n-1}c_0, \; 2.{}^n c_2 = n.{}^{n-1}c_1, \; 3.{}^n c_3 = n.{}^{n-1}c_2 \text{ etc.}$$

Now, $^n c_1 + 2.{}^n c_2 + 3.{}^n c_3 + \ldots\ldots\ldots\ldots + n.{}^n c_n$

$$= n.{}^{n-1}c_0 + n.{}^{n-1}c_1 + n.{}^{n-1}c_2 + \ldots\ldots\ldots\ldots\ldots\ldots + n.{}^{n-1}c_{n-1}$$

$$= n[{}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + \ldots\ldots\ldots\ldots + {}^{n-1}c_{n-1}]$$

$$= n.2^{n-1}$$

(ii) $^n c_0 + 2.{}^n c_1 + 3.{}^n c_2 + \ldots\ldots\ldots\ldots\ldots\ldots + (n+1).{}^n c_n$

$$= \left({}^n c_0 + {}^n c_1 + {}^n c_2 + \ldots\ldots\ldots + {}^n c_n\right) + \left({}^n c_1 + 2.{}^n c_2 + 3.{}^n c_3 + \ldots\ldots\ldots + n.{}^n c_n\right)$$

$$= 2^n + n.2^{n-1} \qquad \text{[using the result (1)]}$$

$$= 2.2^{n-1} + n.2^{n-1}$$

$$= (n+2).2^{n-1}$$

4.8 Simple Applications

In this section, we give some simple applications of Binomial Theorem, as illustrated in the following examples.

Example 10. Compute $(98)^4$ using Binomial Theorem.

Solution : We have

$$(98)^4 = (100-2)^4$$
$$= {}^4c_0.100^4 - {}^4c_1.100^3.2 + {}^4c_2.100^2.2^2 - {}^4c_3.100^1.2^3 + {}^4c_4.2^4$$
$$= 100^4 - 4.100^3.2 + 6.100^2.4 - 4.100.8 + 16$$
$$= 100000000 - 8000000 + 240000 - 3200 + 16$$
$$= 100240016 - 8003200$$
$$= 92236816$$

Example 11. Using Binomial Theorem, prove that $6^n - 5n - 1$ is divisible by 25 for $n \in$ N.

Solution : We have

$$6^n = (1+5)^n$$
$$= 1 + {}^nc_1.5 + {}^nc_2.5^2 + {}^nc_3.5^3 + \cdots\cdots + 5^n$$
$$= 1 + 5n + {}^nc_2.5^2 + {}^nc_3.5^3 + \cdots\cdots + 5^n \qquad (\because {}^nc_1 = n)$$
$$\Rightarrow 6^n - 5n - 1 = {}^nc_2.5^2 + {}^nc_3.5^3 + \cdots\cdots + 5^n$$
$$= 5^2\left({}^nc_2 + {}^nc_3.5 + \cdots\cdots + 5^{n-2}\right)$$
$$= 25 \times \text{(an integer)}$$

$\therefore$ $6^n - 5n - 1$ is divisible by 25 for $n \in$ N.

EXERCISE 4.1

1. How many terms are there in the expansion of

   (i) $(2a - x)^7$ (ii) $(x + 4y)^{10}$ (iii) $(1 - 3x)^{15}$ (iv) $(2 + 5y)^{20}$?

2. Using Pascal's triangle, expand

   (i) $(x + y)^5$ (ii) $(a + 2x)^4$ (iii) $(3 - 2x)^5$

3. Expand the following using Binomial Theorem :

   (i) $(1+x)^5$ (ii) $(a - 2x)^6$ (iii) $(x + 2y)^4$

   (iv) $\left(x + \frac{1}{x}\right)^7$ (v) $(2x - y)^5$ (vi) $\left(\frac{x}{a} + \frac{a}{x}\right)^6$

4. Find the first four terms in the expansion of $(x-2y)^{10}$.
5. Find the 6^{th} term in the expansion of $(1+x)^{10}$.
6. Find the 7^{th} term in the expansion of $\left(x-\frac{1}{x^2}\right)^8$.
7. Find the 11^{th} term in the expansion of $(x+2y)^{15}$.
8. Find the term containing x^6 in the expansion of $(1+x^2)^6$.
9. Find the term containing x^9 in the expansion of $\left(x^2+\frac{1}{x}\right)^9$.
10. Find the coefficient of x^4 in the expansion of $\left(x-\frac{1}{x}\right)^{10}$.
11. Find the term independent of x in the expansion of

    (i) $\left(x^2+\frac{1}{x}\right)^9$ (ii) $\left(x-\frac{1}{x^2}\right)^{12}$ (iii) $\left(x+\frac{2}{x^2}\right)^{15}$
12. Find the middle term in the expansion of

    (i) $(1+x)^6$ (ii) $(x-y)^8$ (iii) $(2x+3y)^{10}$
13. Find the middle terms in the expansion of

    (i) $(x+y)^7$ (ii) $\left(x+\frac{1}{x}\right)^9$ (iii) $(x-2y)^{11}$
14. If c_r denotes the binomial coefficient ${}^nc_r$, prove that

    (i) $c_0+2c_1+4c_2+\ldots\ldots\ldots\ldots\ldots+2^n c_n=3^n$

    (ii) $c_0+3c_1+5c_2+\ldots\ldots\ldots\ldots\ldots\ldots+(2n+1)\,c_n=(n+1)\,2^n$

    (iii) $c_0+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+\ldots\ldots\ldots\ldots\ldots\ldots+n\frac{c_n}{c_{n-1}}=\frac{n(n+1)}{2}$

    (iv) $c_1-2c_2+3c_3-\ldots\ldots\ldots\ldots\ldots\ldots\ldots+(-1)^{n-1}nc_n=0$
15. Evaluate the following by using Binomial Theorem

    (i) 99^4 (ii) 102^5 (iii) 1001^3
16. Show that $\left(2+\sqrt{5}\right)^4+\left(2-\sqrt{5}\right)^4$ is rational.
17. Using Binomial Theorem, prove that 4^n-3n-1 is divisible by 9 for $n\in N$.
18. Using Binomial Theorem, prove that $2^{3n}-7n$ $(n\in N)$ always leaves the remainder 1 when divided by 49.

ANSWER

1. (i) 8 (ii) 11 (iii) 16 (iv) 21

2. (i) $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

   (ii) $a^4+8a^3x+24a^2x^2+32ax^3+16x^4$

   (iii) $243-810x+1080x^2-720x^3+240x^4-32x^5$

3. (i) $1+5x+10x^2+10x^3+5x^4+x^5$

   (ii) $a^6-12a^5x+60a^4x^2-160a^3x^3+240a^2x^4-192ax^5+64x^6$

   (iii) $x^4+8x^3y+24x^2y^2+32xy^3+16y^4$

   (iv) $x^7+7x^5+21x^3+35x+\frac{35}{x}+\frac{21}{x^3}+\frac{7}{x^5}+\frac{1}{x^7}$

   (v) $32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$

   (vi) $\frac{x^6}{a^6}+\frac{6x^4}{a^4}+\frac{15x^2}{a^2}+20+\frac{15a^2}{x^2}+\frac{6a^4}{x^4}+\frac{a^6}{x^6}$

4. $x^{10},\ -20x^9y,\ 180x^8y^2,\ -960x^7y^3$

5. $252x^5$ 6. $\frac{28}{x^{10}}$ 7. ${}^{15}c_{10}2^{10}x^5y^{10}$ 8. $20x^6$ 9. $84x^9$

10. – 120 11. (i) 84 (ii) 495 (iii) ${}^{15}c_5 2^5$

12. (i) $T_4=20x^3$ (ii) $T_5=70x^4y^4$ (iii) $T_6={}^{10}c_5 2^5 3^5 x^5 y^5$

13. (i) $T_4=35x^4y^3,\ T_5=35x^3y^4$

    (ii) $T_5=126x,\ T_6=\frac{126}{x}$

    (iii) $T_6=-{}^{11}c_5 2^5 x^6 y^5,\ T_7={}^{11}c_6 2^6 x^5 y^6$

15. (i) 96059601 (ii) 11040808032 (iii) 1003003001

CHAPTER 5
MATRICES

5.1 Introduction

In 1850, an English mathematician James Joseph Sylvester (1814-1897) used rectangular arrangements of numbers for storing information. He gave the name matrix to such a rectangular arrangement. Later on, other mathematicians recognised how conveniently matrices can be used to write numerical data, system of a large number of equations in several unknowns etc. in a compact form. Among others who have contributed to the development of matrix theory, mention may be made of Arther Cayley (1821-1895), William Rowan Hamilton (1805-1865), Charles Hermite (1822-1901), F.G. Frobenius (1849-1917) and M.E.C. Jordan (1838-1922).

Having originated as mere stores of information, matrices have now found applications not only in Mathematics but also in Physics, Chemistry, Biology, Sociology, Economics, Engineering, Psychology, Statistics etc.

In this chapter, we shall study elementary properties of matrices and basic laws of matrix algebra.

5.2 Definition of a Matrix

Suppose Chaoba has 7 books. We can express this information by the symbol [7] with the understanding that the number written inside the pair of brackets, is in reference to the number of books Chaoba possesses.

However, if Chaoba has 7 books as also 4 pens, we convey the information by [7 4] with the understanding that the first number represents the number of books and the second, the number of pens Chaoba has.

Next suppose Chaoba has 7 books and 4 pens while his friend Ali has 9 books and 5 pens. This information can be displayed in a tabular form as follows :

	Books	Pens
Chaoba	7	4
Ali	9	5

We can further shorten the display as below :

$$\begin{bmatrix} 7 & 4 \\ 9 & 5 \end{bmatrix} \begin{matrix} \rightarrow \text{1st row} \\ \rightarrow \text{2nd row} \end{matrix}$$

$$\begin{matrix} \downarrow & \\ \text{1st} & \text{2nd} \\ \text{column} & \text{column} \end{matrix}$$

Implied in this display, are the following assumptions :

(i) The entries in the first row represent the objects (books and pens) that Chaoba possesses.

(ii) The entries in the second row represent the objects that Ali possesses.

(iii) The entries in the first column represent the number of books.

(iv) The entries in the second column represent the number of pens.

Thus, the entry in the second row and the first column, represents the number of books Ali possesses. Each entry in the display may be interpreted similarly.

We now see how arrangement of numbers in rows and columns can be used conveniently to represent given information. Such an arrangement of numbers is called a *matrix*. A formal definition of matrix is given below :

Definition : A rectangular array of mn numbers, arranged in m rows and n columns, enclosed in a pair of brackets, is called a matrix of order $m \times n$ (read as m by n) or an $m \times n$ matrix.

Each number in a matrix is called an *element* or *entry* of the matrix.

In the matrix

$$\begin{bmatrix} 7 & 4 \\ 9 & 5 \end{bmatrix}$$

there are two rows and two columns. Hence its order is 2×2.

The matrix [7 4] has 1 row and 2 columns so that it is of order 1×2. And the matrix $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$ has 2 rows and 1 column so that its order is 2×1. While specifying the order of a matrix, the number of rows is always written first followed by the mark × and then by the number of columns.

We use capital letters to denote matrices and small letters with two suffixes to denote elements in specific positions.

For instance, a matrix A with *m* rows and *n* columns may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

A notation of this type is known as a double suffix notation. The suffixes *i* and *j* in the element a_{ij}, indicate the row and column in which the element occurs. Here a_{ij} may be called the (*i, j*)th element of the matrix. The above matrix may also be denoted shortly by the symbol $[a_{ij}]$.

As in the case of vectors, in the study of matrices also, numbers are usually referred to as scalars.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

Clearly A is of order 2×3 and

a_{11} = the element in the 1st row and 1st column = (1,1)th element = 2
a_{12} = the element in the 1st row and 2nd column = (1,2)th element = 1
a_{13} = the element in the 1st row and 3rd column = (1,3)th element = 3
a_{21} = the element in the 2nd row and 1st column = (2,1)th element = 4
a_{22} = the element in the 2nd row and 2nd column = (2,2)th element = 7
a_{23} = the element in the 2nd row and 3rd column = (2,3)th element = 5

Example 1. Consider the following information about the result of a monthly test in class X in a school :

	Mathematics	*Science*	*Social Science*
Pass	46	39	44
Fail	4	11	6

Express this information in the form of a matrix and specify its order. What does the entry in the 2nd row and 3rd column represent ?

Solution : The given information may be stored in the following matrix :

$$A = \begin{bmatrix} 46 & 39 & 44 \\ 4 & 11 & 6 \end{bmatrix}$$

This matrix has 2 rows and 3 columns and so it is of order 2×3.

The entry 6 in the 2nd row and 3rd column represents the number of students failed in Social Science.

5.3 Types of Matrices

Rectangular matrix : Any $m \times n$ matrix (where m is not necessarily equal to n) is called a *rectangular matrix.*

For example, $\begin{bmatrix} 2 & 4 & 3 \\ -3 & 5 & 1 \end{bmatrix}$ is a rectangular matrix.

In fact, any $m \times n$ matrix is a rectangular matrix whether $m=n$ or $m \neq n$. However, some authors call an $m \times n$ matrix rectangular only when $m \neq n$.

Square matrix : Any $n \times n$ matrix is called a square matrix of order n or an n-rowed square matrix.

Thus, in a square matrix, the number of rows is equal to the number of columns.

For examples,

$\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ -3 & 4 & 1 \end{bmatrix}$ are square matrices of order 2 and 3 respectively.

Row matrix : A matrix having a single row is called a row matrix.

For example, [1 2 5] is a row matrix.

Every matrix of order $1 \times n$, for some $n \in \mathrm{N}$, is a *row matrix.*

Column matrix : A matrix having a single column is called a column matrix.

For example,

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a column matrix.

The order of a column matrix is $n \times 1$, for some $n \in \mathrm{N}$.

For any matrix A=$[a_{ij}]$, the elements a_{ii} for all possible values of i, are called the *diagonal elements* and the line along which they lie, is called the *principal diagonal* or the *leading diagonal* of the matrix.

Diagonal matrix : A square matrix is said to be a diagonal matrix if all its elements other than the diagonal elements are zero.

For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ are diagonal matrices of order 3.

Scalar matrix : A diagonal matrix whose diagonal elements are all equal, is called a scalar matrix.

Thus, $$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ are scalar matrices.

Identity matrix or Unit matrix : A diagonal matrix whose diagonal elements are unity, is called an *identity matrix* or a *unit matrix.*

For examples,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ are the unit matrices of order 2 and 3 respectively.

Null matrix or Zero matrix : A matrix all of whose elements are zero, is called a *null matrix* or a *zero matrix.*

Examples are

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ etc.}$$

Any matrix having at least one non-zero entry is called a *non-zero matrix.*

Triangular matrices : A square matrix all of whose elements *below* the principal diagonal are zero, is called an *upper triangular matrix*.

A square matrix all of whose elements *above* the principal diagonal are zero, is called a *lower triangular matrix.*

The matrices $\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 5 & 2 \end{bmatrix}$

are examples of upper triangular matrix and lower triangular matrix respectively.

5.4 Equality of Matrices

Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.

Thus, the matrices A=$[a_{ij}]$ and B=$[b_{ij}]$ are equal if

(i) they have the same number of rows and the same number of columns and

(ii) $a_{ij}=b_{ij}$, for all admissible values of i and j.

For example, $\begin{bmatrix} a & b \\ x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & y \end{bmatrix}$ if and only if

$a=2$, $b=5$, $x=1$ and $5=y$.

Example 2. Let

$$A = [a_{ij}] = \begin{bmatrix} 2 & -5 & 6 \\ 7 & 4 & -2 \end{bmatrix}$$

Find the order of A. Also find a_{13} and a_{22}. Does a_{31} exist ?

Solution : There are 2 rows and 3 columns in the given matrix A.

Hence its order is 2×3.

Here,

a_{13}= the element in the 1st row and 3rd column = 6

and a_{22} = the element in the 2nd row and 2nd column = 4

But a_{31} i.e. the element in the 3rd row and 1st column does not exist as there is no 3rd row.

Example 3. A matrix A has 6 elements. Find all possible orders A can have. What if A has 5 elements ?

Solution : Let us find all ordered pairs of positive integers, the product of whose elements is 6. The possible ordered pairs are

(1, 6), (6, 1) (2, 3) and (3, 2).

Hence the possible orders of A are

1×6, 6×1, 2×3 and 3×2.

In case A has 5 elements, the possible orders are 1×5 and 5×1 (as 1 and 5 are the only factors of the prime number 5).

Example 4. Find the 3×3 square matrix $[a_{ij}]$ where $a_{ij} = \frac{i+j}{2}$.

Solution : Here

$$a_{11} = \frac{1+1}{2} = 1,\ a_{12} = \frac{1+2}{2} = \frac{3}{2},\quad a_{13} = \frac{1+3}{2} = 2$$

$$a_{21} = \frac{2+1}{2} = \frac{3}{2},\ a_{22} = \frac{2+2}{2} = 2,\quad a_{23} = \frac{2+3}{2} = \frac{5}{2}$$

$$a_{31} = \frac{3+1}{2} = 2,\ a_{32} = \frac{3+2}{2} = \frac{5}{2},\quad a_{33} = \frac{3+3}{2} = 3$$

$\therefore$ the required matrix $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ \frac{3}{2} & 2 & \frac{5}{2} \\ 2 & \frac{5}{2} & 3 \end{bmatrix}$$

Example 5. Find *x, y, z, p* if $\begin{bmatrix} x & 2y+z \\ y+2p & z+2x \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Solution : Equating the corresponding elements of the two given equal matrices, we get

$x = 3$ .... (1)

$2y + z = 6$ ....(2)

$y + 2p = 2$ ....(3)

and $z + 2x = 4$ ....(4)

From (4), $z = 4 - 2x$

$= 4 - 6$, using (1)

$= -2$

From (2), $y = \frac{6-z}{2}$

$= \frac{6+2}{2} \quad (\because\ z = -2)$

$= 4$

From (3), $p = \frac{1}{2}(2-y)$

$= \frac{1}{2}(2-4) \quad (\because\ y = 4)$

$= -1$

Thus, $x = 3,\ y = 4,\ z = -2$ and $p = -1$.

EXERCISE 5.1

1. Consider the following information regarding the number of boys and girls reading in class X in three schools A, B and C.

Schools	No. of boys	No. of girls
A	26	31
B	35	27
C	42	22

Represent the information in the form of a 3×2 matrix. What does the element in the second row and first column represent ?

2. Let

$$[a_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & -6 \\ -7 & 3 & 8 \\ -2 & -2 & 1 \end{bmatrix}$$

Find the order of the matrix. Find a_{23}, a_{32}, a_{33}, a_{41} and a_{42}. Does a_{24} exist ?

3. State the type of the following matrix :

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4. A matrix A has 24 elements. Find all possible orders A may have. If A has 7 elements, what are the possible types A can be of ?

5. Find the 3×3 matrix $[a_{ij}]$ where

(i) $a_{ij} = 2(i - j)$ (ii) $a_{ij} = 2i + j$ (iii) $a_{ij} = \dfrac{i + 2j}{3}$

(iv) $a_{ij} = \dfrac{(i - j)^2}{2}$ (v) $a_{ij} = 2i - 3j + 1$ (vi) $a_{ij} = (-1)^{i+j}$

6. Is the following equality possible for any values of *x, y, z* ?

$$\begin{bmatrix} x + y & y + z \\ x + 2z & y + 2x \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

7. Find *a, b, c, d* when

$$\begin{bmatrix} a + b & b + c \\ a + c & b + d \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

8. Find the values of *x, y, z* if

(i) $\begin{bmatrix} x + y + z \\ y + z \\ z + x \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ (ii) $\begin{bmatrix} x - y & 2x - y \\ 2z & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} x + y \\ y + z \\ z + x \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$

ANSWER

1. $\begin{bmatrix} 26 & 31 \\ 35 & 27 \\ 42 & 22 \end{bmatrix}$, the number of boys reading class X in school B.

2. 4×3, a_{23}=–6, a_{32}=3, a_{33}=8, a_{41}=–2, a_{42}=–2; a_{24} does not exist.

3. (i) square matrix (ii) diagonal matrix (iii) upper triangular matrix
(iv) column matrix (v) row matrix (vi) zero matrix
(vii) lower triangular matrix (viii) diagonal matrix (ix) scalar matrix.

4. 1×24, 2×12, 3×8, 4×6, 6×4, 8×3, 12×2, 24×1; a row matrix or a column matrix.

5. (i) $\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & \frac{5}{3} & \frac{7}{3} \\ \frac{4}{3} & 2 & \frac{8}{3} \\ \frac{5}{3} & \frac{7}{3} & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & \frac{1}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & \frac{1}{2} & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & -3 & -6 \\ 2 & -1 & -4 \\ 4 & 1 & -2 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

6. No.

7. $a = 2, b = 3, c = 1, d = 1$

8. (i) $x = 1, y = 2, z = 3$ (ii) $x = 3, y = 2, z = \frac{3}{2}$ (ii) $x = 3, y = 2, z = 5$

5.5 Operations on Matrices

(a) *Addition of matrices*

Let A and B be two matrices of the same order. Then the sum of A and B, denoted by A+B is defined as the matrix, each element of which is the sum of the corresponding elements of A and B.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

The sum of two matrices of different orders is undefined. In fact, the sum A+B of two matrices A and B is defined only when they have the same number of rows and the same number of columns. In such a case, the matrices A and B are said to be **conformable for addition.**

(b) *Multiplication of a matrix by a scalar*

For $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ let us calculate A + A. Clearly

$$A + A = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

It is natural to denote A+A by 2A. We observe that each element of 2A is 2 times the corresponding element of A.

Here we define the product kA for any scalar k (integral or not) and for any matrix A as follows :

If k is a scalar and A, a matrix, then the product kA is defined as the matrix obtained on multiplying each element of A by k.

For example, if $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, then for any scalar k,

$$k\,A = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}$$

This operation is often called the multiplication of a matrix by a scalar.

If $k=1$, clearly kA=A and if $k = -1$, then kA= –A where –A is the matrix each element of which is the negative of the corresponding element of A. The matrix –A is called the negative of A.

(c) *Subtraction of matrices*

If A and B are two matrices of the same order, then the difference A–B is defined to be the matrix A+(–B).

Every element of A–B is obtained by subtracting the corresponding element of B from the corresponding element of A.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ then

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \end{bmatrix}$$

Since addition of matrices is based directly on the addition of their elements which are numbers and since addition of numbers is commutative and associative, therefore it follows that

$$A + B = B + A$$

and $$(A + B) + C = A + (B + C)$$

whenever A, B, C are matrices of the same order.

Thus, matrix addition is commutative as well as associative.

Further, the distributive law

$$k(A + B) = kA + kB$$

holds when k is any scalar and A, B are matrices of the same order.

If A is any matrix and O is the null matrix of the same order, then it is easy to see that

$$A + O = A \text{ and } A - A = O$$

Example 6. Verify the associative law of matrix addition for the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}.$$

Solution : We have

$$A+B=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}4 & 1\\2 & 0\end{bmatrix}=\begin{bmatrix}1+4 & 2+1\\3+2 & 4+0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 3\\5 & 4\end{bmatrix}$$

$$\therefore \quad (A+B)+C=\begin{bmatrix}5 & 3\\5 & 4\end{bmatrix}+\begin{bmatrix}1 & 5\\2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & 8\\7 & 7\end{bmatrix} \text{ .... (1)}$$

Again, $$B+C=\begin{bmatrix}4 & 1\\2 & 0\end{bmatrix}+\begin{bmatrix}1 & 5\\2 & 3\end{bmatrix}=\begin{bmatrix}5 & 6\\4 & 3\end{bmatrix}$$

$$\therefore \quad A+(B+C)=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}5 & 6\\4 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & 8\\7 & 7\end{bmatrix} \text{ ....(2)}$$

From (1) and (2), we obtain

$$(A+B)+C= A+(B+C)$$

Hence verified.

Example 7. Find a matrix C such that A+B+C=O (O denotes a null matrix) where

$$A=\begin{bmatrix}2 & 1\\3 & 4\end{bmatrix} \text{ and } B=\begin{bmatrix}-1 & 3\\2 & -5\end{bmatrix}.$$

Solution : Here, $$A+B=\begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}+\begin{bmatrix}-1 & 3\\2 & -5\end{bmatrix}$$

$$=\begin{bmatrix}2-1 & 1+3\\3+2 & 4-5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 4\\5 & -1\end{bmatrix}$$

By the given condition

$$A+B+C=O$$

$$\therefore \quad C=-(A+B)+O$$

$$=-(A+B)$$

$$=\begin{bmatrix}-1 & -4\\ -5 & 1\end{bmatrix}$$

Example 8. If $A=\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & -2\\ 1 & 0 & -3\end{bmatrix}$, find 3A+2B.

Solution : Here,

$$3A=3\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & 4\end{bmatrix}=\begin{bmatrix}3 & 6 & 9\\ 0 & -9 & 12\end{bmatrix}$$

and $$2B=2\begin{bmatrix}2 & 1 & -2\\ 1 & 0 & -3\end{bmatrix}=\begin{bmatrix}4 & 2 & -4\\ 2 & 0 & -6\end{bmatrix}$$

$$\therefore \quad 3A+2B=\begin{bmatrix}3 & 6 & 9\\ 0 & -9 & 12\end{bmatrix}+\begin{bmatrix}4 & 2 & -4\\ 2 & 0 & -6\end{bmatrix}$$

$$=\begin{bmatrix}3+4 & 6+2 & 9-4\\ 0+2 & -9+0 & 12-6\end{bmatrix}$$

$$=\begin{bmatrix}7 & 8 & 5\\ 2 & -9 & 6\end{bmatrix}$$

Example 9. Find the matrices A and B, if

$$A+B=\begin{bmatrix}7 & 3\\ 2 & 1\end{bmatrix} \text{ and } A-B=\begin{bmatrix}1 & 5\\ 0 & -7\end{bmatrix}$$

Solution : Here,

$$A+B=\begin{bmatrix}7 & 3\\ 2 & 1\end{bmatrix} \text{ .... } (1)$$

and $$A-B=\begin{bmatrix}1 & 5\\ 0 & -7\end{bmatrix} \text{ ....} (2)$$

Adding (1) and (2)

$$2A = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 2 & -6 \end{bmatrix}$$

$\therefore$ $$A = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix}$$

Then $B = A - A + B$

$= A - (A - B)$

$$= \begin{bmatrix} 4 & 4 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 5 \\ 0 & -7 \end{bmatrix} \quad \text{[using (2)]}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$$

Example 10. Find x and y such that

$$x\begin{bmatrix} 2 \\ -3 \end{bmatrix} + y\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

Solution : We have

$$x\begin{bmatrix} 2 \\ -3 \end{bmatrix} + y\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 2x \\ -3x \end{bmatrix} + \begin{bmatrix} -3y \\ 2y \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 2x - 3y \\ -3x + 2y \end{bmatrix} = \begin{bmatrix} 13 \\ -12 \end{bmatrix}$$

Comparing elements in corresponding positions, we get

$2x - 3y = 13$ .... (1)

and $-3x + 2y = -12$ .... (2)

Adding (1) and (2),

$-x - y = 1$ .... (3)

Subtracting (2) from (1),

$5x - 5y = 25$

$\Rightarrow$ $x - y = 5$ .... (4)

Solving (3) and (4), we obtain $x = 2$, $y = -3$.

(d) *Multiplication of matrices*

Let A and B be two matrices such that the number of columns of A is equal to the number of rows of B. Then the matrices A and B are said to be *conformable* for the product AB. And the product AB is defined only when A and B are conformable for this product.

As an example, let us take

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Here, the number of columns of A=3= the number of rows of B. So A and B are conformable for the product AB.

The product AB is defined as the matrix

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + b_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

To get the product AB, we proceed by taking the 1st row of A and the 1st column of B and obtain the product of each element in the row with the corresponding element in the column. The sum of the products so obtained, is called the inner product of the row and column under consideration. We take this inner product as the element in the 1st row and the 1st column i.e. (1,1)th element of the product AB. Next we form the inner product of the first row of A and the second column of B and take this product as the (1,2)th element of the product AB. In general, we find the inner product of the ith row of A and the jth column of B and take it as the (i,j)th element of the product AB. In this way, we compute each element of the product AB.

We refer to this process of finding the product AB as "row by column rule".

Here, observe that A is 3×3 matrix and B, 3×2 matrix and the product AB is 3×2 matrix (where 3 indicates the number of rows of A and 2 indicates the number of columns of B). In general, if A is $m \times n$ matrix and B, $n \times p$ matrix, then the product AB is an $m \times p$ matrix.

Two given matrices A and B conformable for the product AB, may not be conformable for the product BA and even if they are, $AB \neq BA$ in general. This means that matrix multiplication is *not commutative.*

For example, if $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4+10 \\ 4+15 \end{bmatrix} = \begin{bmatrix} 14 \\ 19 \end{bmatrix}$$ whereas the product

BA is undefined.

Again, if $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4+10 & 1+2 \\ 4+15 & 1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 \\ 19 & 4 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4+1 & 8+3 \\ 5+1 & 10+3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 11 \\ 6 & 13 \end{bmatrix}$$

so that $AB \neq BA$.

When A is a square matrix, we can form the product A×A which we denote by A^2. Inductively we can form the products denoted by A^3, A^4 etc.

Again if A, B, C are matrices such that A and B are conformable for the product AB and B and C are conformable for the product BC, then it can be shown that

$$(AB)C = A(BC).$$

It means that matrix multiplication is *associative.*

It may also be shown that the *distributive law,*

$$A(B+C) = AB + AC$$

holds for matrices A, B, C provided they are conformable for the products and the sum.

Example 11. If A is any square matrix of order 3 and I, the unit matrix of the same order, show that AI=IA=A.

Solution : Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then

$$AI = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1.1 + b_1.0 + c_1.0 & a_1.0 + b_1.1 + c_1.0 & a_1.0 + b_1.0 + c_1.1 \\ a_2.1 + b_2.0 + c_2.0 & a_2.0 + b_2.1 + c_2.0 & a_2.0 + b_2.0 + c_2.1 \\ a_3.1 + b_3.0 + c_3.0 & a_3.0 + b_3.1 + c_3.0 & a_3.0 + b_3.0 + c_3.1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = A$$

and

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + 0 + 0 & b_1 + 0 + 0 & c_1 + 0 + 0 \\ 0 + a_2 + 0 & 0 + b_2 + 0 & 0 + c_2 + 0 \\ 0 + 0 + a_3 & 0 + 0 + b_3 & 0 + 0 + c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = A$$

Thus, $AI = IA = A$.

Example 12. Form the products AB and BA when

(i) $A = [3 \quad 2 \quad 1]$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

Solution :

(i) $AB = [3 \ 2 \ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= [3\times1+ 2\times2+1\times3]$

$= [10]$

$$BA = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \ 2 \ 1]$$

$$= \begin{bmatrix} 1\times3 & 1\times2 & 1\times1 \\ 2\times3 & 2\times2 & 2\times1 \\ 3\times3 & 3\times2 & 3\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{bmatrix}$$

(ii) $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & 2+0 & 0+0 \\ 0+0 & 0+2 & 0+1 \\ 1+0 & 2+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+2+0 \\ 0+0+1 & 0+2+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Example 13. If $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$, show that

$$A^2 - B^2 \neq (A+B)(A-B).$$

Solution : We have

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-2 \\ -1+2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 9+4 & 12-4 \\ 3-1 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix}$$

$$\therefore \quad A^2 - B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 13 & 8 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -13 & -9 \\ -1 & -2 \end{bmatrix}$$

Again, $$A + B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 0 & -3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -2 & -1 \end{bmatrix}$$

$$\therefore \quad (A + B)(A - B) = \begin{bmatrix} 4 & 5 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8-10 & -12-5 \\ 0+6 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & -17 \\ 6 & 3 \end{bmatrix}.$$

Thus, $A^2 - B^2 \neq (A + B)\ (A - B)$.

Example 14. Find a 2×2 matrix A such that

$$A\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 3 & 8 \end{bmatrix}.$$

Solution : Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 3 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a+b & 3a-2b \\ 2c+d & 3c-2d \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 3 & 8 \end{bmatrix}$$

$\therefore \quad 2a + b = -11$ .... (1)

$3a - 2b = -6$ .... (2)

$2c + d = 3$ .... (3)

and $3c - 2d = 8$ .... (4)

From (i) and (2), we get a=–4, b= –3

And from (3) and (4), we get $c=2$, $d=-1$

Hence $$A=\begin{bmatrix}-4 & -3\\ 2 & -1\end{bmatrix}.$$

Example 15. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$, prove that $A^2-4A=I$, where I is the unit matrix of order 2.

Solution : Here,

$$A^2=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+4 & 2+6\\ 2+6 & 4+9\end{bmatrix}=\begin{bmatrix}5 & 8\\ 8 & 13\end{bmatrix}$$

and $$4A=4\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}=\begin{bmatrix}4 & 8\\ 8 & 12\end{bmatrix}$$

$$\therefore \quad A^2-4A=\begin{bmatrix}5 & 8\\ 8 & 13\end{bmatrix}-\begin{bmatrix}4 & 8\\ 8 & 12\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= I.$$

Example 16. Solve the matrix equation :

$$\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}7\\ 4\end{bmatrix}.$$

Solution : We have

$$\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2x+3y\\ 3x-2y\end{bmatrix}.$$

$\therefore$ the equation becomes

$$\begin{bmatrix} 2x+3y \\ 3x-2y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Equating elements in corresponding positions, we obtain

$2x+3y=7$ .... (1)

and $3x-2y=4$ .... (2)

Solving (1) and (2), we find that x=2 and y=1.

Example 17. Show that for the non-zero matrices

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix},$$

(i) AB=O and (ii) $BA \neq O$.

Solution : We have

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2-2 & 3-3 \\ -2+2 & -3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2-3 & -2+3 \\ 2-3 & -2+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \neq O.$$

Remark : There are non-zero matrices A and B (as in the above example) such that AB=O.

EXERCISE 5.2

1. Find 2A – 3B when

(i) $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$

(iv) $A = \begin{bmatrix} \frac{3}{2} & 2 & \frac{5}{2} \\ 2 & \frac{5}{2} & 3 \\ \frac{7}{2} & 4 & \frac{9}{2} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & \frac{4}{3} & \frac{5}{3} \\ \frac{5}{3} & 2 & \frac{7}{3} \end{bmatrix}$

2. Find a matrix C such that

(i) $2A + B + C = 0$ where $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ -3 & 3 \end{bmatrix}$

(ii) $2A - 3B + C = 0$ where $A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

(iii) $A + B + 2C = 0$ where $A = \begin{bmatrix} -3 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -2 & -5 \end{bmatrix}$

(iv) $3A - 2B + C = 0$ where $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$

(v) $2A - 5B + 3C = 0$ where $A = \begin{bmatrix} 1 & -3 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \\ 6 & 3 \end{bmatrix}$

3. Find the matrices A and B if

(i) $A + B = \begin{bmatrix} 3 & 4 \\ 0 & 7 \end{bmatrix}$ and $A - B = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$

(ii) $2A + B = \begin{bmatrix} 3 & 3 \\ 1 & 8 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 3 & 0 \\ -4 & 7 \end{bmatrix}$

(iii) $2A + 3B = \begin{bmatrix} -4 & 8 & 7 \\ 7 & 0 & 2 \end{bmatrix}$ and $3A + 2B = \begin{bmatrix} -1 & 9 & 7 \\ 8 & 5 & 8 \end{bmatrix}$

4. Find the values of x and y if

(i) $x\begin{bmatrix} 5 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

(ii) $2\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} + 3\begin{bmatrix} y & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 3 & 10 \end{bmatrix}$

(iii) $3\begin{bmatrix} x & y \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} x & -5 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ 4 & 2 \end{bmatrix}$

5. If

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -4 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ 4 & 2 & 5 \end{bmatrix}$$

and $C = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$,

find (i) 2A + B–C (ii) A + 2B – 2C.

6. Compute the following products

(i) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & -2 \\ 3 & -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 3 & 1 \\ 1 & 2 & -4 \\ 1 & -2 & 2 \end{bmatrix}$

(ix) $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 1 & -1 \\ 3 & -4 & 2 \\ -2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

7. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, show that

   $A^2B + BA^2 = A$.

8. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$,

   verify that (i) $(AB)C = A(BC)$

   (ii) $A(B+C) = AB + AC$.

9. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, verify that

   $(A+B)^2 \neq A^2 + 2AB + B^2$.

10. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, show that

    $AB = AC$ although $B \neq C$.

11. Show that $AB \neq BA$ when

    (i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

12. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, show that

$A^3 - 2A^2 + A = 0$.

13. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, show that $A^2 - 6A + 5I = 0$.

where I is the unit matrix of order 2.

14. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$, verify that $A^2 - 5A + 8I = 0$.

15. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ find $A^2 - 3A + 2I$.

16. If $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, show that $(aI + bA)^3 = a^3I + 3a^2bA$

where I is the 2×2 unit matrix.

17. If $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$, show that $A^2 - 4A - I = 0$. Hence

find a matrix B such that $AB = I$.

18. If $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$, show that

(i) $A^2 = I$ (ii) $B^2 = B$ (iii) $C^2 = 0$.

19. If $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, find a matrix B such that $AB = I$.

20. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 3 & 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$,

verify that (AB)C = A(BC).

21. If $A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$, find k such that $A^2 - kA + 14I = 0$.

22. Find x and y, when

(i) $\begin{bmatrix} 2 & 5 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 7 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$.

ANSWER

1. (i) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} -4 & 2 & 1 \\ 1 & -6 & -3 \\ -1 & 7 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

2. (i) $\begin{bmatrix} 0 & 3 \\ -1 & -9 \end{bmatrix}$ (ii) $\begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -5 \\ -16 & -4 \end{bmatrix}$

(v) $\begin{bmatrix} 6 & 2 \\ -1 & 2 \\ 8 & 5 \end{bmatrix}$

3. (i) $A=\begin{bmatrix}4 & 3\\ 3 & 5\end{bmatrix},\ B=\begin{bmatrix}-1 & 1\\ -3 & 2\end{bmatrix}$

(ii) $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix},\ B=\begin{bmatrix}1 & -1\\ -3 & 2\end{bmatrix}$

(iii) $A=\begin{bmatrix}1 & 1 & 2\\ 2 & 3 & 4\end{bmatrix},\ B=\begin{bmatrix}-2 & 3 & 1\\ 1 & -2 & -2\end{bmatrix}$

4. (i) $x=1,\ y=2$ (ii) $x=2,\ y=2$ (iii) $x=3,\ y=-1$

5. (i) $\begin{bmatrix}5 & -3 & 4\\ 3 & 3 & -8\\ 7 & -1 & 9\end{bmatrix}$ (ii) $\begin{bmatrix}7 & -3 & 7\\ 0 & 6 & -4\\ 5 & -5 & 15\end{bmatrix}$

6. (i) [1] (ii) $\begin{bmatrix}2 & 4 & 6\\ -2 & -4 & -6\\ 1 & 2 & 3\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 5\\ -5 & -8\end{bmatrix}$

(iv) $\begin{bmatrix}-1 & 3 & 7\\ 5 & -7 & -3\\ -3 & 4 & 1\end{bmatrix}$ (v) $\begin{bmatrix}a^2+b^2 & 0\\ 0 & a^2+b^2\end{bmatrix}$ (vi) $\begin{bmatrix}4 & 7 & 7\\ -1 & 0 & 7\end{bmatrix}$

(vii) $\begin{bmatrix}2 & 4\\ -4 & -5\\ 1 & 2\end{bmatrix}$ (viii) $\begin{bmatrix}11 & 1 & -1\\ 35 & 10 & -4\\ 15 & 4 & 6\end{bmatrix}$

(ix) $[ax^2+by^2+cz^2+2hxy+2fyz+2gzx]$ (x) $\begin{bmatrix}10 & 5\\ 12 & 6\\ 6 & 3\end{bmatrix}$

15. $\begin{bmatrix}2 & -1 & -2\\ 5 & -3 & -5\\ -5 & 2 & 1\end{bmatrix}$ 17. $B=\begin{bmatrix}-2 & 1\\ 5 & -2\end{bmatrix}$ 19. $B=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$ 21. $k=3$

22. (i) $x=2,\ y=1$ (ii) $x=-2,\ y=3$ (iii) $x=3,\ y=5$.

5.7 Transpose of a Matrix

The matrix obtained from a given matrix A, by changing the rows into columns and vice versa, is called the transpose of A and is denoted by A' or A^t.

For example, the transpose of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

It is readily seen that if A is an $m \times n$ matrix, then its transpose A' is an $n \times m$ matrix. Further, the element in the ith row-jth column of A i.e. the (i,j)th element of A, becomes the (j,i)th element of A'.

Theorem 5.1 For any matrix A, (A')'=A.

Proof : Let the order of A be $m \times n$. Then A' will be an $n \times m$ matrix and hence (A')' will be an $m \times n$ matrix. Thus A and (A')' are matrices of the same order.

Further, (i,j)th element of (A')' = (j,i)th element of A'

= (i,j)th element of A

It follows that (A')'=A (both being of the same order and corresponding elements being equal).

Theorem 5.2 If A and B are matrices of the same order, then (A+B)'=A'+B'.

Proof : If A and B are of order $m \times n$, then both (A+B)' and A'+B' are of order $n \times m$. Thus (A+B)' and A'+B' are matrices of the same order.

Further, (i,j)th element of (A+B)' = (j,i)th element of A+B

= (j,i)th element of A+(j,i)th element of B

= (i,j)th element of A'+(i,j)th element of B'

= (i,j)th element of (A'+B')

Hence (A+B)' = A'+B'

Theorem 5.3 If k is a scalar, then $(kA)' = kA'$ for any matrix A.

Proof : If A is of order $m \times n$, then both the matrices $(kA)'$ and kA' are of the same order $n \times m$.

Now, (i,j)th element of $(kA)'$ = (j,i)th element of kA

= $k\times(j,i)$th element of A

= $k\times(i,j)$th element of A'

= (i,j)th element of kA'

Thus $(kA)'$ and kA' are matrices of the same order whose corresponding elements are equal. Hence $(kA)'=kA'$.

Theorem 5.4 If A and B are conformable for the product AB, then (AB)'=B'A'.

Proof : If A is $m\times n$ matrix and B, $n\times p$ matrix, then AB is $m\times p$ matrix so that (AB)' is $p\times m$ matrix.

Again B' is $p\times n$ matrix while A' is $n\times m$ matrix so that the product B'A' is $p\times m$ matrix.

Hence (AB)' and B'A' are matrices of the same order.

Now, (i,j)th element of (AB)' = (j,i)th element of AB

= inner product of jth row of A and ith column of B

= inner product of ith column of B and jth row of A

= inner product of ith row of B' and jth column of A'

= (i,j)th element of B'A'

Thus (AB)' and B'A' are matrices of the same order and their corresponding elements are equal. Hence (AB)'=B'A'.

Remark : Here, by the inner product of ith row of A and jth column of B we mean the sum of the products formed by multiplying each element in the ith row of A by the corresponding element in the jth column of B. For example, if $[a_1\ b_1\ c_1]$ is a row and $\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ is a column, then their inner product is the sum $a_1a_2+b_1b_2+c_1c_2$.

5.8 Symmetric and Skew-symmetric Matrices

A square matrix $A=[a_{ij}]$ is said to be (i) symmetric if $a_{ij}=a_{ji}$

(ii) skew-symmetric if $a_{ij}=-a_{ji}$

For examples,

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

are symmetric matrices, whereas

$$\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

are skew-symmetric matrices.

Obviously, a square matrix A is symmetric if and only if it coincides with its transpose i.e. $A'=A$ and skew-symmetric if and only if $A'=-A$.

For a skew-symmetric matrix $A=[a_{ij}]$, we should have by definition $a_{ii}=-a_{ii}$ (for any i) i.e. $a_{ii}=0$. Thus, every diagonal element of a skew-symmetric matrix is necessarily zero.

Theorem 5.5 Every square matrix can be expressed in one and only way, as a sum of symmetric matrix and a skew-symmetric matrix.

Proof : Let A be a given square matrix and let

$$B=\frac{1}{2}(A+A') \text{ and } C=\frac{1}{2}(A-A')$$

Then $$B'=\left[\frac{1}{2}(A+A')\right]'=\frac{1}{2}(A+A')'$$

$$=\frac{1}{2}[A'+(A')']=\frac{1}{2}(A'+A)$$

$$=\frac{1}{2}(A+A')$$

$$= B$$

and $\quad C' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}(A - A')'$

$$= \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

$$= -C.$$

Thus, B is symmetric and C is skew-symmetric.

Further, $\quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$$= B + C$$

In this way, A has been expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Let A=P+Q be another such representation of A,

where P is symmetric and Q is skew-symmetric.

Then we have

$$A' = (P + Q)'$$

$$= P' + Q'$$

$$= P - Q \quad (\because \; P' = P \text{ while } Q' = -Q)$$

So, $A + A' = (P+Q) + (P-Q) = 2P$

$$\therefore \quad P = \frac{1}{2}(A + A') = B$$

And $\; A - A' = (P + Q) - (P - Q) = 2Q$

$$\therefore \quad Q = \frac{1}{2}(A - A') = C$$

Hence, the representation A=B+C is unique.

Example 18. Find the transpose of

(i) $[2 \quad 3 \quad -5]$ (ii) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 4 \\ 4 & 5 & 2 \end{bmatrix}$

Solution : (i) $\begin{bmatrix}2 & 3 & -5\end{bmatrix}' = \begin{bmatrix}2\\3\\-5\end{bmatrix}$ (ii) $\begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix}' = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$

(iii) $\begin{bmatrix}1 & 2 & 3\\2 & -1 & 4\end{bmatrix}' = \begin{bmatrix}1 & 2\\2 & -1\\3 & 4\end{bmatrix}$ (iv) $\begin{bmatrix}1 & 0 & -1\\-2 & 3 & 4\\4 & 5 & 2\end{bmatrix}' = \begin{bmatrix}1 & -2 & 4\\0 & 3 & 5\\-1 & 4 & 2\end{bmatrix}$

Example 19. If $A = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$, verify that

(i) $(A')' = A$ (ii) $(A+B)' = A' + B'$

(iii) $(5A)' = 5A'$ (iv) $(AB)' = B'A'$

Solution : (i) $(A')' = \begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix}' = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix} = A$

(ii) $(A+B)' = \begin{bmatrix}2+1 & -1+2\\1+2 & 3-1\end{bmatrix}' = \begin{bmatrix}3 & 1\\3 & 2\end{bmatrix}'$

$= \begin{bmatrix}3 & 3\\1 & 2\end{bmatrix}$

and $A' + B' = \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}' + \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}' = \begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix} + \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$

$= \begin{bmatrix}2+1 & 1+2\\-1+2 & 3-1\end{bmatrix}$

$= \begin{bmatrix}3 & 3\\1 & 2\end{bmatrix}$

Hence (A+B)' = A'+B'.

(iii) $(5A)' = \begin{bmatrix} 10 & -5 \\ 5 & 15 \end{bmatrix}' = \begin{bmatrix} 10 & 5 \\ -5 & 15 \end{bmatrix} = 5\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = 5\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}'$

$= 5A'$.

(iv) $AB = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2-2 & 4+1 \\ 1+6 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 7 & -1 \end{bmatrix}$

$\therefore$ $(AB)' = \begin{bmatrix} 0 & 7 \\ 5 & -1 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}'\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-2 & 1+6 \\ 4+1 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 5 & -1 \end{bmatrix}$

Hence (AB)' = B'A'.

Example 20. If A be a square matrix, prove that AA' and A'A are both symmetric matrices.

Solution : Let AA'= B and A'A=C

Then $B' = (AA')'$

= (A')A' $\quad\because\ (AB)' = B'A'$

= AA' $\quad\because\ (A')' = A$

= B

$\therefore$ B is symmetric

Again, C'= (A'A)'

= A'(A')'

= A'A

= C

$\therefore$ C is symmetric

Thus, both AA' and A'A are symmetric for any square matrix A.

Example 21. If A and B be symmetric matrices of the same order, show that AB – BA is a skew-symmetric matrix.

Solution : Let C = AB – BA

Then $C' = (AB - BA)'$

$= (AB)' - (BA)'$ $\quad \because \quad (P-Q)' = P' - Q'$

$= B'A' - A'B'$ $\quad \because \quad (PQ)' = Q'P'$

$= BA - AB$ $\quad \because \quad A' = A \quad B' = B$

$= -AB + BA$

$= -(AB - BA)$

$= -C$

Hence C is skew symmetric.

EXERCISE 5.3

1. Write the transpose of the following matrix :

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 \\ 3 & -4 \\ 4 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 5 & -2 \\ 6 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

2. Verify that (A+B)' = A'+B', when

(i) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 4 & -2 & 0 \end{bmatrix}$

3. Verify that (AB)' = B'A', when

(i) $A=\begin{bmatrix}0 & -1 & 4\\ 5 & 2 & -3\end{bmatrix}$, $B=\begin{bmatrix}3 & 1\\ 1 & 0\\ 6 & -2\end{bmatrix}$

(ii) $A=\begin{bmatrix}-1 & 1 & 2\\ 3 & 1 & -2\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ -2 & 3\\ 3 & 0\end{bmatrix}$

(iii) $A=\begin{bmatrix}-1 & 1 & 2\\ 3 & 1 & -2\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 1 & 2\\ 0 & 3\end{bmatrix}$

(iv) $A=\begin{bmatrix}1 & -4\\ 1 & 3\\ -3 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & -1\\ 1 & 0 & -2\end{bmatrix}$

(v) $A=\begin{bmatrix}3 & 4\\ 2 & 5\end{bmatrix}$, $A=\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 2\end{bmatrix}$

4. If $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$, verify the following:

(i) $(A')' = A$ (ii) $(A+B)' = A' + B'$ (iii) $(2A)' = 2A'$ (iv) $(AB)' = B'A'$

5. Express the following matrix as a sum of a symmetric matrix and a skew-symmetric matrix.

(i) $\begin{bmatrix}2 & 3\\ 5 & 6\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 1 & -1\\ 2 & 3 & 4\\ -1 & 2 & 3\end{bmatrix}$ (iii) $\begin{bmatrix}1 & -2 & 3\\ 3 & 4 & 2\\ 5 & 0 & -1\end{bmatrix}$

6. For any square matrix A, prove that both the products AA' and A'A are symmetric.

7. If A and B are symmetric matrices of the same order, show that

(i) A+B is symmetric. (ii) A–B is symmetric.

(iii) AB+BA is symmetric. (iv) AB–BA is skew-symmetric.

8. Show that the matrix ABA' (when A, B are square matrices of the same order) is symmetric or skew-symmetric according as B is symmetric or skew-symmetric.

9. If A and B are symmetric matrices of the same order, show that AB is symmetric if and only if AB=BA.

ANSWER

1. (i) $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & -4 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 3 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 6 & 1 \\ 5 & 4 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

5. (i) $\begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \frac{3}{2} & -1 \\ \frac{3}{2} & 3 & 3 \\ -1 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & \frac{1}{2} & 4 \\ \frac{1}{2} & 4 & 1 \\ 4 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} & -1 \\ \frac{5}{2} & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

CHAPTER 6

FACTORISATION (Harder Type) AND IDENTITIES (Conditional and Unconditional)

6.1 Introduction

We have learnt how simple algebraic expressions of types $a^2 - b^2$, $a^3 + b^3$, $a^3 - b^3$, $ax^2 + bx + c$, etc. are resolved into factors. Here we shall discuss factorisation of a harder type. But, factorisation of expressions of the forms : $a^3 + b^3 + c^3 - 3abc$, $a^2(b - c) + b^2(c - a) + c^2(a - b)$, etc. will not be considered as they are discussed in the Mathematics (general course) for Class X. We may however use the results. For occasional reference, the following results are given :

(i) $a^2 - b^2 = (a + b)(a - b)$

(ii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(iii) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(iv) $x^2 + (p + q)x + pq = (x + p)(x + q)$

(v) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

(vi) $a^2(b - c) + b^2(c - a) + c^2(a - b)$
$= bc(b - c) + ca(c - a) + ab(a - b) = -(b - c)(c - a)(a - \mathrm{b})$

(vii) $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 2abc$
$= a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$
$= bc(b + c) + ca(c + a) + ab(a + b) + 2abc = (b + c)(c + a)(a + b)$

(viii) $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3abc$
$= a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc$
$= bc(b + c) + ca(c + a) + ab(a + b) + 3abc = (a + b + c)(bc + ca + ab)$

(ix) $(a + b + c)^3 - a^3 - b^3 - c^3 = 3(b + c)(c + a)(a + b)$

(x) $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = (a + b + c)(a + b - c)$
$(b + c - a)(c + a - b)$

6.2 Factorisation by Trial

The factor theorem on polynomials states that 'A polynomial $f(x)$ is exactly divisible by $(x - a)$ if and only if $f(a) = 0$'. We may, very well, use this theorem in resolving polynomials into factors. Given a polynomial $f(x)$, we first find by inspection suitable value or values of x for which $f(x)$ vanishes. If $f(x)$ vanishes for a value, say a of x then the terms of $f(x)$ may be grouped into parts each of which is divisible by $x - a$ and the factor $x - a$ be taken common. In this way, we find a polynomial $g(x)$ of degree one less than that of $f(x)$, such that $f(x) = (x - a)$ $g(x)$. We next proceed to factorise $g(x)$ by the same process or any other process known to us. The process is illustrated in the following examples.

Example 1. Factorise $x^3 - 7x^2 + 14x - 8$

Solution : Let $f(x) = x^3 - 7x^2 + 14x - 8$

Then, $f(1) = 1 - 7 + 14 - 8 = 0$ and so $x - 1$ is a factor of $f(x)$.

Now we write, by grouping the terms into parts divisible by $x - 1$;

and then take out $(x - 1)$ as a factor :

$$f(x) = x^3 - x^2 - 6x^2 + 6x + 8x - 8$$
$$= \underline{x^2(x - 1) - 6x(x - 1) + 8(x - 1)}$$
$$= \underline{(x - 1)(x^2 - 6x + 8)}$$
$$= (x - 1)(x - 2)(x - 4)$$

(Here $x^2 - 6x + 8$ being a quadratic expression, is factorised by the usual method.)

[Here, the factorisation can also be effected by grouping the terms as $x^3 - 8 - (7x^2 - 14x)$, (Sec §6.5)]

Example 2. Resolve into factors, $x^4 - 4x^3 + 5x^2 - 4x + 4$.

Solution : Let $f(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$. Then

$$f(1) = 1 - 4 + 5 - 4 + 4 \neq 0$$
$$f(-1) = 1 + 4 + 5 + 4 + 4 \neq 0$$

$$f(2) = 16 - 32 + 20 - 8 + 4 = 0$$

So, $x - 2$ is a factor ; we now group the terms into parts each of which is divisible by $x - 2$; and then take out $(x - 2)$ as a factor.

$$\begin{aligned} f(x) &= x^4 - 2x^3 - 2x^3 + 4x^2 + x^2 - 2x - 2x + 4 \\ &= x^3 (x - 2) - 2x^2(x - 2) + x (x - 2) - 2(x - 2) \\ &= (x - 2) (x^3 - 2x^2 + x - 2) \end{aligned}$$

Writing $g(x) = x^3 - 2x^2 + x - 2$, we find $g(2) = 0$

and so $g(x) = x^3 - 2x^2 + x - 2 = x^2 (x - 2) + (x - 2)$

$$= (x - 2) (x^2 + 1)$$

Hence, the given expression $= (x - 2)^2 (x^2 + 1)$.

(Here the quadratic factor $x^2 + 1$ is not factorisable into real linear factors.)

Remarks : (i) If the sum of coefficients in any polynomial $f(x)$ is zero, then $f(1) = 0$ and hence $x - 1$ is a factor of $f(x)$.

(ii) If the sum of coefficients of odd powers of x in $f(x)$ is equal to the sum of the remaining coefficients, then $f(-1) = 0$ and hence $x + 1$ is a factor of $f(x)$.

Example 3. Resolve into factors $x^4 + 5x^3 + 5x^2 - 5x - 6$.

Solution : The sum of the coefficients of the polynomial $= 1 + 5 + 5 -5 - 6 = 0$

Hence $x - 1$ is a factor and so grouping the terms into parts each of which is divisible by $x - 1$, we have

$$\begin{aligned} \text{the given expression} &= x^4 - x^3 + 6x^3 - 6x^2 + 11x^2 - 11x + 6x - 6 \\ &= x^3 (x - 1) + 6x^2 (x - 1) + 11x (x - 1) + 6 (x - 1) \\ &= (x - 1) (x^3 + 6x^2 + 11x + 6) \end{aligned}$$

Further, in the polynomial $x^3 + 6x^2 + 11x + 6$, the sum of the coefficients of odd powers of x is $1 + 11 = 12$ which is equal to the sum of the remaining coefficients. Hence $x + 1$ is a factor. Thus,

$$\begin{aligned} x^3 + 6x^2 + 11x + 6 &= x^3 + x^2 + 5x^2 + 5x + 6x + 6 \\ &= x^2 (x + 1) + 5x (x + 1) + 6 (x + 1) \end{aligned}$$

$= (x + 1)(x^2 + 5x + 6)$

$= (x + 1)(x + 2)(x + 3)$ (on factorising the quadratic polynomial by usual method)

$\therefore$ the given expression $= (x - 1)(x + 1)(x + 2)(x + 3)$

6.3 Factorisation of Reciprocal Expressions

A polynomial of degree n in x is said to be in its complete form if it involves all powers x^r of x for $0 \le r \le n$. For example, the polynomial $x^4 - 3x^3 + x^2 - 5x - 2$ is in complete form. The polynomial $x^4 + x^2 + 5x - 2$ is not in its complete form but may be written in complete form as $x^4 + 0.x^3 + x^2 + 5x - 2$.

Definition : A complete polynomial is said to be a *reciprocal* or *recurring expression* if the coefficients of the terms equidistant from the beginning and the end are equal (the terms being in descending or ascending order of their degrees).

For example, $x^6 - 3x^5 + 5x^4 + x^3 + 5x^2 - 3x + 1$ is a reciprocal expression whereas $x^5 + 3x^3 + x^2 + 3x + 1$ is not.

A reciprocal expression of even degree can be factorised by grouping terms with equal coefficients. The process is illustrated in the following examples.

Example 4. Factorise $x^4 + 3x^3 + 4x^2 + 3x + 1$.

Solution : The expression is a reciprocal expression of even degree. So, the given expression $= (x^4 + 1) + (3x^3 + 3x) + 4x^2$

$= \{(x^2 + 1)^2 - 2x^2\} + 3x(x^2 + 1) + 4x^2$

$= (x^2 + 1)^2 + 3x(x^2 + 1) + 2x^2$

$= y^2 + 3xy + 2x^2$ where $y = x^2 + 1$

$= y^2 + xy + 2xy + 2x^2$

$= y(y + x) + 2x(y + x) = (y + x)(y + 2x)$

$= (x^2 + x + 1)(x^2 + 2x + 1)$ (restoring value of y)

$= (x + 1)^2 (x^2 + x + 1)$

Example 5. Factorise $2x^4 - 5x^3 + 4x^2 - 5x + 2$.

Solution : The given expression $= (2x^4 + 2) - (5x^3 + 5x) + 4x^2$

$= 2(x^4 + 1) - 5x(x^2 + 1) + 4x^2$

$= 2\{(x^2+1)^2 - 2x^2\} - 5x\,(x^2+1) + 4x^2$

$= 2(x^2+1)^2 - 5x\,(x^2+1)$

$= (x^2+1)\,\{2(x^2+1) - 5x\}$

$= (x^2+1)\,(2x^2 - 5x + 2)$

$= (x^2+1)\,(2x^2 - 4x - x + 2)$

$= (x^2+1)\,\{2x\,(x-2) - (x-2)\}$

$= (x^2+1)\,(x-2)\,(2x-1)$

A reciprocal expression of odd degree in x has in general $(x+1)$ as a factor (for, the expression vanishes when $x = -1$). Dividing the expression by $x+1$, we obtain a quotient which is a reciprocal expression of even degree, and which may be factorised by grouping terms with equal coefficients.

Example 6. Factorise $2x^5 + 3x^4 - 5x^3 - 5x^2 + 3x + 2$.

Solution : The expression vanishes when $x = -1$ and so $x+1$ is a factor.

Given expression $= 2x^5 + 2x^4 + x^4 + x^3 - 6x^3 - 6x^2 + x^2 + x + 2x + 2$

$=$

$= (x+1)\,(2x^4 + x^3 - 6x^2 + x + 2)$

$= (x+1)\,(2x^4 + 2 + x^3 + x - 6x^2)$

$= (x+1)\,\{2\,(x^4+1) + x\,(x^2+1) - 6x^2\}$

$= (x+1)\,\{2\,(x^2+1)^2 + x\,(x^2+1) - 10x^2\}$

Now, $2\,(x^2+1)^2 + x\,(x^2+1) - 10x^2 = 2y^2 + xy - 10x^2$, where $y = x^2 + 1$

$= 2y^2 + 5xy - 4xy - 10x^2$

$= y\,(2y + 5x) - 2x\,(2y + 5x)$

$= (y - 2x)\,(2y + 5x)$

$= (x^2 - 2x + 1)\,(2x^2 + 5x + 2)$ (restoring the value of y)

$= (x-1)^2\,(2x^2 + 4x + x + 2)$

$= (x-1)^2\,(2x+1)\,(x+2)$

Hence, the given expression = $(x + 1)(x - 1)^2(2x + 1)(x + 2)$.

6.4 Factorisation of a polynomial expression in which the coefficients of the terms equidistant from the beginning and end are equal in magnitude but opposite in sign

If such an expression in x is of odd degree, the sum of coefficients will be zero and hence it has $x - 1$ as a factor. Division by $x - 1$ gives as quotient a reciprocal expression of even degree and this can be factorised by using the method discussed already. And if the degree of the expression is even, say $2m$, then the coefficient of x^m is zero and so as can be seen both $x - 1$ and $x + 1$ are its factors. Division by $x^2 - 1$ will give as quotient a reciprocal expression of even degree, which can further be factorised.

Example 7. Resolve into factors with integral coefficients $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$

Solution : As the sum of the coefficients is zero, $x - 1$ is a factor of the expression.

On division by $x - 1$ we see that, the expression

$$= (x - 1)(x^4 - 4x^3 + 5x^2 - 4x + 1)$$

Further, $x^4 - 4x^3 + 5x^2 - 4x + 1 = (x^4 + 2x^2 + 1) - 4x(x^2 + 1) + 3x^2$

$$= (x^2 + 1)^2 - 4(x^2 + 1)x + 3x^2$$

$$= y^2 - 4yx + 3x^2, \text{ where } y = x^2 + 1$$

$$= (y - x)(y - 3x)$$

$$= (x^2 - x + 1)(x^2 - 3x + 1)$$

$\therefore$ the given expression = $(x - 1)(x^2 - x + 1)(x^2 - 3x + 1)$

Note : In this section, we are concerned with factors involving integral coefficients only and so the quadratic factor $x^2 - 3x + 1$ is left as it is, although it can further be factorised as $\left(x - \frac{3+\sqrt{5}}{2}\right)\left(x - \frac{3-\sqrt{5}}{2}\right)$.

Example 8. Factorise $2x^6 - 3x^5 - 3x^4 + 3x^2 + 3x - 2$

Solution : It is evident that both $x - 1$ and $x + 1$ are factors of the given expression.

Division by $x^2 - 1$ gives $2x^4 - 3x^3 - x^2 - 3x + 2$ as quotient. Also

$$2x^4 - 3x^3 - x^2 - 3x + 2 = 2(x^4 + 1) - 3(x^3 + x) - x^2$$

$$= 2(x^2 + 1)^2 - 3x(x^2 + 1) - 5x^2$$

$= 2y^2 - 3yx - 5x^2$, where $y = x^2 + 1$

$= 2y^2 - 5yx + 2yx - 5x^2$

$= (2y - 5x)(y + x)$

$= (2x^2 - 5x + 2)(x^2 + x + 1)$

$= (2x^2 - 4x - x + 2)(x^2 + x + 1)$

$= (x - 2)(2x - 1)(x^2 + x + 1)$

$\therefore$ The given expression $= (x + 1)(x - 1)(x - 2)(2x - 1)(x^2 + x + 1)$

6.5 Factorisation by suitable arrangement and grouping of terms

Some algebraic expressions may be resolved into factors by suitable arrangement of the terms. However, there is no specific rule to be followed while arranging the terms ; unless a factor is predetermined, and the method is handy only in certain cases. The following examples will make the process clear.

Example 9. Factorise $a^2 + ab - bc - c^2$.

Solution : $a^2 + ab - bc - c^2 = (a^2 - c^2) + (ab - bc)$

$= (a + c)(a - c) + b(a - c)$

$= (a - c)(a + b + c)$

Example 10. Factorise $(p^2 + q^2)xy + pq(x^2 + y^2)$.

Solution : The expression $= p^2xy + q^2xy + pqx^2 + pqy^2$

$= (p^2xy + pqx^2) + (pqy^2 + q^2xy)$

$= px(py + qx) + qy(py + qx)$

$= (px + qy)(py + qx)$

$= (px + qy)(qx + py)$

Example 11. Factorise $x^4 - ab^3 + b^3x - ax^3$.

Solution : The expression $= (x^4 + b^3x) - (ax^3 + ab^3)$

$= x(x^3 + b^3) - a(x^3 + b^3)$

$= (x^3 + b^3)(x - a)$

$= (x + b)(x^2 - bx + b^2)(x - a)$

$= (x - a)(x + b)(x^2 - bx + b^2)$

Example 12. Factorise $a^3 - 7a^2 - 21a + 27$.

Solution : The expression

$$= (a^3 + 27) - (7a^2 + 21a)$$
$$= (a^3 + 3^3) - 7a\,(a + 3)$$
$$= (a + 3)\,(a^2 - 3a + 9) - 7a\,(a + 3)$$
$$= (a + 3)\,(a^2 - 3a + 9 - 7a)$$
$$= (a + 3)\,(a^2 - 10a + 9)$$
$$= (a + 3)\,(a - 1)\,(a - 9)$$

Example 13. Factorise $x^4 + 2x^3y - 2xy^3 - y^4$.

Sloution : The expression

$$= (x^4 - y^4) + (2x^3y - 2xy^3)$$
$$= (x^2 + y^2)\,(x^2 - y^2) + 2xy\,(x^2 - y^2)$$
$$= (x^2 - y^2)\,(x^2 + 2xy + y^2)$$
$$= (x - y)\,(x + y)\,(x + y)^2$$
$$= (x - y)\,(x + y)^3$$

Example 14. Factorise $(a + b + c)\,(bc + ca + ab) - abc$.

Solution : The expression

$$= \{a + (b + c)\}\,\{a\,(b + c) + bc\} - abc$$
$$= a^2(b + c) + abc + a\,(b + c)^2 + bc\,(b + c) - abc$$
$$= a^2\,(b+c) + a\,(b + c)^2 + bc\,(b + c)$$

(arranging in descending powers of a)

$$= (b + c)\,[a^2 + a\,(b + c) + bc]$$
$$= (b + c)\,[(a^2 + ac) + (a + c)\,b]$$
$$= (b + c)\,[a\,(c + a) + b\,(c + a)]$$
$$= (b + c)\,(c + a)\,(a + b)$$

6.6 Factorisation of expressions of second degree in two variables

The process is illustrated in the following examples :

Example 15. Factorise $x^2 + xy - 2y^2 + x + 5y - 2$.

Solution : Arranging in descending powers of one of the letters say x, we have

$$x^2 + xy - 2y^2 + x + 5y - 2 = x^2 + x\,(y + 1) - (2y^2 - 5y + 2)$$
$$= x^2 + x\,(y + 1) - (2y - 1)\,(y - 2)$$

This can be treated as quadratic in x. So, following the usual method, we can factorise it. Here we split the coefficient in the middle term. Hence, the given expression

$= x^2 + x\{(2y-1)-(y-2)\}-(2y-1)(y-2)$

$= x^2 + x(2y-1)-x(y-2)-(2y-1)(y-2)$

$= x(x+2y-1)-(y-2)(x+2y-1)$

$= (x+2y-1)(x-y+2)$

Example 16. Factorise $4x^2-4xy+y^2-6x+3y$.

Solution : Arranging in descending powers of x, we find that the expression

$= 4x^2-2x(2y+3)+y(y+3)$

We now split $4y(y+3)$ into two factors viz, $-2y$ and $-2(y+3)$, whose sum is the coefficient of the middle term. Hence, the expression

$= 4x^2-\{2y+2(y+3)\}x+y(y+3)$

$= 4x^2-2yx-2(y+3)x+y(y+3)$

$= 2x(2x-y)-(y+3)(2x-y)$

$= (2x-y)(2x-y-3)$

[Factorisation can also be effected as follows :

Given expression $= (4x^2-4xy+y^2)-(6x-3y)$

$= (2x-y)^2-3(2x-y)$

$= (2x-y)(2x-y-3)$]

6.7 Factorisation of homogeneous expressions of second degree

We may proceed as in the previous article, by arranging in descending (or ascending) powers of one of the letters involved. See the following examples.

Example 17. Factorise $x^2+3xy+2y^2+xz+2yz$.

Solution : The expression $= x^2+x(3y+z)+2y^2+2yz$

(on arranging in descending powers of x)

$= x^2 + x\{2y + (y+z)\} + 2y(y+z)$

[splitting the coefficient of x in the middle term into two parts whose product is equal to $2y(y+z)$]

$= (x^2 + 2xy) + \{x(y+z) + 2y(y+z)\}$

$= x(x+2y) + (y+z)(x+2y)$

$= (x+2y)(x+y+z)$

Example 18. Factorise $a^2 + 2b^2 - 2c^2 + 3ab + 3bc + ca$.

Solution : The given expression $= a^2 + a(3b+c) + (2b^2 + 3bc - 2c^2)$

$= a^2 + a(3b+c) + (2b^2 + 4bc - bc - 2c^2)$

$= a^2 + a(3b+c) + (b+2c)(2b-c)$

$= a^2 + a\{(b+2c) + (2b-c)\} + (b+2c)(2b-c)$

$= a(a+b+2c) + (2b-c)(a+b+2c)$

$= (a+b+2c)(a+2b-c)$

Miscellaneous Examples

Example 19. Resolve into two quadratic factors $x^4 - 4x^3 - x^2 + 10x + 4$.

Solution : The expression $= (x^4 - 4x^3 + 4x^2) - 5x^2 + 10x + 4$

$= (x^2 - 2x)^2 - 5(x^2 - 2x) + 4$

$= y^2 - 5y + 4$ (writing y for $x^2 - 2x$)

$= (y-4)(y-1)$

$= (x^2 - 2x - 4)(x^2 - 2x - 1)$ (restoring the value of y)

Example 20. Resolve into factors $x^4 - 5x^3y + 6x^2y^2 - 5xy^3 + y^4$.

Solution : The expression $= (x^4 + y^4) - 5xy(x^2 + y^2) + 6x^2y^2$

$= (x^2 + y^2)^2 - 5xy(x^2 + y^2) + 4x^2y^2$

$= u^2 - 5uv + 4v^2$ where $u = x^2 + y^2$, $v = xy$

$= u^2 - 4uv - uv + 4v^2$

$= u(u - 4v) - v(u - 4v)$

$= (u - 4v)(u - v)$

$= (x^2 + y^2 - 4xy)(x^2 + y^2 - xy)$

$= (x^2 - 4xy + y^2)(x^2 - xy + y^2)$

Example 21. Resolve into two quadratic factors $x^4 - 7x^3y + 14x^2y^2 - 14xy^3 + 4y^4$.

Solution : The expression $= (x^4 + 4y^4) - 7x^3y - 14xy^3 + 14x^2y^2$

$= (x^2 + 4x^2y^2 + 4y^4) - 7xy(x^2 + 2y^2) + 10x^2y^2$

$= (x^2 + 2y^2)^2 - 7xy(x^2 + 2y^2) + 10x^2y^2$

$= u^2 - 7uv + 10v^2$, where $u = x^2 + 2y^2$ and $v = xy$

$= u^2 - 2uv - 5uv + 10v^2$

$= (u - 2v)(u - 5v)$

$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 - 5xy)$

$= (x^2 - 2xy + 2y^2)(x^2 - 5xy + 2y^2)$

Example 22. Factorise $8x^3 + 8x^2 - 3$.

Solution : The expression $= (2x)^3 + 2(2x)^2 - 3$

$= y^3 + 2y^2 - 3$, where $y = 2x$

$= (y^3 - 1) + (2y^2 - 2)$

$= (y - 1)(y^2 + y + 1) + 2(y - 1)(y + 1)$

$= (y - 1)(y^2 + 3y + 3)$

$= (2x - 1)(4x^2 + 6x + 3)$

Example 23. Resolve into two quadratic factors $(x - 1)(x - 2)(x + 3)(x + 4) + 4$.

Solution : The expression $= (x - 1)(x - 2)(x + 3)(x + 4) + 4$

$= \{(x - 1)(x + 3)\}\{(x - 2)(x + 4)\} + 4$

$= (x^2 + 2x - 3)(x^2 + 2x - 8) + 4$

$= (y - 3)(y - 8) + 4$, where $y = x^2 + 2x$

$= y^2 - 11y + 28$

$= (y - 7)(y - 4)$

$= (x^2 + 2x - 7)(x^2 + 2x - 4)$

Note : In multiplying together the four binomials $x-1$, $x-2$, $x+3$, $x+4$, we combine $x-1$ with $x+3$ and $x-2$ with $x+4$, so that in the resulting products the terms containing x^2 and x may remain the same.

Example 24. Factorise $x^2(y^2-z^2)+4xyz-y^2+z^2$.

Solution : The expression $= x^2y^2 - x^2z^2 + 4xyz - y^2 + z^2$

$$= (x^2y^2 + 2xyz + z^2) - (x^2z^2 - 2xyz + y^2)$$

$$= (xy+z)^2 - (xz-y)^2$$

$$= (xy + z + xz - y)(xy + z - xz + y)$$

$$= \{x(y+z) - y + z\}\{x(y-z) + y + z\}$$

EXERCISE 6.1

Resolve into factors :

1. $x^3 - 2x^2 - 5x + 6$
2. $x^3 + 2x^2 - 5x - 6$
3. $x^3 + 4x^2 - 2x - 20$
4. $x^3 + x^2 - 5x + 3$
5. $x^3 + 3x^2 + 4x + 2$
6. $6x^3 - 11x^2 + 6x - 1$
7. $x^3 - 5x^2 - 2x + 4$
8. $x^3 - 6x^2 + 3x + 10$
9. $x^3 + 2x^2 - 4x + 1$
10. $x^3 - 2x^2 + x - 2$
11. $x^3 - 6x + 4$
12. $x^3 - 3x^2 + 4$
13. $x^3 - 7x^2 + 36$
14. $x^3 - 3x^2 - 6x + 8$
15. $8x^3 + 8x^2 - 1$
16. $8x^3 + 24x - 13$
17. $27x^3 - 9x + 2$
18. $27x^3 + 3x - 10$
19. $x^4 - 2x^3 - 3x^2 - 2x + 1$
20. $x^4 - 5x^3 - 12x^2 - 5x + 1$
21. $x^4 - 6x^2 + 8x - 3$
22. $x^4 - 10x^3 + 26x^2 - 10x + 1$
23. $x^4 - 3x^3 + 4x^2 - 3x + 3$
24. $x^4 - 7x^3 + 10x^2 - 35x + 25$
25. $x^4 - 6x^3 + 12x^2 - 2x - 21$
26. $x^5 + 4x^4 - 13x^3 - 13x^2 + 4x + 1$
27. $2x^5 - 7x^4 - x^3 - x^2 - 7x + 2$
28. $2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2$
29. $x^4 - 6x^3 + 15x^2 - 18x + 5$
30. $6a^2 + 7ab + 2b^2 + 11a + 7b + 3$
31. $a^2 - 4b^2 - 9c^2 + 12bc + 4a - 8b + 12c$
32. $x^2 - y^2 - z^2 - 2yz + x - y - z$
33. $9x^2 - 4y^2 - 24zx + 16z^2 - 15x + 10y + 20z$
34. $6x^2 + 7xy + 2y^2 + 11xz + 7yz + 3z^2$
35. $a^2 - 3ab + 2b^2 - 2bc - 4c^2$

36. $2x^2 + 5yz + zx - 10xy - z^2$

37. $x^2 - 2xy + y^2 - 5x + 5y$

38. $4x^2 - 4xy + y^2 - 6x + 3y$

39. $4x^2 - 12xy + 9y^2 + 2x - 3y - 2$

40. $x^2 - 3x(2y - 1) + 4y(2\text{y} - 3)$

41. $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$

42. $(a^2 + b^2)x^2 - a^2b(2a + b) + a(2bx^2 - a^3)$

43. $(2x^2 + 3b^2)a - (2a^2 + 3x^2)b$

44. $a^4 - b^3c + a^2b^2 - b^2c^2$

45. $a^3 - 7a^2b + 14ab^2 - 8b^3$

46. $a^3 + 6a^2 - 24a - 64$

47. $3x^3 - (5a + 3b)x^2 + (3a + 5ab)x - 5a^2$

48. $x^4 + 4x^3y + 10x^2y^2 + 4xy^3 + y^4$

49. $x^4 - 5x^3y + 6x^2y^2 - 5xy^3 + y^4$

50. $a^4b^4 + a^2b^2 - c^2 + 2abc + 1$

51. $a^3(b - c) + b^3(c - a) + c^3(a - b)$

52. (i) $(x + 1)(x + 2)(x - 3)(x - 4) + 6$

(ii) $(x - 1)(x - 2)(x + 4)(x + 5) + 8$

(iii) $(x - 1)(x - 3)(x + 4)(x + 6) + 13$

(iv) $(x + 1)(x + 2)(x + 3)(x + 4) - 3$

(v) $x(x - 2)(2x + 1)(2x - 3) - 63$

53. $2x^3 - x^2y - y^3$

54. $x^3 - 6xy^2 + 9y^3$

55. $x^2 + bx - (a^2 - 3ab + 2b^2)$

56. $x^2 + 2xy - 5zx - 4yz + 6z^2$

57. $a^2x^2 - b^2y^2 - bcyz + cazx$

58. $(a^2 + b^2)(x^2 - y^2) + 2ab(x^2 + y^2)$

59. Find the value of $x^4 - x^3 + x^2 + 2$, when $x^2 + 2 = 2x$

60. Find the value of $xy(x + y) + yz(y + z) + zx(z + x) + 3xyz$, when $x = a(b - c)$, $y = b(c - a)$, $z = c(a - b)$.

ANSWER

1. $(x - 1)(x + 2)(x - 3)$

2. $(x + 1)(x - 2)(x + 3)$

3. $(x - 2)(x^2 + 6x + 10)$

4. $(x - 1)^2(x + 3)$

5. $(x + 1)(x^2 + 2x + 2)$

6. $(x - 1)$ (2

7. $(x + 1)(x^2 - 6x + 4)$

8. $(x + 1)(x - 2)(x - 5)$

9. $(x - 1)$ (x

10. $(x - 2)(x^2 + 1)$

11. $(x - 2)(x^2 + 2x - 2)$

12. $(x + 1)(x - 2)^2$

13. $(x + 2)(x - 3)(x - 6)$

14. $(x - 1)(x + 2)(x - 4)$

15. $(2x + 1)(4x^2 + 2x - 1)$

16. $(2x - 1)(4x^2 + 2x + 13)$ 17. $(3x - 1)^2 (3x + 2)$ 18. $(3x - 2)(9x^2 + 6x + 5)$

19. $(x^2 - x + 1)^2$ 20. $(x + 1)^2 (x^2 - 7x + 1)$ 21. $(x - 1)^3 (x + 3)$

22. $(x^2 - 4x + 1)(x^2 - 6x + 1)$ 23. $(x^2 + 1)(x^2 - 3x + 3)$

24. $(x^2 + 5)(x^2 - 7x + 5)$ 25. $(x + 1)(x - 3)(x^2 - 4x + 7)$

26. $(x + 1)(x^2 - 3x + 1)(x^2 + 6x + 1)$ 27. $(x + 1)(x^2 - 4x + 1)(2x^2 - x + 2)$

28. $(x - 1)(x - 2)(2x - 1)(x^2 - 4x + 1)$ 29. $(x^2 - 3x + 1)(x^2 - 3x + 5)$

30. $(2a + b + 3)(3a + 2b + 1)$ 31. $(a - 2b + 3c)(a + 2b - 3c + 4)$

32. $(x - y - z)(x + y + z + 1)$ 33. $(3x - 2y - 4z)(3x + 2y - 4z - 5)$

34. $(2x + y + 3z)(3x + 2y + z)$ 35. $(a - b + 2c)(a - 2b - 2c)$

36. $(2x - z)(x - 5y + z)$ 37. $(x - y)(x - y - 5)$

38. $(2x - y)(2x - y - 3)$ 39. $(2x - 3y - 1)(2x - 3y + 2)$

40. $(x - 4y)(x - 2y + 3)$ 41. $(x - y)^2 (x^2 + y^2)$

42. $(a + b)^2 (x - a)(x + a)$ 43. $(2a - 3b)(x^2 - ab)$

44. $(a^2 - bc)(a^2 + bc + b^2)$ 45. $(a - b)(a - 2b)(a - 4b)$

46. $(a + 2)(a - 4)(a + 8)$ 47. $(3x - 5a)(x^2 - bx + a)$

48. $(x - y)^2 (x^2 + 6xy + y^2)$ 49. $(x^2 - xy + y^2)(x^2 - 4xy + y^2)$

50. $(a^2b^2 + ab - c + 1)(a^2b^2 - ab + c + 1)$ 51. $-(b - c)(c - a)(a - b)(a + b + c)$

52. (i) $(x^2 - 2x - 5)(x^2 - 2x - 6)$ (ii) $(x^2 - 3x - 6)(x^2 + 3x - 8)$

(iii) $(x^2 + 3x - 5)(x^2 + 3x - 17)$ (iv) $(x^2 + 5x + 3)(x^2 + 5x + 7)$

(v) $(x - 3)(2x + 3)(2x^2 - 3x + 7)$

53. $(x - y)(2x^2 + xy + y^2)$ 54. $(x + 3y)(x^2 - 3xy + 3y^2)$

55. $(x + a - b)(x - a + 2b)$ 56. $(x + 2y - 3z)(x - 2z)$

57. $(ax - by)(ax + by + cz)$

58. $\{(a + b)x + (a - b)y\}\{(a + b)x - (a - b)y\}$

59. 0. 60. 0

6.8 Identities

We are already familiar with simple algebraic and trigonometric identities. In fact, an algebraic identity is a statement that two algebraic expressions are equal for all values of the letters or variables involved. For example, $a^2 - b^2 = (a + b)(a - b)$ is an identity, for the statement is true for all values of a and b ; whereas $2x + 3 = 3x - 1$ is simply an equation but not an identity ; for it holds only when $x = 4$.

To prove an identity, we are to establish the equality of its two sides. The following procedures may be noted for proving an identity :

(i) Reduce one of the sides (preferably, the more complex side) to the form of the other by simplification using known formulae.

(ii) If both sides are complex, reduce each side to its simplest form and establish their equality.

(iii) Sometimes an identity follows easily by transposition of terms or addition of terms to both sides.

(iv) Sometimes an identity becomes trivial when new letter(s) are substituted for a group of letters occuring in the identity. Make such substitutions whenever necessary.

The following examples will illustrate the process :

Example 25. Show that $(x - a)^2 (b - c) + (x - b)^2 (c - a) + (x - c)^2 (a - b)$
$= -(b - c)(c - a)(a - b)$

Solution : Putting $x - a = p$, $x - b = q$ and $x - c = r$, we have

$$q - r = (x - b) - (x - c) = -(b - c)$$

$$r - p = (x - c) - (x - a) = -(c - a)$$

and $p - q = (x - a) - (x - b) = -(a - b)$

$\therefore$ L.H.S. $= -[p^2 (q - r) + q^2 (r - p) + r^2 (p - q)]$

$= (q - r)(r - p)(p - q)$ (Result VI of § 2.1)

$= (-1)^3 (b - c)(c - a)(a - b)$

$= -(b - c)(c - a)(a - b) =$ R.H.S.

Example 26. Prove that $27(x + y + z)^3 - (x + 2y)^3 - (y + 2z)^3 - (z + 2x)^3$
$= 3(x + 3y + 2z)(2x + y + 3z)(3x + 2y + z)$

Solution : Putting $x + 2y = a$, $y + 2z = b$ and $z + 2x = c$ we have

$$a + b + c = 3x + 3y + 3z = 3(x + y + z)$$

Now $(a + b + c)^3 - a^3 - b^3 - c^3 = 3\,(b + c)\,(c + a)\,(a + b)$

$\therefore\quad 27(x + y + z)^3 - (x + 2y)^3 - (y + 2z)^3 - (z + 2x)^3$

$= 3\{(y + 2z) + (z + 2x)\}\ \{(z + 2x) + (x + 2y)\}\ \{(x + 2y) + (y + 2z)\}$

$= 3(2x + y + 3z)\,(3x + 2y + z)\,(x + 3y + 2z)$

$= 3(x + 3y + 2z)\,(2x + y + 3z)\,(3x + 2y + z)$

Example 27. Prove that $(x + y + 2z)\,(x + 2y + z)\,(2x + y + z) - (y + z)\,(z + x)\,(x + y)$
$= 2(x + y + z)^3 + 2xyz$

Solution : We have

$(a + b + c)^3 = a^3 + b^3 + c^3 + 3(b + c)\,(c + a)\,(a + b)$ ....(1)

$\Rightarrow\quad 3(b + c)\,(c + a)\,(a + b) - (a + b + c)^3 = -\,(a^3 + b^3 + c^3)$ ....(2)

Let $a = y + z$, $b = z + x$ and $c = x + y$. Then

$b + c = 2x + y + z$, $c + a = x + 2y + z$, $a + b = x + y + 2z$ and $a + b + c$
$= 2(x + y + z)$.

$\therefore$ (2) becomes $3(2x + y + z)\,(x + 2y + z)\,(x + y + 2z) - 8(x + y + z)^3$

$= -\,[(y + z)^3 + (z + x)^3 + (x + y)^3]$

$= -\,[2\,(x^3 + y^3 + z^3) + 3\{yz\,(y + z) + zx\,(z + x) + xy\,(x + y)\}]$

$= -\,2\,(x^3 + y^3 + z^3) - 3\{yz(y + z) + zx\,(z + x) + xy\,(x + y) + 2xyz\} + 6xyz$

$= -2\,(x^3 + y^3 + z^3) - 3\,(y + z)\,(z + x)\,(x + y) + 6xyz$ (Result VII of § 6.1)

$= -2\,[(x + y + z)^3 - 3\,(y + z)(z + x)(x + y)] - 3\,(y + z)(z + x)(x + y) + 6xyz$
[by using (1)]

$= -\,2(x + y + z)^3 + 3\,(y + z)\,(z + x)\,(x + y) + 6xyz$

$\therefore$ By transposition,

$3\,(x + y + 2z)\,(x + 2y + z)\,(2x + y + z) - 3\,(y + z)\,(z + x)\,(x + y) =$
$6\,(x + y + z)^3 + 6xyz$

Dividing both sides by 3, we get the desired identity.

Example 28. Prove that $(a + b + c)\,(x + y + z) + (a + b - c)\,(x + y - z) +$
$(b + c - a)\,(y + z - x) + (c + a - b)\,(z + x - y) = 4\,(ax + by + cz)$

Solution : Let $b + c - a = l$, $c + a - b = m$ and $a + b - c = n$ Then,

$$m + n = 2a,\ n + l = 2b,\ l + m = 2c \text{ and } l + m + n = a + b + c$$

$\therefore$ L.H.S. $= (l+m+n)(x+y+z)+n\,(x+y-z)+l\,(y+z-x)+m\,(z+x-y)$

$=$

$= 2\,(m + n)\,x + 2\,(n + l)\,y + 2\,(l + m)\,z$

$= 4ax + 4by + 4cz = 4\,(ax + by + cz)$ =R.H.S.

Example 29. Prove that $(y - z)(1 + xy)(1 + xz) + (z - x)(1 + yz)(1 + yx) + (x - y)(1 + zx)(1 + zy) = (y - z)(z - x)(x - y)$

Solution : L.H.S.

$= (y - z)(xy + 1)(xz + 1) + (1 + yz)[(z - x)(1 + xy) + (x - y)(1 + zx)]$

$= (y - z)[x^2yz + x(y + z) + 1] + (1 + yz)[-x^2(y - z) - \{y - z\}]$

(arranging in descending powers of x)

$= (y - z)[x^2yz + x(y + z) + 1 - (1 + yz)(x^2 + 1)]$

$= (y - z)[-x^2 + x(y + z) - yz]$

$= (y - z)[(zx - x^2) - (yz - xy)]$

$= (y - z)[x(z - x) - y(z - x)]$

$= (y - z)(z - x)(x - y)$ = R.H.S.

6.9 Conditional Identities

Let us consider the relation

$(a + b + c)^3 - a^3 - b^3 - c^3 = 3\,(a + b)(b + c)(c + a)$.

This relation is true for all values of a, b, c and therefore it is an identity. If we impose a condition on a, b, c say, $a + b + c = 0$, then the above relation becomes

$$-a^3 - b^3 - c^3 = 3\,(a + b)(b + c)(c + a)$$

i.e. $a^3 + b^3 + c^3 = -3\,(a + b)(b + c)(c + a)$.

Thus, the relation $a^3 + b^3 + c^3 = -3\,(a + b)(b + c)(c + a)$ holds only when $a + b + c = 0$. Such relations which hold under some condition(s) imposed on the symbols (variables) involved, are called conditional identities. We shall now establish some conditional identities.

6.10 If $a + b + c = 0$, then

(i) $a^2 + b^2 + c^2 = -2(ab + bc + ca)$

(ii) $a^3 + b^3 + c^3 = 3abc$

(iii)

(iv)

To establish the conditional identity (i), we have

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$0 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$ [$a + b + c = 0$]

$a^2 + b^2 + c^2 = -2(ab + bc + ca)$

To establish (ii), we have

$a^3 + b^3 + c^3 = 3abc$ [$a + b + c = 0$]

[Alternatively, we can also establish as follows :

$a + b + c = 0$

$a + b = -c$

$(a + b)^3 = (-c)^3$

$a^3 + b^3 + 3ab(a + b) = -c^3$

$a^3 + b^3 + 3ab(-c) = -c^3$

$a^3 + b^3 + c^3 = 3abc$]

To prove (iii), we have

$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) = a^2b^2 + b^2c^2 + c^2a^2$

[$a + b + c = 0$]

Also, $-2(ab + bc + ca) = a^2 + b^2 + c^2$ [from (i)]

Hence, .

To prove (iv), we have

$= 0 \quad [\; a + b + c = 0\,]$

$a^4 + b^4 + c^4 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2$

$= 2\,(a^2b^2 + b^2c^2 + c^2a^2)$

$=$ [from (iii)]

$=$

Hence, .

Example 30. If $a + b + c = 0$, prove that

$(b + c - a)^3 + (c + a - b)^3 + (a + b - c)^3$
$= 3\,(b + c - a)\,(c + a - b)\,(a + b - c) = -\,24abc.$

Solution : Putting $p = b + c - a$, $q = c + a - b$ and $r = a + b - c$, we get

$p + q + r = a + b + c = 0$

Hence, $p^3 + q^3 + r^3 = 3pqr$

i.e. $(b + c - a)^3 + (c + a - b)^3 + (a + b - c)^3$

$= 3(b + c - a)\,(c + a - b)\,(a + b - c)$

Further, $b + c - a = a + b + c - 2a = -\,2a$

$c + a - b = a + b + c - 2b = -2b$

and $a + b - c = a + b + c - 2c = -\,2c$, so that

$(b + c - a)\,(c + a - b)\,(a + b - c) = -\,8abc$

Thus we have,

$(b + c - a)^3 + (c + a - b)^3 + (a + b - c)^3$

$= 3\,(b + c - a)\,(c + a - b)\,(a + b - c) = -\,24abc$

Example 31. If $a + b + c = 0$, prove that

$b^2 + bc + c^2 = c^2 + ca + a^2 = a^2 + ab + b^2 = -\,(bc + ca + ab).$

Solution : $b^2 + bc + c^2 = (b + c)^2 - bc$

$= (b + c)(b + c) - bc$

$= -a(b + c) - bc \qquad (\because b + c = -a)$

$= -ab - ca - bc$

$= -(bc + ca + ab)$

Similarly, $c^2 + ca + a^2 = (c + a)^2 - ca = -b(c + a) - ca$

$= -(bc + ca + ab)$

and $a^2 + ab + b^2 = (a + b)^2 - ab = -c(a + b) - ab$

$= -(bc + ca + ab)$

$\therefore \quad b^2 + bc + c^2 = c^2 + ca + a^2 = a^2 + ab + b^2 = -(bc + ca + ab)$

Example 32. If $a + b + c = 0$, prove that

$bc - a^2 = ca - b^2 = ab - c^2 = bc + ca + ab.$

Solution : $bc - a^2 = bc + a(-a) = bc + a(b + c) \qquad (\because b + c = -a)$

$= bc + ca + ab$;

$ca - b^2 = ca + b(-b) = ca + b(c + a) = bc + ca + ab$;

$ab - c^2 = ab + c(-c) = ab + c(a + b) = bc + ca + ab$

$\therefore \quad bc - a^2 = ca - b^2 = ab - c^2 = bc + ca + ab$

Example 33. If $a + b + c = 0$, prove that

$(b + c)^3 + (c + a)^3 + (a + b)^3 = 3(b + c)(c + a)(a + b) = -3abc.$

Solution : Putting $b + c = p$, $c + a = q$ and $a + b = r$, we get

$p + q + r = 2(a + b + c) = 0$

$\therefore \ p^3 + q^3 + r^3 = 3pqr$

i.e. $(b + c)^3 + (c + a)^3 + (a + b)^3 = 3(b + c)(c + a)(a + b)$

Also, since $a + b + c = 0$, $b + c = -a$, $c + a = -b$ and $a + b = -c$

$\therefore \ 3(b + c)(c + a)(a + b) = 3(-a)(-b)(-c) = -3abc$

Thus, $(b + c)^3 + (c + a)^3 + (a + b)^3 = 3(b + c)(c + a)(a + b) = -3abc.$

Example 34. If $a + b + c = 0$, prove that

$(a + 2b + 3c)^3 + (2a + 3b + c)^3 + (3a + b + 2c)^3$
$= 3\,(b + 2c)\,(a + 2b)\,(c + 2a)$.

Solution : Let $a + 2b + 3c = p,\ 2a + 3b + c = q$ and $3a + b + 2c = r$

Then $p + q + r = 6\,(a + b + c) = 0$

$p^3 + q^3 + r^3 = 3pqr$

i.e. $(a + 2b + 3c)^3 + (2a + 3b + c)^3 + (3a + b + 2c)^3$

$= 3\,(a + 2b + 3c)\,(2a + 3b + c)\,(3a + b + 2c)$

$= 3\,(a + b + c + b + 2c)\,(a + b + c + a + 2b)\,(a + b + c + c + 2a)$

$= 3\,(b + 2c)\,(a + 2b)\,(c + 2a)$

Example 35. Prove that $2(s - a)\,(s - b)\,(s - c) + a\,(s - b)\,(s - c) + b\,(s - c)\,(s - a)$
$+ c\,(s - a)\,(s - b) = abc$, if $2s = a + b + c$

Solution : We have

$2\,(s - a)\,(s - b)\,(s - c) = 2\,[s^3 - s^2\,(a + b + c) + s\,(bc + ca + ab) - abc]$

$= 2\,[s^3 - s^2\,.\,2s + s\,(bc + ca + ab) - abc]$

$= -\,2s^3 + 2s\,(bc + ca + ab) - 2abc$ .... (i)

Also, $a\,(s - b)\,(s - c) + b\,(s - c)\,(s - a) + c\,(s - a)\,(s - b)$

$= a\,[s^2 - (b + c)\,s + bc] + b\,[s^2 - (c + a)\,s + ca] + c\,[s^2 - (a + b)\,s + ab]$

$= s^2\,(a + b + c) - s\,[a\,(b + c) + b\,(c + a) + c\,(a + b)] + 3abc$

$= 2s^3 - 2s\,(bc + ca + ab) + 3abc$ .... (ii)

Adding (i) and (ii), we obtain

EXERCISE 6.2

Prove that (1 – 15)

1. $(y - z)^3 + (z - x)^3 + (x - y)^3 = 3\,(y - z)\,(z - x)\,(x - y)$

2. $(b + c - a)^3 + (c + a - b)^3 + (a + b - c)^3 + 24abc$
$= (2a + b - c)^3 + (b + c)^3 - (a + b - c)^3 - 6a\,(a + b)\,(a - 2c)$

3. $ax + by + cz = (a + b + c)\,(x + y + z)$ if $x = a^2 - bc,\ y = b^2 - ca,\ z = c^2 - ab$.

4. $x^3 + y^3 + z^3 - 3xyz = (a^3 + b^3 + c^3 - 3abc)^2$ if $x = a^2 - bc,\ y = b^2 - ca,\ z = c^2 - ab$.

5. $a^2x + b^2y + c^2z = (x + y + z)(a^2 + b^2 + c^2)$,
if $a^2 = x^2 - yz,\ b^2 = y^2 - zx$ and $c^2 = z^2 - xy$

6. $s(s - a)(s - b) + s(s - c)(s - a) + s(s + a)(s - c) + c(s + a)(s + b)$
$= (s + a)(s + b)(s + c)$, if $s = a + b + c$

7. $(s - a)^3 + (s - b)^3 + (s - c)^3 - 3(s - a)(s - b)(s - c)$
= ______, if $2s = a + b + c$

8. $a^3(b - c)^3 + b^3(c - a)^3 + c^3(a - b)^3 = 3abc(b - c)(c - a)(a - b)$

9. $(x - y)(x + y - 2z)^3 + (y - z)(y + z - 2x)^3 + (z - x)(z + x - 2y)^3 = 0$

10. $(s - a)^3 + (s - b)^3 + (s - c)^3 = s^3 - 3abc$, if $2s = a + b + c$

11. $2a(b + c - a) + (c + a - b)(a + b - c)$
$= 2b(c + a - b) + (a + b - c)(b + c - a)$
$= 2c(a + b - c) + (b + c - a)(c + a - b)$
$= (c + a - b)(a + b - c) + (a + b - c)(b + c - a) + (b + c - a)(c + a - b)$

12. $a(b - c)^3 + b(c - a)^3 + c(a - b)^3 = (b - c)(c - a)(a - b)(a + b + c)$

13. $(a^2 + b^2 + c^2)(p^2 + q^2 + r^2) - (ap + bq + cr)^2 = (aq - bp)^2 + (br - cq)^2 + (cp - ar)^2$

14. $x(y - z)(1 + xy)(1 + zx) + y(z - x)(1 + yz)(1 + yx)$
$+ z(x - y)(1 + zx)(1 + zy) = xyz(y - z)(z - x)(x - y)$

15. $(b - c)(1 + a^2b)(1 + a^2c) + (c - a)(1 + b^2c)(1 + b^2a) + (a - b)(1 + c^2a)(1 + c^2b)$
$= -abc(a + b + c)(b - c)(c - a)(a - b)$

16. If $x + y + z = a,\ yz + zx + xy = b$ and $xyz = c$, prove that
$a^3 - 3ab + 3c = x^3 + y^3 + z^3$

17. If $2s = a + b + c$ and $2t^2 = a^2 + b^2 + c^2$, show that
$(t^2 - a^2)(t^2 - b^2) + (t^2 - b^2)(t^2 - c^2) + (t^2 - c^2)(t^2 - a^2) = 4s(s - a)(s - b)(s - c)$

18. If $s = a + b + c$, prove that

$(s - 3a)^2 + (s - 3b)^2 + (s - 3c)^2 = 3\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$

19. If $a + b + c = 1$, prove that

$(a + bc)(b + c) = (b + ca)(c + a) = (c + ab)(a + b) = (1 - a)(1 - b)(1 - c)$

20. If $s = a + b + c$, show that $(s - a)(s - b)(s - c) = (a + b + c)(bc + ca + ab) - abc$

21. If $a + b + c = 0$, prove that

(i) $a\,(a + b)\,(a + c) = b\,(b + c)\,(b + a) = c\,(c + a)\,(c + b) = abc$

(ii) $a^2\,(b + c) + b^2\,(c + a) + c^2\,(a + b) = 3\,(b + c)\,(c + a)\,(a + b)$

(iii) $a\,(b - c)^3 + b\,(c - a)^3 + c\,(a - b)^3 = 0$

(iv)

(v)

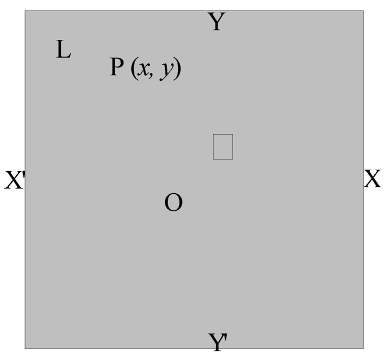
CHAPTER 7
TRIGONOMETRY

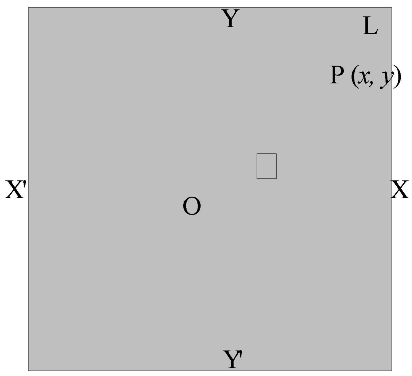
7.1 Introduction

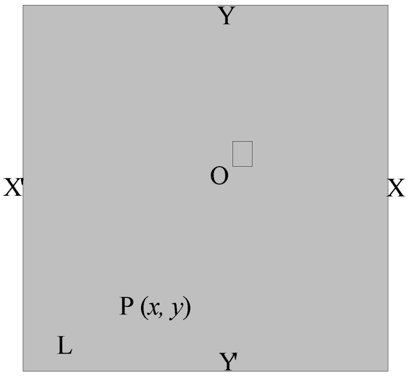
You are already familiar with the trigonometric ratios of an acute angle. The definitions of the trigonometric ratios were given with reference to a right triangle. But in Trigonometry, you know that angles can be of any sign i.e. positive or negative and of any magnitude. Indeed, we can talk about angles like – 45°, 390°, – 215°, 7200°, 980° etc. When it comes to the definitions of trigonometric ratios of angles like these, it is not that simple as the ones for an acute angle as given in the previous class. In this chapter, we shall give general definitions of trigonometric ratios of angles of any sign and magnitude. We shall then discuss associated or allied angles and their trigonometric ratios.

7.2 Trigonometric ratios of angles of any sign and magnitude

Let a revolving line OL start from the initial position OX and trace out an angle θ. The revolution is anti-clockwise or clockwise according as θ +ve or – ve. Also, the final position of the revolving line can be anywhere around O in the plane of the rectangular cartesian coordinate system according to the magnitude of θ (Fig. 7.1)







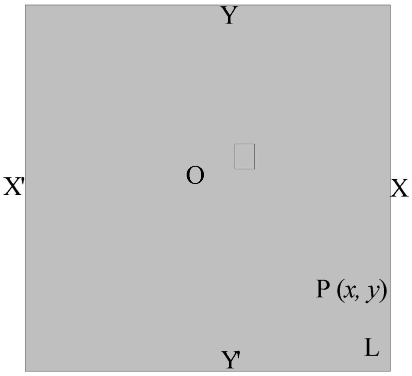


Fig. 7.1

Let us take any point P (x, y) other than the origin on the final position of the revolving line OL and let OP = r (> 0). Then the trigonometric ratios of the angle are defined as

and

If $x = 0$, then tan and sec are not defined and if $y = 0$, then cot and cosec are not defined.

Here, r is always positive but x and y can be positive, negative or zero according to the final position of the revolving line OL. Also, we say that the angle lies in the 1st quadrant, 2nd quadrant, 3rd quadrant, 4th quadrant according as the final position of the revolving line is in the 1st quadrant, 2nd quadrant, 3rd quadrant, 4th quadrant respectively. Further, you see for yourself that these general definitions of trigonometric ratios of agree with the definitions given in previous class when is acute.

7.3 Signs of Trigonometric ratios

In the previous section, we have defined the trigonometric ratios of an angle . As you see, each of the trigonometric ratios of is a ratio of two of r, x and y. Since r is always positive, the sign of a trigonometric ratio (whether +ve or –ve) depends upon the signs of the coordinates x and y of the point P, which in turn depend upon the quadrant in which lies. Let us now check the signs of the trigonometric ratios of an angle when lies in different quadrants.

(i) If □ $x > 0$, $y > 0$, $r > 0$.

□ all the trigonometric ratios of □ are positive.

(ii) If □ lies in the 2nd quadrant, then $x < 0$, $y > 0$, $r > 0$.

□ only sin □ □ are positive and others negative.

(iii) If □ $x < 0$, $y < 0$, $r > 0$.

□ only tan □ and cot □ are positive and others negative.

(iv) If □ lies in the 4th quadrant, then $x > 0$, $y < 0$, $r > 0$.

□ only cos □ and sec □

The results can be easily remembered by the quadrant rule : "all, sin, tan, cos" (see *Fig.* 7.2)

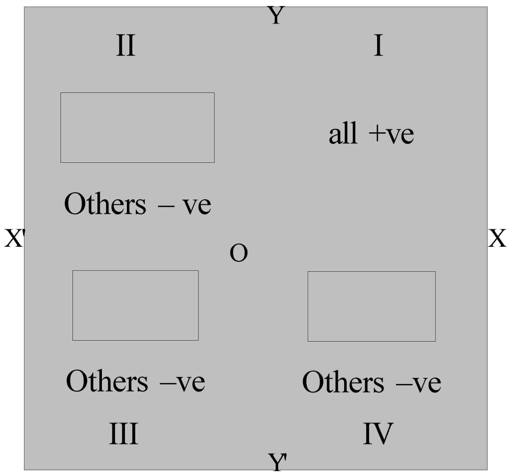


Fig. 7.2

7.4 Allied (or Associated) Angles

Two angles are said to be allied to (or associated with) each other if their sum or difference is a multiple of 90°

Thus, the angles – □, 90° □ □, 180° □ □, 270° □ □, 360° □ □, etc. are angles allied to □ (measured in degrees). In general, any angle of the form $n \times 90°$ □ □, n□Z, is allied to (associated with) □. In this chapter, we shall find the trigonometric ratios of angles allied to □ in terms of those of □.

7.5 Trigonometric ratios of (–□) in terms of those of □

Let a revolving line OA starting from the initial position OX trace out an acute angle □ in the anti-clockwise sense. Let there be another revolving line OA′ which starts from the initial position OX and traces out angle – □ in the clockwise sense.

Let P (x, y) be any point on OA such that $OP = r$. We draw PM produce it to meet OA′ at P′.

Then, the coordinates of P′ are $(x, -y)$ and $OP' = r$ (′M).

and

Examples :

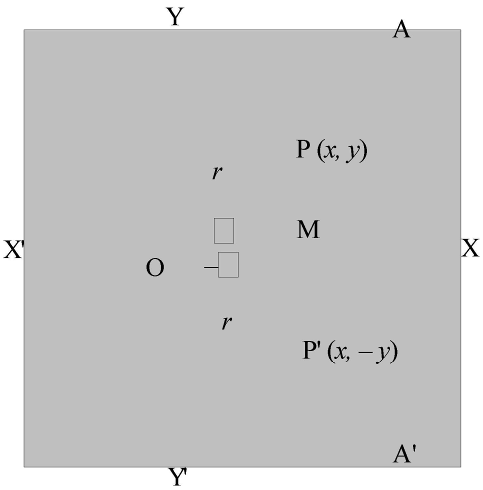


Fig. 7.3

7.6 Trigonometric ratios of (90°–) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another revolving line OA′ starting from the initial position OX first trace out an angle 90° and then revolves backwards through an angle position of the line, ′ = 90° – .

Let P (x, y) and P′ (x', y') be two points on OA and OA′ respectively such that $OP = OP' = r$. We draw PM OX and P′M′ OX. Then, clearly triangle OPM and P′OM′ are congruent.

So, we have

$x' = y$ and $y' = x$ (same sign)

Now,

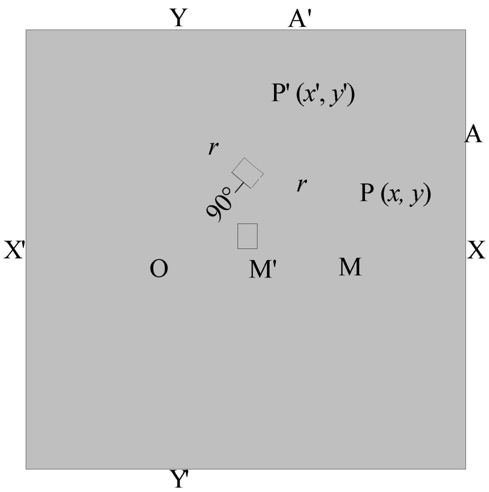


Fig. 7.4

and

Examples :

7.7 Trigonometric ratios of (90°+) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another line OA′ starting from the initial position OX first trace out an angle 90° and further revolve through an angle in the anti-clockwise direction so that in the final position of the line, ′ = 90° + .

Let P(x, y) and P′(x', y') be two points on OA and OA′ respectively such that OP = OP′ = r. We draw PM OX and P′M′ OX.

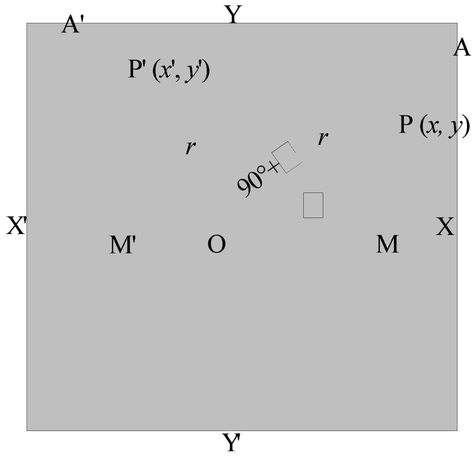


Fig. 7.5

Now, in the right triangles OPM and OP′M′, PMO = P′M′O = 90°, POM = OP′M′ and OP = OP′.

OPM P′OM′

So, we have

$x' = -y$ (x' and y have opposite signs)

and $y' = x$ (x and y' have the same sign)

Now,

and

Examples :

7.8 Trigonometric ratios of (180° –) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another line OA′ starting from the initial position OX first trace out an angle 180° and then revolves backwards through an angle so that in the final position of the line, ′ = 180 – .

Ler P (x, y) and P′ (x', y') be two points on OA and OA′ respectively such that OP = OP′ = r. We draw PM OX and P′M′ OX.

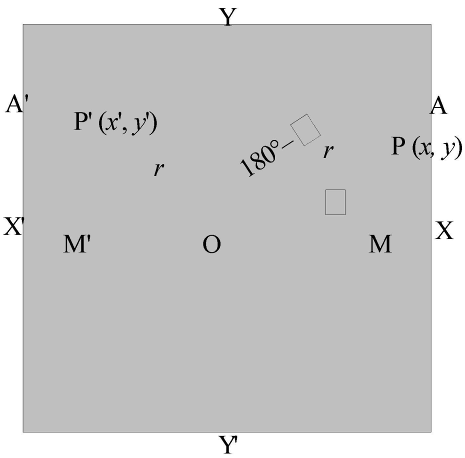


Fig. 7.6

Now, in the right triangles OPM and OP′M′,

PMO = P′M′O = 90°,

POM = P′OM′ and OP = OP′.

OPM OP′M′

So, we have

$x' = -x$ (x and x' have opposite signs)

and $y' = y$ (y and y' have the same sign)

Now,

and

Examples :

7.9 Trigonometric ratios of (180°+) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another line OA′ starting from the initial position OX first trace out an angle 180° and further revolve through an angle in the anti-clockwise direction so that in the final position of the line, ′ = 180° + .

Ler P (x, y) and P′ (x', y') be two points on OA and OA′ respectively such that OP = OP′ = r. We draw PM OX and P′M′ OX.

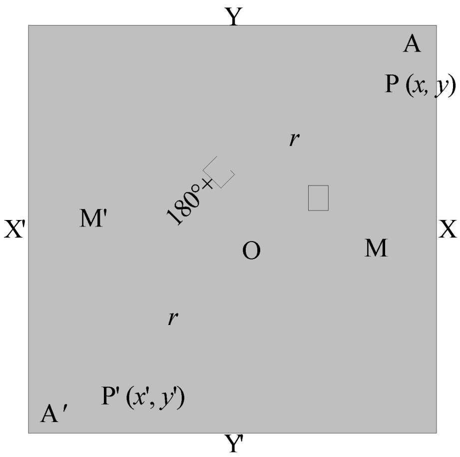


Fig. 7.8

Now, in the right triangles OPM and OP′M′,

PMO = P′M′O = 90°,

POM = P′OM′ and OP = OP′.

OPM OP′M′

So, we have

$x' = -x$ (x and x' have opposite signs)

and $y' = -y$ (y and y' have opposite signs)

Now,

and

Examples :

7.10 Trigonometric ratios of (270°–) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another line OA′ starting from the initial position OX first trace out an angle 270° and revolves backwards through an angle so that in the final position of the line, XOA′ = 270° – .

Ler P (x, y) and P′ (x', y') be two points on OA and OA′ respectively such that OP = OP′ = r. We draw PM OX and P′M′ OX.

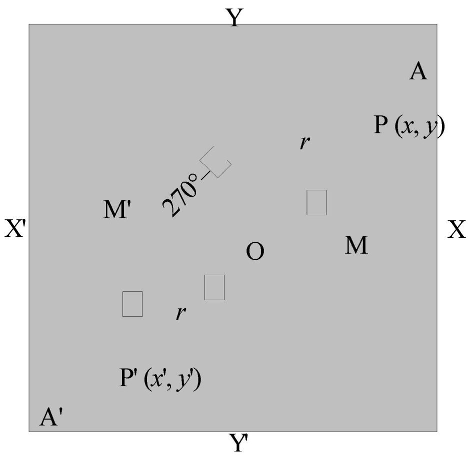


Fig. 7.9

Now, in the right triangles OPM and OP′M′,

PMO = P′M′O = 90°,

POM = OP′M′ and OP = OP′.

OPM P′OM′

So, we have

$x' = -y$ (x' and y have opposite signs)

and $y' = -x$ (x and y' have opposite signs)

Now,

and

Examples :

7.11 Trigonometric ratios of (270°+) in terms of those of

Let a revolving line OA starting from the initial position OX trace out an acute angle . Let another line OA′ starting from the initial position OX first trace out an angle 270° and further revolves through an angle
the final position of the line, ′ = 270° + .

Let P(x, y) and P′(x', y') be two points on OA and OA′ respectively such that OP = OP′ = r. We draw PM OX and P′M′ OX.

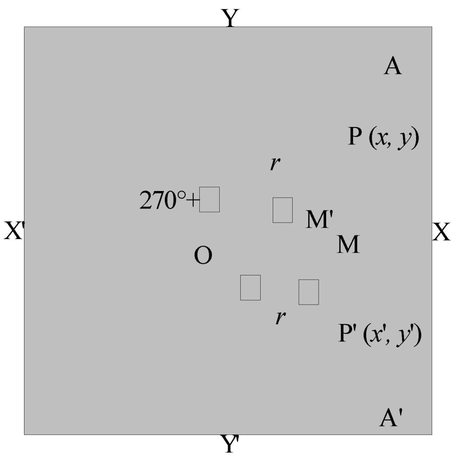


Fig. 7.10

Now, in the right triangles OPM and OP′M′,

PMO = P′M′O = 90°,

POM = OP′M′ and OP = OP′.

OPM P′OM′

So, we have

$x' = y$ (x' and y have the same sign)

and $y' = -x$ (x and y' have opposite signs)

Now,

and

Examples :

7.12 Trigonometric ratios of (360°) in terms of those of

When a revolving line OA starting from the initial position OX makes a complete revolution in the anti-clockwise direction, it traces out an angle 360° and the line returns to the initial position OX. So, when the revolving line traces out an angle (360°+), the final position of the line will be the same as that of the line tracing out an angle . Hence, the trigonometric ratios of (360° +) are the same as those of . By a similar argument, the trigonometric ratios of (360° –) are the same as those of (–).

sin (360° –) = sin (–) = – sin

cos (360° –) = cos (–) = cos

tan (360° –) = tan (–)= – tan

cot (360° –) = cot (–) = – cot

sec (360° –) = sec (–) = sec

cosec (360°–) = cosec (–

And, sin (360° +) = sin

cos (360° +) = cos

tan (360° +) = tan

cot (360° +) = cot

sec (360° +) = sec

cosec (360° +) = cosec

Here, it can be easily seen that any multiple of 360° can be added or subtracted from an angle without altering the trigonometric ratios of the angle.

Thus, for any integer n, sin ($n \times 360°$) = sin ( ) and so on.

Note : The results established for allied angles based on the assumption that is acute are also valid for any value of without restriction.

Example 1. Show that

(i) (ii) (iii)

Solution : (i)

[or,]

(ii) cos 420°

$= \cos (360° + 60°)$

$= \cos 60° =$

(iii) tan (– 1050°) = – tan 1050°, (tan (–

$= - \tan (3 \times 360° - 30°)$

$= - \tan (- 30°)$

$= - (- \tan 30°) = \tan 30°$

=

Example 2. Find the values of the sine, cosine and tangent of the following angles :

(i) 120° (ii) – 480° (iii) 495°

Solution : (i)

(ii) $\sin(-480^\circ) = -\sin 480^\circ$ ($\sin(-$ $) = -\sin$

$= -\sin(360^\circ + 120^\circ)$

$= -\sin 120^\circ = -\sin(90^\circ + 30^\circ)$

$=$

$\cos(-480^\circ) = \cos 480^\circ$ ($\cos(-$

$= \cos(360^\circ + 120^\circ)$

$= \cos 120^\circ = \cos(90^\circ + 30^\circ)$

$= -\sin 30^\circ = -$

$\tan(-480^\circ) = -\tan 480^\circ$

$= -\tan(360^\circ + 120^\circ)$

$= -\tan 120^\circ = -\tan(90^\circ + 30^\circ)$

$= -(-\cot 30^\circ) = \cot 30^\circ =$

(iii) $\sin 495^\circ = \sin(360^\circ + 135^\circ) = \sin 135^\circ$

$= \sin(90^\circ + 45^\circ)$

$=$

$\cos 495^\circ = \cos(360^\circ + 135^\circ) = \cos 135^\circ$

$= \cos(90^\circ + 45^\circ)$

$=$

$\tan 495^\circ = \tan(360^\circ + 135^\circ) = \tan 135^\circ$

$= \tan(90^\circ + 45^\circ)$

$= -\cot 45^\circ = -1$

Example 3. Simplify :

(i) $\sin 420^\circ \cos 390^\circ + \cos(-660^\circ)\sin(-330^\circ)$

(ii)

(iii)

Solution : (i) The given expression

= $\sin 420° \cos 390° + \cos(-660°) \sin(-330°)$

= $\sin(360° + 60°) \cos(360° + 30°) + \cos(-2 \times 360° + 60°)$
$\sin(-360° + 30°)$

= $\sin 60° \cos 30° + \cos 60° \sin 30°$

=

=

(ii) The given expression

=

=

=

=

=

= .

(iii) We have

and

the given expression

=

=

=

7.13 Solution of Trigonometric Equations

An equation involving one or more trigonometric functions (ratios) of a variable (angle) is called a trigonometric equation. A value of the unknown angle which satisfies a trigonometric equation is called a solution or a root of the equation. And, to solve a trigonometric equation means to find the solutions (roots) of the equation.

In the previous articles, we have seen that the values of a trigonometric ratio of coterminus angles (angles whose terminal sides are the same) are the same. Thus, in general, a trigonometric equation has infinitely many solutions for if is a solution of a trigonometric equation, then the infinitely many angles which are coterminus with are also solutions of the equation. The solutions
$0°$ $< 360°$ are referred to as principal solutions. And, the infinitely many solutions of a trigonometric equation constitute the general solution of the equation which in most cases can be represented by an expression in terms of n Z.

Let us consider the equation .

We know that and i.e.

Thus, $= 30°$ and

Now, the angles coterminus with 30° are given by $n \times 360° + 30°$, n Z and they are .... – 690°, – 330°, 30°, 390°, 750°, ... (i)

And, the angles coterminus with 150° are given by $n \times 360° + 150°$, n Z and they are .... – 570°, –210°, 150°, 510°, 870°, ...(ii)

Now, all the angles in (i) and (ii) are the solutions of the given equation. They constitute the general solution of the equation and they can be represented by the expression $n \times 180° + (-1)^n 30°$, n Z (as you can check out).

In this chapter, we shall not be concerned with the general solution of a trigonometric equation which will be dealt with in higher classes. Here, we shall be concerned with only solutions in a given range.

Example 4. Solve (i)

(ii) cot + tan =2 cosec , (0° < < 360°)

Solution : (i)

Case I :

Now, we have ,

Also,

i.e.

Here, the principal solutions are 30° and 150°.

And, the angles in the range – 360° < < 360° and coterminus with 30° and 150° are – 360° + 30°, – 360° + 150°, 30° and 150° i.e. – 330°, –210°, 30° and 150°.

(***Note :*** To find angles coterminus with a given angle, add integral multiples of 360° to the angle.)

Case II :

We have,

i.e.

Also,

i.e.

The principal solutions are 210° and 330°.

And, the angles in the range – 360° < <360° and coterminus with 210° and 330° are – 360° + 310°, – 360° + 330°, 210° and 330° i.e. – 150°, – 30°, 210° and 330°.

Combining the two cases, we have the solutions are = 30°, 150°, and

(ii) cot + tan = 2 cosec (0° < < 360°)

[illegible] 2 sin [illegible] cos [illegible] – sin [illegible] = 0

[illegible] sin [illegible] (2 cos [illegible] – 1) = 0

[illegible] Either sin [illegible] = 0 or 2 cos [illegible] – 1 = 0

Here, sin [illegible] = 0 is neglected because if sin [illegible] = 0, then [illegible] and [illegible] are undefined.

[illegible] We have, 2 cos [illegible] – 1 = 0

[illegible] [illegible]

= cos (360° – 60°)

= cos 300°.

Here, the principal solutions are 60° and 300° and they are the only solutions in the given range 0° < [illegible] < 360°.

[illegible] the solutions are [illegible]

Remark : To find solutions of a trigonometric equation in a given range, find the principal solutions first and then find the angles in the given range which are coterminus with the principal solutions.

EXERCISE 7.1

1. Write down the values of the trigonometric ratios of the following angles :

   (i) – 150° (ii) 690° (iii) 840° (iv) – 1530°

2. Find the value of :

   (i) sin 4620° (ii) cos 870° (iii) [illegible] (iv) [illegible]

3. Show that :

   (i) sin (540° + [illegible]) = – sin [illegible] (ii) cot ([illegible] – 630°) = – tan [illegible]

   (iii) [illegible] (iv) sin (– 360° – [illegible]) = – sin [illegible]

4. Simplify :

   (i) sin 405° cos 300° – cos 420° sin 225°

   (ii) sin 420° cos 390° + cos (– 660°) sin (–330°)

(iii)

(iv)

(v)

(vi) cos 24° + cos 55° + cos 125° + cos 204° + cos 300°

5. Show that :

(i)

(ii)

(iii)

(iv) cos n = $(-1)^n$, where n is any integer (positive or negative or zero)

(v) , where n is zero, or any integer (positive or negative)

(vi) , where n is any integer.

6. Solve for

(i) $\sec^2$ = 2

(ii)

(iii) $\tan^2$ + $\cot^2$ = 2

(iv) 2 $\cos^2$ – 3 cos + 1 = 0

(v)

7. If A, B, C denote the angles of a triangle, prove that

(i)

(ii)

(iii)

(iv)

(v)

8. (a) If A, B, C, D are the angles of a quadrilateral, then show that

(i)

(ii)

(iii)

(b) If the quadrilateral ABCD is cyclic, then prove that

(i) $\sin B = \sin D$

(ii) $\cos A + \cos B + \cos C + \cos D = 0$

(iii) $\tan A + \tan B + \tan C + \tan D = 0$.

9. Show that

(i)

(ii)

(iii)

ANSWER

1. (i)

(ii)

(iii)

(iv) –1, 0, undefined, 0, undefined, –1.

2. (i) (ii) (iii) (iv) –1

4. (i) (ii) 1 (iii) 2 (iv) 9 (v) 1 (vi)

6. (i) 45º, 135º, 225º, 315º

(ii) 60º, – 300°

(iii) 45º, 135º, 225º, 315º

(iv) 60º, 300º

(v) 30º, 120º, 210º, 300º, – 60°, –150°, – 240°, – 330°

CHAPTER 8
STATICS

8.1 Introduction

We first get ourselves familiar with some fundamental concepts and terms.

Matter is anything that occupies space and can be perceived by our senses.

A **body** is a portion of matter limited in all directions, having a definite shape and size and occupying some definite space.

A **force** is that which changes or tends to change, the state of rest or of uniform motion of a body.

A **rigid body** is one whose size and shape do not alter when acted on by any forces whatsoever, so that the distance between any pair of particles in it remains invariable.

A **perfectly rigid body** does not exist in nature. Bodies do change their shape as well as size to some extent under great pressure. But, under ordinary forces, however, the alterations are very slight and in many cases can be ignored. Thus, in problems where action of forces on bodies are concerned, unless otherwise stated, we assume bodies to be perfectly rigid.

A **particle** is a body of infinitely small dimensions. When we speak of a body as a particle, we mean that we are not concerned with its actual dimensions and that we can represent its position simply by a mathematical point.

Mechanics is that branch of Science which deals with the action of forces on bodies. When acted upon by forces, a body may move or remain at rest relative to its surroundings and accordingly Mechanics has two parts namely Dynamics and Statics.

Dynamics is that part of Mechanics which deals with forces which do not neutralise and therefore cause non-uniform motion and this has been studied in class IX.

Statics is that part of Mechanics which deals with bodies at rest when acted on by forces or with the relations between the forces which keep a rigid body (or a system of bodies) at rest.

Equilibrium

If a system of forces acting on a body keeps it at rest, then the forces are said to be in equilibrium.

8.2 Representation of a force

A force has a given magnitude and acts at a particular point of a body in a definite direction. In other words, a force has a definite magnitude and direction and as such it is a vector quantity.

Now, a line segment has also a length and a direction and can be drawn through a particular point. Thus, a line segment drawn through the point of application of a force can represent the force completely in magnitude, direction and position, the magnitude of the force being represented on a suitably chosen scale by the length of the line segment drawn, the direction of the line segment representing the direction of the force, the extremity of the line segment being at the point of application of the force.

For example, if the line segment AB represents a force R, the direction of the line segment from A to B represents the direction of R and the length AB of the segment represents the magnitude of R on some suitable scale.

Fig. 8.1

If instead of AB, we write BA, it represents the same force with its direction reversed. The force represented by the line segment AB is denoted in vector notation by

Note : 1. The sense of direction of a force represented by a line segment AB is from A to B.

2. The magnitude of a force is proportional to the length of the line segment representing it.

8.3 The principle of transmissibility of a force

The effect of a force acting on a rigid body at any point is unaltered if its point of application is transferred to any other point on its line of action, provided the two points are in the body.

Suppose a force P acts at a point A of a rigid body along the line AX. At any other point B of the body in AX, introduce along the same line, two equal and opposite forces each equal to P. The force P along AB and the force P along BA, being equal and opposite, balance each other and we are left with the force P acting at B along BX, which is thus equivalent to the original force P at A.

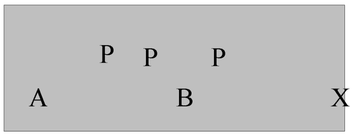


Fig. 8.2

8.4 Some special forces

(i) Weight

The weight of a body is the force with which the earth attracts the body. It is proportional to the mass of the body, i.e., the quantity of matter in the body and its direction is vertically downwards.

(ii) Reaction

According to Newton's third law of motion, to every action there corresponds an equal and opposite reaction. Thus in a system of two bodies, A and B if A exerts a force P (action) on B, then the body B also exerts an equal force P (reaction) in the opposite direction on A.

(iii) Tension

When a string is used to support a weight or to drag a body, the force exerted is transmitted to the body through the string. Such a force exerted by means of a string is called Tension.

If the string is of negligible weight, the tension is the same throughout its length and is unchanged even when a portion of the string passes over a smooth surface, say, a smooth peg or pulley.

If however, a string be knotted at any of its points to other strings or to weights, the tension will not in general, be the same in the different portions separated by the knots.

8.5 Resultant and Components

If two or more forces act simultaneously on a rigid body and if a single force can be obtained whose effect on the body is the same as the joint effect of the given forces (i.e. produces exactly the same state of motion of the body), then this single force is known as the resultant of the given forces, and the given forces, in turn, are called the components of the single resultant force.

Now, we shall proceed to find the resultant of two forces acting at a point on a rigid body.

8.6 Parallelogram of forces

Statement : If two forces acting at a point on a body be represented in magnitude, direction and sense by the two adjacent sides of a parallelogram drawn from an angular point, then their resultant is represented in magnitude, direction and sense by the diagonal of the parallelogram drawn from that point.

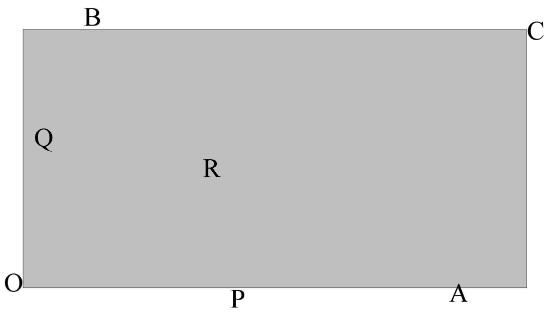


Fig. 8.3

Thus, if two forces P and Q, acting on a body at a point O, be represented in magnitude, direction and sense by the two lines OA and OB respectively both drawn from O, and the parallelogram OACB be completed with OA and OB as adjacent sides, then the resultant force say R, will be represented in magnitude, direction and sense by the diagonal OC drawn from O. Using vector notation, this law can be stated as

8.7 Analytical expression for the resultant of two given forces

Let the two forces P and Q acting at a point O at an angle to each other be represented by OA and OB respectively. Complete the parallelogram OACB and join the diagonal OC, which then, by parallelogram of forces, represents the resultant R. Let which will give the direction of the resultant. Now, draw (produced if necessary).

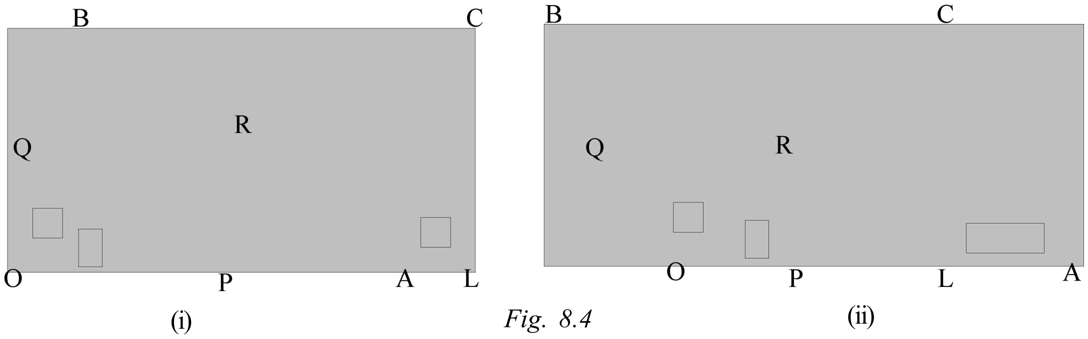


Fig. 8.4

Then, in fig. 8.4 (i), we have

Now,

= P+Q (

Also, from we have

CL=AC

Again, in fig. 8.4 (ii), we have

Now, OL=OA–LA=OA–AC

= OA+AC

= P+Q

Also, from we have

=

Thus, in both the figures, we have

and

Now, from the right triangle OCL, we have

$OC^2 = OL^2 + CL^2$

R^2 = (P+Q ^{2} +

.... (i)

And, in we have

.... (ii)

Thus, (i) and (ii) give respectively the magnitude and direction of the resultant.

Observation : In both the figures, we have

and

Thus,

Corollary 1. If i.e. if the two forces P and Q are perpendicular to each other, then

and (

Corollary 2. The greatest and the least values of the resultant

Since R will be the greatest when is the greatest i.e., when =1 or when

Then,

Also, R will be least when is least i.e., when =–1 or when

Then,

Thus, the greatest value of the resultant R is P+Q and the least value is P–Q or Q–P according as P>Q or Q>P.

8.8 Resolution (breaking up) of a given force into two components

A given force may be resolved into two components in an infinite number of ways, for, by parallelogram of forces, if with the line segment representing the given force as diagonal, we construct any parallelogram, the two adjacent sides of this parallelogram will represent the two component forces having the given force as their resultant and we can construct infinitely many such parallelograms.

If, however, with a given force, both the directions are definitely given in which we are to break it up into components, then these components can be uniquely determined.

Let OC represent the given force R and OX and OY be two given directions (not necessarily perpendicular) making angles and respectively with OC, on opposite sides of it, along which we are to find the components of R.

Complete the parallelogram OACB with diagonal OC and sides along OX and OY. Then, by parallelogram of forces, OA and OB represent the required components P and Q, having R as their resultant.

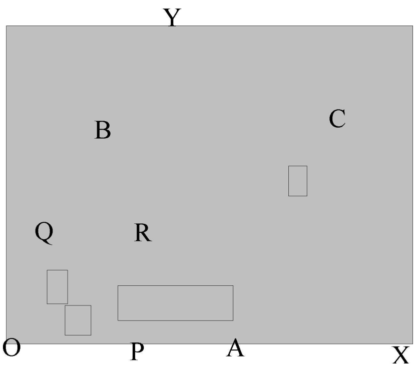


Fig. 8.5

Now, in the we have

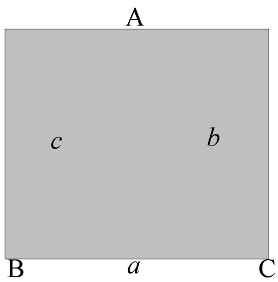
and

By sine formula*, we have

and

Thus, the components of R along OX and OY are and respectively.

* **Sine formula :** In a triangle ABC,



where a=BC, b=AC and c=AB.

You will be dealing with this formula in Trigonometry in higher classes.

8.9 Resolution of a force into two perpendicular components

If the two components of a given force are along two perpendicular directions, then these components are called the resolved parts of the force along those directions.

If we have in the last article, then P and Q are the resolved parts of the force R. Then from the result obtained in the previous article, we have

(

and

Thus, the resolved parts of R along and perpendicular to OX are respectively and angle between R and OX.

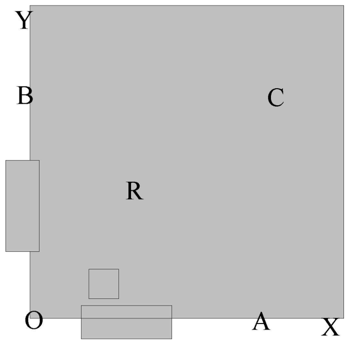


Fig. 8.6

Observe that the resolved part of a force in any direction = the force × the cosine of the angle which the force makes with the given direction.

Also, the resolved part of a force R in a direction at right angles to itself = R cos 90º = 0. Thus, a force has no effect in a direction perpendicular to itself.

Hence, any given force R is mathematically equivalent to (and accordingly, can be replaced, whenever needed, by) two resolved parts, one along a direction OX at an angle to it, and another perpendicular to OX. This mode of replacing, a given force by its two equivalent resolved parts in two suitable perpendicular directions is particularly useful in finding the resultant of several forces simultaneously acting at a point, as is shown in section 8.13.

Note : Resolved part of the force R represented by OC, along the direction OX is represented by OA where A is the foot of perpendicular from C upon OX.

Theorem. The algebraic sum of the resolved parts of any two forces acting at a point, along any direction, is equal to the resolved part of their resultant, in that direction.

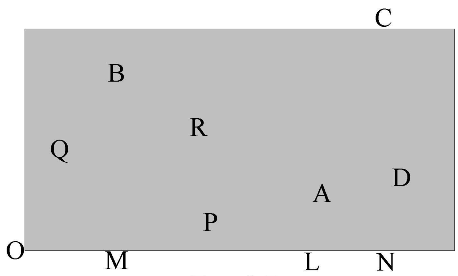


Fig. 8.7

Let OA and OB represent the two forces P and Q acting at a point O. Complete the parallelogram OACB. Then the diagonal OC represents the resultant R.

Let OX be a line drawn in any direction through O and let AL, BM and CN be the perpendiculars drawn on it from A, B and C respectively so that OL, OM and ON represent the resolved parts of P, Q and R respectively along OX.

From Fig. 8.7, we have

$$OM = AD = LN$$

Now, $ON = OL + LN = OL + OM$

Thus, the resolved part of the resultant R along OX is equal to the algebraic sum of the resolved parts of P and Q along OX.

Corollary : The above theorem may be generalised as follows :

If any number of forces act at a point, the algebraic sum of their resolved parts in any direction is equal to the resolved part of their resultant in that direction.

8.10 Equilibrium of Concurrent Forces

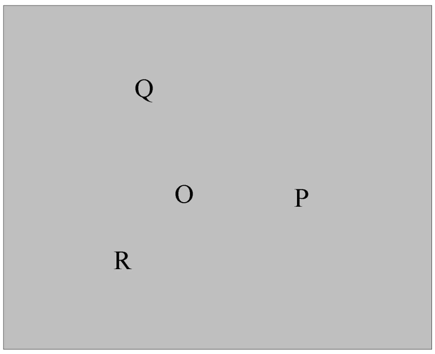
Recall that if a number of forces acting upon a body (or a particle) keep it at rest, then the forces are said to be in equilibrium.

We know that a number of forces acting on a body may be compounded into a single force, called the resultant of the forces, by the parallelogram law of forces or by the method of resolved parts. If the body is to be at rest, then the resultant of all the forces acting on it must vanish.

When three concurrent forces are in equilibrium, we have a useful law called "Triangle of forces" which can be applied in such a situation. This law along with its converse, is discussed below.

8.11 Triangle of Forces

Statement : If three forces acting at a point be such as can be represented in magnitude, direction and sense (but not in position) by the three sides of a triangle taken in order, then the forces are in equilibrium.



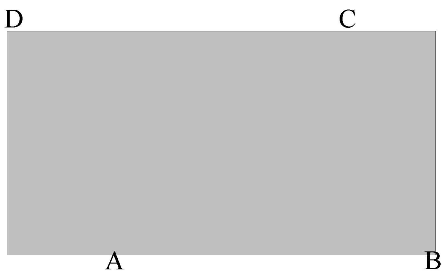


Fig. 8.8

Let the three forces P, Q, R acting at a point O be represented in magnitude, direction and sense respectively by the sides AB, BC, CA taken in order of the triangle ABC. Complete the parallelogram ABCD. Since AD is equal and parallel to BC, the force Q which is represented by BC can as well be represented in magnitude and direction by AD.

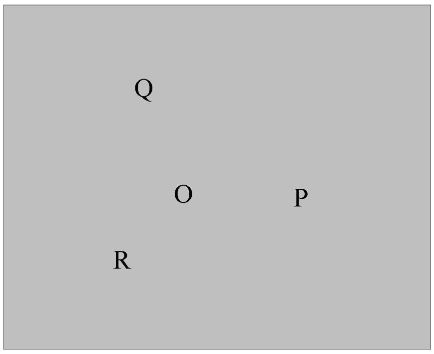
By parallelogram of forces, resultant of P and Q is represented by AC. Thus, we are left with two forces acting at the point A, represented by AC and CA. But, AC and CA are equal in magnitude and opposite in direction and hence they balance each other. Thus, the resultant of P, Q, R must vanish.

Hence, the three forces are in equilibrium.

8.12 Converse of Triangle of Forces

Statement : If three forces acting at a point be in equilibrium, then they can be represented in magnitude, direction and sense by the three sides of a triangle, taken in order.

Let the three forces P, Q, R acting at O be in equilibrium. Draw the line segments AB, BC, parallel to the directions of P and Q, to represent these forces respectively in magnitude, direction and sense, on any chosen scale. Complete the parallelogram ABCD and join the diagonal AC.



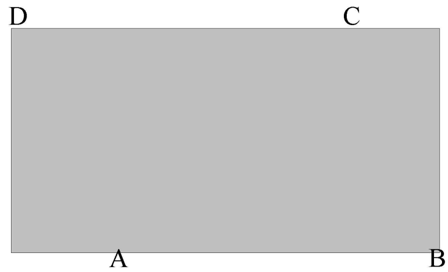


Fig. 8.9

Now, AD being equal and parallel to BC, represents the force Q as well in magnitude, direction and sense.

Since the three forces P, Q, R are in equilibrium, R is equal and opposite to the resultant of P and Q and therefore must be represented in magnitude, direction and sense by CA.

Thus, we have a triangle ABC whose sides AB, BC and CA taken in order represent the forces P, Q and R respectively and this proves the theorem.

Note 1 : If we draw any other triangle having its sides parallel to the directions of the forces P, Q and R, then this triangle will be similiar to and accordingly the corresponding sides will be proportional. And, as such the three forces in this case may as well be represented in magnitude, direction and sense by the sides of that triangle, taken in order.

Note 2 : If three forces acting at a point be such that the sum of two of them is less than the third, then they can never be in equilibrium, for they cannot be represented by the sides of a triangle.

8.13 Resultant of several coplanar forces simultaneously acting at a point

Let a number of coplanar forces P_1, P_2, P_3, P_4, etc. be simultaneously acting at a point O and let their directions make respectively angles .... with a suitably chosen direction OX in the plane. And, let OY be perpendicular to OX.

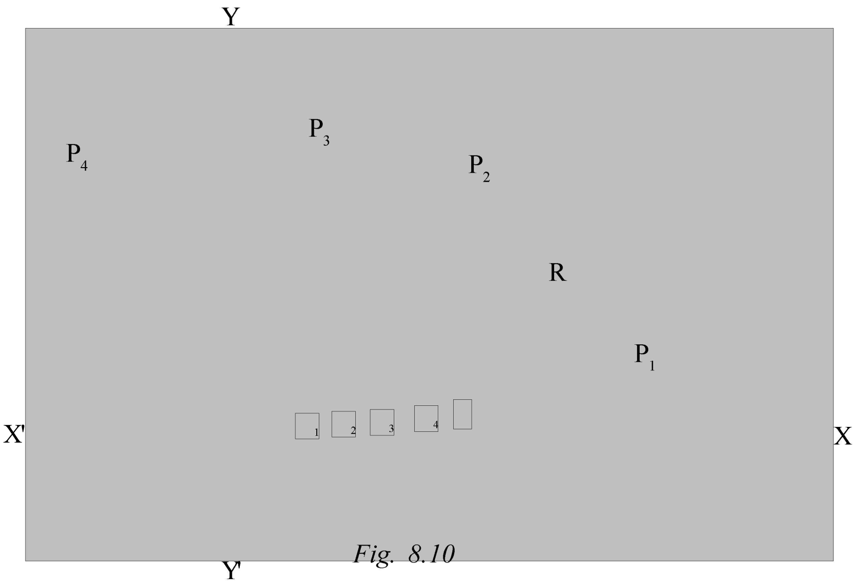


Fig. 8.10

We can now replace the force P_1 by its resolved parts along OX and along OY. Similarly, P_2 may be replaced by along OX and along OY and so on for each of the forces.

Let R be the resultant of the given forces and let it make an angle with OX.

Since the resolved parts of R along OX and OY are equal to the algebraic sum of the resolved parts of the component forces along the same directions, we have

and

Hence, or .... (i)

and .... (ii)

Equations (i) and (ii) give respectively the magnitude and direction of the resultant.

Corollary : Conditions of equilibrium of concurrent and coplanar forces.

When X=O and Y=O, then R=O.

Therefore, the forces are in equilibrium if the sum of their resolved parts along two perpendicular directions OX and OY vanish separately.

Conversely, if the forces are in equilibrium i.e. R=O, then it follows from (i) that X=O and Y=O.

Thus, the necessary and sufficient condition for the equilibrium of the concurrent and coplanar forces are X=O and Y=O.

Example 1. Find the magnitude of the resultant of two forces 8 kgwt and 7 kg wt acting at an angle of 60º to each other.

Solution : Let R be the resultant of the two forces.

Then,

= 13 kg wt.

Example 2. Two forces whose magnitudes are P and act on a particle in directions inclined at an angle of 135º to each other. Find the magnitude and direction of the resultant.

Solution : Let R be the resultant of the two forces and let it make an angle with the direction of the force P.

Then, we have

(

$= -\cos 45^{o}$

)

And,

Hence, the resultant is a force equal to P at right angles to the direction of the first component.

Example 3. Find the greatest and the least resultants of two forces whose magnitudes are 12 kg wt and 8 kg wt.

Solution : The greatest resultant $= 12+8$

$= 20$ kg wt

and the least resultant $=12-8$

$= 4$ kg wt.

Example 4. Two forces acting at a point have got their resultant 10 when acting at right angles and their least resultant is 2. Find their greatest resultant and also the resultant when they act at an angle 60º to each other.

Solution : Let P and Q be the two forces (P>Q).

Then, while acting perpendicularly, we have

resultant = 10

$P^2 + Q^2 = 100$ .... (i)

Also, their least resultant $= 2$

$P-Q=2$

$(P-Q)^2=2^2$

$P^2+Q^2-2PQ=4$

$100 - 2PQ = 4$ (using (i))

$2PQ = 100 - 4$

........................(ii)

Now, the greatest resultant =

(using (i) & (ii))

Also, when they act at an angle of 60º,

their resultant

Example 5. A force of 10 kg wt is inclined at an angle of 30º to the horizontal. Find its resolved parts along the horizontal and the vertical directions.

Solution : The resolved parts along the horizontal and the vertical directions are 10 cos 30º kg wt and 10 sin 30º kg wt respectively i.e. kg wt and 5 kg wt.

Example 6. Forces of magnitudes 2, , 5, and 2 kg wt respectively act at one angular point of a regular hexagon towards the five other angular points. Find the magnitude and direction of their resultant.

Solution : Let ABCDEF be the regular hexagon and let the given forces act at the point A as shown in the figure.

Let the resultant R make an angle with the side AB.

Now, resolving the given forces along two perpendicular lines AB and AE, we have

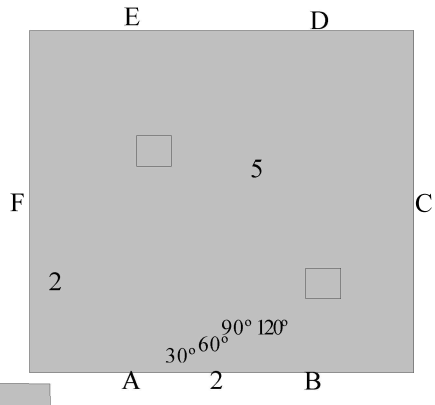


Fig. 8.11

and

R=10 kg wt.

Also,

Hence, the resultant is 10 kg wt and acts along AD.

Example 7. Two forces of magnitudes 3P and 2P respectively have a resultant R. If the first one is doubled, the magnitude of the resultant is doubled. Find the angle between the forces.

Solution : Let

Then, we have

And,

.... (ii)

From (i) and (ii), we have

.

Example 8. The sum of two forces is 18N and the resultant, whose direction is perpendicular to the lesser of the two forces, is 12N. Find the magnitudes of the two forces.

Solution : Let P and Q be the two forces (Q>P).

Then, P + Q =18 .... (i)

Also, from the figure 8.12, we have in the rt. BAC,

$BC^2 = AB^2 + AC^2$

$Q^2 = P^2 + 12^2$

$Q^2 - P^2 = 144$

$(Q + P)(Q - P) = 144$

$18(Q - P) = 144$ [using (i)]

$Q - P = 8$ .... (ii)

Solving (i) and (ii), we have Q=13 and P = 5

the magnitudes of the two forces are 13N and 5N.

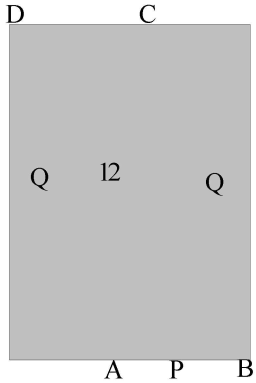


Fig. 8.12

EXERCISE 8.1

1. Find the magnitude of the resultant of the following pair of forces inclined to each other at the given angle.
   - (i) 3N and 4N, 60°
   - (ii) 10 kg wt and kg wt, 45°
   - (iii) 24N and 7N, 90°
   - (iv) 5N and 9N, 120°
   - (v) 8 kg wt and kg wt, 150°.
2. Find the resolved parts of each of the following forces whose inclination to one of the resolved parts is given alongside :

   (i) 16N, 30° (ii) 50kg wt, 60° (iii) 20N, 120° (iv) kg wt, 135°.
3. Find the angle between two equal forces P when their resultant is equal to P.

   (Equal forces means forces with equal magnitude).
4. Two forces acting at an angle of 60° have a resultant equal to kg wt. If one of the forces is 2 kg wt, find the other.

5. Two equal forces act on a particle. Find the angle between them if the square of their resultant is equal to three times their product.

6. Find the magnitudes of two forces such that, if they act at right angles, their resultant is and if they act at an angle of 60º, their resultant is

7. The greatest and the least resultants of two forces of given magnitudes acting at a point are 16 kg wt and 4 kg wt respectively. Find the magnitude of their resultant when they are at an angle of 60º with one another.

8. Two forces act at an angle of 120º. If the greater force is 80N and the resultant is at right angles to the smaller, find the smaller force.

9. Two forces equal to 2P and P respectively act on a particle. If the first is doubled and the second is increased by 12, the direction of the resultant remains unaltered. Find the value of P.

10. If the resultant of two forces acting on a particle be at right angles to one of them and its magnitude be one-third of the magnitude of the other, show that the ratio of the larger force to the smaller is .

11. The resultant of forces P and Q is R; if Q is doubled, R is doubled and if Q is reversed, R is again doubled. Show that

12. Resolve a force of 50N into two forces making angles of 90º and 45º with it on opposite sides.

13. If a force P is resolved into two component forces and if one component is at right angles to the force and equal to it in magnitude, find the magnitude and direction of the other component.

14. Forces P and Q, whose resultant is R, act at a point O. If any transversal cuts the line of action of the forces P, Q, R at the points L, M, N respectively, then show that

.

15. Forces equal to 3N, 4N, 5N and 6N act on a particle in directions respectively due north, south, east and west. Find the magnitude and direction of the resultant.

16. Forces equal to 1 kg wt, 2 kg wt and kg wt act at a point A in the directions AP, AQ and AR respectively. If and find the magnitude and direction of the resultant.

17. Forces 2P, 3P and 4P act at a point in directions parallel to the sides of an equilateral triangle taken in order. Show that the magnitude of the resultant is

18. Forces of magnitudes 1, 2, 3, 4, 5 respectively act at the angular point A of a regular hexagon ABCDEF towards the other angular points taken in order. Show that the magnitude of the resultant is $2\sqrt{19+10\sqrt{3}}$ and $\tan\theta = \frac{5+4\sqrt{3}}{\sqrt{3}}$, where θ is the angle which the resultant makes with AB.

19. The resultant of two forces P and Q is $\sqrt{3}Q$ at an angle 30° with P. Show that either P=Q or P=2Q.

ANSWERS

1. (i) $\sqrt{37}$N (ii) $5\sqrt{10}$ kg wt (iii) 25N (iv) $\sqrt{61}$N (v) $4\sqrt{13}$ kg wt.

2. (i) $8\sqrt{3}$N, 8N (ii) 25 kg wt, $25\sqrt{3}$ kg wt
   (iii) –10N, $10\sqrt{3}$N (iv) –5kg wt, 5 kg wt.

3. 120° 4. 2 kg wt 5. 60°

6. 3N and 1N 7. 14 kg wt 8. 40 kg wt

9. 12 12. 50N, $50\sqrt{2}$N

13. $\sqrt{2}$P at an angle of 45° with the direction of P.

15. $\sqrt{2}$N due south-west. 16. 4 kg wt in the direction AQ.