CHAPTER – 3
FACTORISATION

❖ ***Some Basic Algebraic Identities***

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $a^2 - b^2 = (a-b)(a+b)$
4. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
5. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
6. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
7. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$
   $= (a+b)(a^2 - ab + b^2)$
8. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$
   $= (a-b)(a^2 + ab + b^2)$

❖ ***Cyclic Expression***

An algebraic expression which remains unchanged under cyclical replacement of the letters involved is called a cyclic expression.

❖ ***Cyclic factors***

An algebraic expression is said to have cyclic factors if it has as its factors all the expressions obtained by cyclical replacement in any one of the factors.

❖ ***Factorization of cyclic expressions***

In many cases, cyclic expressions can be factorised by using the following steps:

1. Write the terms of the expression according to the ascending or descending powers of one of the letters involved in the expression.
2. Take out the factor(s) common to each coefficient.
3. Write the terms of the other factor according to the ascending or descending powers of any letters other than the previous.
4. Repeat the process till the factorization is completed.

   There are cyclic expressions which cannot be factorised by the above method (process).

➢ ***Some standard results:***

1. $a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc = (a+b)(b+c)(c+a)$
2. $a^2(b-c) + b^2(c-a) + c^2(a-b) = -(a-b)(b-c)(c-a)$
3. $a^3(b^2-c^2) + b^3(c^2-a^2) + c^3(a^2-b^2) = -(a-b)(b-c)(c-a)(ab+bc+ca)$

4. $(a+b+c)^3 - a^3 - b^3 - c^3 = 3(a+b)(b+c)(c+a)$
5. $a^2(b+c) + b^2(c+a) + c^2(a+b) + 3abc = (a+b+c)(ab+bc+ca)$
6. $a^3 + b^3 + c^3 - 3abc = (a+b+c)\{a^2+b^2+c^2-ab-bc-ca\}$

$$= \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

7. $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 = (a+b+c)(a+b-c)(b+c-a)(c+a-b)$

SOLUTIONS

EXERCISE 3.1

1. Factorise the following:

(i) $\boldsymbol{x^3 + y^3 - z^3 + 3xyz}$

Solution:-

$x^3 + y^3 - z^3 + 3xyz$

$= (x+y)^3 - 3xy(x+y) - z^3 + 3xyz \quad [\because a^3+b^3 = (a+b)^3 - 3ab(a+b)]$

$= (x+y)^3 - z^3 - 3xy(x+y) + 3xyz$

$= \{(x+y)-z\}\{(x+y)^2 + (x+y)z + z^2\} - 3xy(x+y-z)$

$[\because a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

$= (x+y-z)(x^2+2xy+y^2+zx+yz+z^2-3xy)$

$= (x+y-z)(x^2+y^2+z^2-xy+yz+zx)$

(ii) $\boldsymbol{a^3 - b^3 + 9ab + 27}$

Solution:-

$a^3 - b^3 + 9ab + 27$

$= (a-b)^3 + 3ab(a-b) + 9ab + 3^3 \ [\because a^3 - b^3 = (a-b)^3 + 3ab(a-b)]$

$= (a-b)^3 + 3^3 + 3ab(a-b) + 9ab$

$= \{(a-b)+3\}\{(a-b)^2-(a-b)\times 3+3^2\}+3ab(a-b+3)$

$[\because x^3+y^3=(x+y)(x^2-xy+y^2)]$

$= (a-b+3)(a^2-2ab+b^2-3a+3b+9+3ab)$

$= (a-b+3)(a^2+b^2+9+ab+3b-3a)$

(iii) $\mathbf{8a^3+27b^3+64c^3-72abc}$

Solution:-

$8a^3+27b^3+64c^3-72abc$

$= (2a)^3+(3b)^3+(4c)^3-72abc$

$= (2a+3b)^3-3\times 2a\times 3b(2a+3b)+(4c)^3-72abc$

$[\because x^3+y^3=(x+y)^3-3xy(x+y)]$

$= (2a+3b)^3+(4c)^3-18ab(2a+3b)-72abc$

$= \{(2a+3b)+4c\}\{(2a+3b)^2-(2a+3b)4c+(4c)^2\}-18ab(2a+3b+4c)$

$[\because x^3+y^3=(x+y)(x^2-xy+y^2)$

$= (2a+3b+4c)\{(2a)^2+2\times 2a\times 3b+(3b)^2-8ca-12bc+16c^2\}$

$-18ab(2a+3b+4c)$

$= (2a+3b+4c)(4a^2+12ab+9b^2-8ca-12bc+16c^2-18ab)$

$= (2a+3b+4c)(4a^2+9b^2+16c^2-6ab-12bc-8ca)$

(iv) $\mathbf{x^3-y^3-125z^3-15xyz}$

Solution:-

$x^3-y^3-125z^3-15xyz$

$= (x-y)^3+3xy(x-y)-(5z)^3-15xyz \quad [\because a^3-b^3=(a-b)^3+3ab(a-b)]$

$= (x-y)^3-(5z)^3+3xy(x-y)-15xyz$

$= \{(x-y)-5z\}\{(x-y)^2+(x-y)5z+(5z)^2\}+3xy(x-y-5z)$

$[\because a^3-b^3=(a-b)(a^2+ab+b^2)$

$= (x-y-5z)(x^2-2xy+y^2+5zx-5yz+25z^2)+3xy(x-y-5z)$

$= (x-y-5z)(x^2-2xy+y^2+5zx-5yz+25z^2+3xy)$

$= (x-y-5z)(x^2+y^2+25z^2+xy-5yz+5zx)$

(v) $\boldsymbol{a^6 + 5a^3 + 8}$

Solution:-

$$a^6 + 5a^3 + 8$$
$$= (a^2)^3 + (6-1)a^3 + 8$$
$$= (a^2)^3 + 6a^3 - a^3 + 2^3$$
$$= (a^2)^3 - a^3 + 2^3 + 6a^3$$
$$= (a^2 - a)^3 + 3.a^2a(a^2 - a) + 2^3 + 6a^3 \qquad [\because x^3 - y^3 = (x-y)^3 + 3xy(x-y)]$$
$$= (a^2 - a)^3 + 2^3 + 3.a^3(a^2 - a) + 6a^3$$
$$= \{(a^2 - a) + 2\}\{(a^2 - a)^2 - (a^2 - a).2 + 2^2\} + 3a^3(a^2 - a + 2)$$
$$[\because x^3 + y^3 = (x+y)(x^2 - xy + y^2)]$$
$$= (a^2 - a + 2)\{(a^2)^2 - 2.a^2.a + a^2 - 2a^2 + 2a + 4\} + 3a^3(a^2 - a + 2)$$
$$= (a^2 - a + 2)(a^4 - 2a^3 + a^2 - 2a^2 + 2a + 4 + 3a^3)$$
$$= (a^2 - a + 2)(a^4 + a^3 - a^2 + 2a + 4)$$

(vi) $\boldsymbol{x^6 + 8x^3 + 27}$

Solution:-

$$x^6 + 8x^3 + 27$$
$$= (x^2)^3 + (9-1)x^3 + 27$$
$$= (x^2)^3 + 9x^3 - x^3 + 3^3$$
$$= (x^2)^3 - x^3 + 3^3 + 9x^3$$
$$= (x^2 - x)^3 + 3.x^2.x(x^2 - x) + 3^3 + 9x^3 \qquad [\because a^3 - b^3 = (a-b)^3 + 3ab(a-b)]$$
$$= (x^2 - x)^3 + 3^3 + 3x^3(x^2 - x) + 9x^3$$
$$= \{(x^2 - x) + 3\}\{(x^2 - x)^2 - (x^2 - x).3 + 3^2\} + 3x^3(x^2 - x + 3)$$
$$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$$
$$= (x^2 - x + 3)\{(x^2)^2 - 2.x^2.x + x^2 - 3x^2 + 3x + 9 + 3x^3\}$$
$$= (x^2 - x + 3)(x^4 - 2x^3 + x^2 - 3x^2 + 3x + 9 + 3x^3)$$
$$= (x^2 - x + 3)(x^4 + x^3 - 2x^2 + 3x + 9)$$

2. Factorise the following:

(i) $\boldsymbol{yz(y-z)+zx(z-x)+xy(x-y)}$

Solution: $yz(y-z)+zx(z-x)+xy(x-y)$

$= yz(y-z)+z^2x-zx^2+x^2y-xy^2$

$= (x^2y-zx^2)-(xy^2-z^2x)+yz(y-z)$ [arranging in descending powers of x]

$= x^2(y-z)-x(y^2-z^2)+yz(y-z)$

$= x^2(y-z)-x(y+z)(y-z)+yz(y-z)$

$= (y-z)\{x^2-x(y+z)+yz\}$

$= (y-z)(x^2-xy-zx+yz)$

$= (y-z)\{(yz-xy)-(zx-x^2)\}$ [arranging in descending powers of y]

$= (y-z)\{y(z-x)-x(z-x)\}$

$= (y-z)(z-x)(y-x)$

$= -(x-y)(y-z)(z-x)$

(ii) $\boldsymbol{yz(y+z)+zx(z+x)+xy(x+y)+2xyz}$

Solution: $yz(y+z)+zx(z+x)+xy(x+y)+2xyz$

$= yz(y+z)+z^2x+zx^2+x^2y+xy^2+2xyz$

$= (x^2y+zx^2)+(xy^2+2xyz+z^2x)+yz(y+z)$ [arranging in descending powers of x]

$= x^2(y+z)+x(y^2+2yz+z^2)+yz(y+z)$

$= x^2(y+z)+x(y+z)^2+yz(y+z)$

$= (y+z)\{x^2+x(y+z)+yz\}$

$= (y+z)(x^2+xy+zx+yz)$

$= (y+z)\{(yz+xy)+(zx+x^2)\}$ [arranging in descending powers of y]

$= (y+z)\{y(z+x)+x(z+x)\}$

$= (y+z)(z+x)(y+x)$

$= (x+y)(y+z)(z+x)$

(iii) $\boldsymbol{bc(b+c)+ca(c+a)+ab(a+b)+3abc}$

Solution: $bc(b+c)+ca(c+a)+ab(a+b)+3abc$

$= bc(b+c)+ca(c+a)+ab(a+b)+abc+abc+abc$

$= \{bc(b+c)+abc\}+\{ca(c+a)+abc\}+\{ab(a+b)+abc\}$

$= bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$

$= bc(a+b+c)+ca(a+b+c)+ab(a+b+c)$

$= (a+b+c)(ab+bc+ca)$

(iv) $\boldsymbol{x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz}$

Solution: $x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz$

$= xy^2+z^2x+yz^2+x^2y+zx^2+y^2z+2xyz$

$= (x^2y+zx^2)+(xy^2+2xyz+z^2x)+(y^2z+yz^2)$ [arranging in descending powers of x]

$= x^2(y+z)+x(y^2+2yz+z^2)+yz(y+z)$

$= x^2(y+z)+x(y+z)^2+yz(y+z)$

$= (y+z)\{x^2+x(y+z)+yz\}$

$= (y+z)(x^2+xy+zx+yz)$

$= (y+z)\{(yz+xy)+(zx+x^2)\}$ [arranging in descending powers of y]

$= (y+z)\{y(z+x)+x(z+x)\}$

$= (y+z)(z+x)(y+x)$

$= (x+y)(y+z)(z+x)$

(v) $\boldsymbol{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc}$

Solution: $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$

$= ab^2+c^2a+bc^2+a^2b+ca^2+b^2c+abc+abc+abc$

$= (a^2b+ab^2+abc)+(abc+b^2c+bc^2)+(ca^2+abc+c^2a)$

$= ab(a+b+c)+bc(a+b+c)+ca(a+b+c)$

$= (a+b+c)(ab+bc+ca)$

(vi) $\boldsymbol{x^4(y-z)+y^4(z-x)+z^4(x-y)}$

Solution: $x^4(y-z)+y^4(z-x)+z^4(x-y)$

$=x^4(y-z)+y^4z-xy^4+z^4x-yz^4$

$=x^4(y-z)-(xy^4-z^4x)+(y^4z-yz^4)$ [arranging in descending powers of x]

$=x^4(y-z)-x(y^4-z^4)+yz(y^3-z^3)$

$=x^4(y-z)-x\{(y^2)^2-(z^2)^2\}+yz(y-z)(y^2+yz+z^2)$

$=x^4(y-z)-x(y^2-z^2)(y^2+z^2)+yz(y-z)(y^2+yz+z^2)$

$=x^4(y-z)-x(y-z)(y+z)(y^2+z^2)+yz(y-z)(y^2+yz+z^2)$

$=(y-z)\{x^4-x(y+z)(y^2+z^2)+yz(y^2+yz+z^2)\}$

$=(y-z)\{x^4-x(y^3+yz^2+y^2z+z^3)+yz(y^2+yz+z^2)\}$

$=(y-z)(x^4-xy^3-xyz^2-xy^2z-z^3x+y^3z+y^2z^2+yz^3)$

$=(y-z)\{(y^3z-xy^3)+(y^2z^2-xy^2z)+(yz^3-xyz^2)-(z^3x-x^4)\}$

[arranging in descending powers of y]

$=(y-z)\{y^3(z-x)+y^2z(z-x)+yz^2(z-x)-x(z^3-x^3)\}$

$=(y-z)\{y^3(z-x)+y^2z(z-x)+yz^2(z-x)-x(z-x)(z^2+zx+x^2)\}$

$=(y-z)(z-x)\{y^3+y^2z+yz^2-x(z^2+zx+x^2)\}$

$=(y-z)(z-x)(y^3+y^2z+yz^2-xz^2-zx^2-x^3)$

$=(y-z)(z-x)\{-(xz^2-yz^2)-(zx^2-y^2z)-(x^3-y^3)$ [arranging in descending powers of z]

$=(y-z)(z-x)\{-z^2(x-y)-z(x^2-y^2)-(x-y)(x^2+xy+y^2)\}$

$=(y-z)(z-x)\{-z^2(x-y)-z(x-y)(x+y)-(x-y)(x^2+xy+y^2)\}$

$=(x-y)(y-z)(z-x)\{-z^2-z(x+y)-(x^2+xy+y^2)\}$

$=(x-y)(y-z)(z-x)(-z^2-zx-yz-x^2-xy-y^2)$

$=-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$

(vii) $\boldsymbol{yz(y^3 - z^3) + zx(z^3 - x^3) + xy(x^3 - y^3)}$

Solution: $yz(y^3 - z^3) + zx(z^3 - x^3) + xy(x^3 - y^3)$

$= yz(y^3 - z^3) + z^4x - zx^4 + x^4y - xy^4$

$= (x^4y - zx^4) - (xy^4 - z^4x) + yz(y^3 - z^3)$ [arranging in descending powers of x]

$= x^4(y - z) - x(y^4 - z^4) + yz(y^3 - z^3)$

$= x^4(y - z) - x\{(y^2)^2 - (z^2)^2\} + yz(y - z)(y^2 + yz + z^2)$

$= x^4(y - z) - x(y^2 + z^2)(y^2 - z^2) + yz(y - z)(y^2 + yz + z^2)$

$= x^4(y - z) - x(y^2 + z^2)(y + z)(y - z) + yz(y - z)(y^2 + yz + z^2)$

$= (y - z)\{x^4 - x(y^2 + z^2)(y + z) + yz(y^2 + yz + z^2)\}$

$= (y - z)\{x^4 - x(y^3 + y^2z + yz^2 + z^3) + y^3z + y^2z^2 + yz^3\}$

$= (y - z)(x^4 - xy^3 - xy^2z - xyz^2 - z^3x + y^3z + y^2z^2 + yz^3)$

$= (y - z)\{(y^3z - xy^3) + (y^2z^2 - xy^2z) + (yz^3 - xyz^2) - (z^3x - x^4)\}$

[arranging in descending powers of y]

$= (y - z)\{y^3(z - x) + y^2z(z - x) + yz^2(z - x) - x(z^3 - x^3)\}$

$= (y - z)\{y^3(z - x) + y^2z(z - x) + yz^2(z - x) - x(z - x)(z^2 + zx + x^2)\}$

$= (y - z)(z - x)\{y^3 + y^2z + yz^2 - x(z^2 + zx + x^2)\}$

$= (y - z)(z - x)(y^3 + y^2z + yz^2 - z^2x - zx^2 - x^3)$

$= (y - z)(z - x)\{-(z^2x - yz^2) - (zx^2 - y^2z) - (x^3 - y^3)\}$ [arranging in descending powers of z]

$= (y - z)(z - x)\{-z^2(x - y) - z(x^2 - y^2) - (x^3 - y^3)\}$

$= (y - z)(z - x)\{-z^2(x - y) - z(x + y)(x - y) - (x - y)(x^2 + xy + y^2\}$

$= (x - y)(y - z)(z - x)\{-z^2 - z(x + y) - (x^2 + xy + y^2\}$

$= (x - y)(y - z)(z - x)(-z^2 - zx - yz - x^2 - xy - y^2)$

$= -(x - y)(y - z)(z - x)(x^2 + y^2 + z^2 + xy + yz + zx)$

(viii) $\boldsymbol{b^2c^2(b^2 - c^2) + c^2a^2(c^2 - a^2) + a^2b^2(a^2 - b^2)}$

Solution: $b^2c^2(b^2 - c^2) + c^2a^2(c^2 - a^2) + a^2b^2(a^2 - b^2)$

$= b^2c^2(b^2 - c^2) + c^4a^2 - c^2a^4 + a^4b^2 - a^2b^4$

$= (a^4b^2 - c^2a^4) - (a^2b^4 - c^4a^2) + b^2c^2(b^2 - c^2)$ [arranging in descending powers of a]

$= a^4(b^2 - c^2) - a^2(b^4 - c^4) + b^2c^2(b^2 - c^2)$

$= a^4(b^2 - c^2) - a^2\{(b^2)^2 - (c^2)^2\} + b^2c^2(b^2 - c^2)$

$= a^4(b^2 - c^2) - a^2(b^2 + c^2)(b^2 - c^2) + b^2c^2(b^2 - c^2)$

$= (b^2 - c^2)\{a^4 - a^2(b^2 + c^2) + b^2c^2\}$

$= (b^2 - c^2)(a^4 - a^2b^2 - c^2a^2 + b^2c^2)$

$= (b^2 - c^2)\{(b^2c^2 - a^2b^2) - (c^2a^2 - a^4)\}$ [arranging in descending powers of b]

$= (b^2 - c^2)\{b^2(c^2 - a^2) - a^2(c^2 - a^2)\}$

$= (b^2 - c^2)(c^2 - a^2)(b^2 - a^2)$

$= -(a^2 - b^2)\,(b^2 - c^2)(c^2 - a^2)$

$= -(a - b)(a + b)(b - c)(b + c)(c - a)(c + a)$

$= -(a - b)(b - c)(c - a)(a + b)(b + c)(c + a)$

(ix) $\boldsymbol{x^2y^2(x - y) + y^2z^2(y - z) + z^2x^2(z - x)}$

Solution: $x^2y^2(x - y) + y^2z^2(y - z) + z^2x^2(z - x)$

$= x^3y^2 - x^2y^3 + y^2z^2(y - z) + z^3x^2 - z^2x^3$

$= (x^3y^2 - z^2x^3) - (x^2y^3 - z^3x^2) + y^2z^2(y - z)$

[arranging in descending powers of x]

$= x^3(y^2 - z^2) - x^2(y^3 - z^3) + y^2z^2(y - z)$

$= x^3(y + z)(y - z) - x^2(y - z)(y^2 + yz + z^2) + y^2z^2(y - z)$

$= (y - z)\{x^3(y + z) - x^2(y^2 + yz + z^2) + y^2z^2\}$

$= (y - z)(x^3y + zx^3 - x^2y^2 - x^2yz - z^2x^2 + y^2z^2)$

$= (y - z)\{(y^2z^2 - x^2y^2) - (x^2yz - x^3y) - (z^2x^2 - zx^3)\}$

[arranging in descending powers of y]

$= (y - z)\{y^2(z^2 - x^2) - x^2y(z - x) - zx^2(z - x)\}$

$= (y - z)\{y^2(z + x)(z - x) - x^2y(z - x) - zx^2(z - x)\}$

$= (y - z)(z - x)\{y^2(z + x) - x^2y - zx^2\}$

$= (y - z)(z - x)(y^2z + xy^2 - x^2y - zx^2)$

$= (y - z)(z - x)\{-(zx^2 - y^2z) - (x^2y - xy^2)\}$ [arranging in descending powers of z]

$= (y - z)(z - x)\{-z(x^2 - y^2) - xy(x - y)\}$

$= (y - z)(z - x)\{-z(x + y)(x - y) - xy(x - y)\}$

$= (y - z)(z - x)(x - y)\{-z(x + y) - xy\}$

$= (y - z)(z - x)(x - y)(-zx - yz - xy)$

$= -(x - y)(y - z)(z - x)(xy + yz + zx)$

(x) $\boldsymbol{8z^3-(x-y)^3-(y+z)^3-(z-x)^3}$

Solution: Let $a=x-y, b=y+z$ and $c=z-x$.

Then $a+b+c=x-y+y+z+z-x=2z$

Now, $8z^3-(x-y)^3-(y+z)^3-(z-x)^3$

$=(2z)^3-(x-y)^3-(y+z)^3-(z-x)^3$

$=(a+b+c)^3-a^3-b^3-c^3$

$=3(a+b)(b+c)(c+a)$

$=3(x-y+y+z)(y+z+z-x)(z-x+x-y)$ [Restoring the values of a, b and c]

$=3(z+x)(y+2z-x)(z-y)$

(xi) $\boldsymbol{x^6(y^4-z^4)+y^6(z^4-x^4)+z^6(x^4-y^4)}$

Solution: $x^6(y^4-z^4)+y^6(z^4-x^4)+z^6(x^4-y^4)$

$=x^6(y^4-z^4)+y^6z^4-x^4y^6+z^6x^4-y^4z^6$

$=x^6(y^4-z^4)-(x^4y^6-z^6x^4)+(y^6z^4-y^4z^6)$ [arranging in descending powers of x]

$=x^6(y^4-z^4)-x^4(y^6-z^6)+y^4z^4(y^2-z^2)$

$=x^6\{(y^2)^2-(z^2)^2\}-x^4\{(y^2)^3-(z^2)^3+y^4z^4(y^2-z^2)$

$=x^6(y^2+z^2)(y^2-z^2)-x^4(y^2-z^2)\{(y^2)^2+y^2z^2+(z^2)^2\}+y^4z^4(y^2-z^2)$

$=x^6(y^2+z^2)(y^2-z^2)-x^4(y^2-z^2)(y^4+y^2z^2+z^4)+y^4z^4(y^2-z^2)$

$=(y^2-z^2)\{x^6(y^2+z^2)-x^4(y^4+y^2z^2+z^4)+y^4z^4\}$

$=(y^2-z^2)(x^6y^2+z^2x^6-x^4y^4-x^4y^2z^2-z^4x^4+y^4z^4)$

$=(y^2-z^2)\{(y^4z^4-x^4y^4)-(x^4y^2z^2-x^6y^2)-(z^4x^4-z^2x^6)\}$

[arranging in descending powers of y]

$=(y^2-z^2)\{y^4(z^4-x^4)-x^4y^2(z^2-x^2)-z^2x^4(z^2-x^2)\}$

$=(y^2-z^2)[y^4\{(z^2)^2-(x^2)^2\}-x^4y^2(z^2-x^2)-z^2x^4(z^2-x^2)]$

$= (y^2 - z^2)[y^4(z^2 + x^2)(z^2 - x^2) - x^4y^2(z^2 - x^2) - z^2x^4(z^2 - x^2)]$

$= (y^2 - z^2)(z^2 - x^2)\{y^4(z^2 + x^2) - x^4y^2 - z^2x^4\}$

$= (y^2 - z^2)(z^2 - x^2)(y^4z^2 + x^2y^4 - x^4y^2 - z^2x^4)$

$= (y^2 - z^2)(z^2 - x^2)\{-(z^2x^4 - y^4z^2) - (x^4y^2 - x^2y^4)\}$ [arranging in descending powers of z]

$= (y^2 - z^2)(z^2 - x^2)\{-z^2(x^4 - y^4) - x^2y^2(x^2 - y^2)\}$

$= (y^2 - z^2)(z^2 - x^2)[-z^2\{(x^2)^2 - (y^2)^2\} - x^2y^2(x^2 - y^2)]$

$= (y^2 - z^2)(z^2 - x^2)\{-z^2(x^2 + y^2)(x^2 - y^2) - x^2y^2(x^2 - y^2)\}$

$= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)\{-z^2(x^2 + y^2) - x^2y^2\}$

$= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)(-z^2x^2 - y^2z^2 - x^2y^2)$

$= -(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)(x^2y^2 + y^2z^2 + z^2x^2)$

$= -(x - y)(x + y)(y - z)(y + z)(z - x)(z + x)(x^2y^2 + y^2z^2 + z^2x^2)$

$= -(x - y)(y - z)(z - x)(x + y)(y + z)(z + x)(x^2y^2 + y^2z^2 + z^2x^2)$

(xii) $(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})(\boldsymbol{bc} + \boldsymbol{ca} + \boldsymbol{ab}) - \boldsymbol{abc}$

Solution: $(a + b + c)(bc + ca + ab) - abc$

$= abc + ca^2 + a^2b + b^2c + abc + ab^2 + bc^2 + c^2a + abc - abc$

$= ca^2 + a^2b + b^2c + ab^2 + bc^2 + c^2a + 2abc$

$= (a^2b + ca^2) + (ab^2 + 2abc + c^2a) + (b^2c + bc^2)$ [arranging in descending powers of a]

$= a^2(b + c) + a(b^2 + 2bc + c^2) + bc(b + c)$

$= a^2(b + c) + a(b + c)^2 + bc(b + c)$

$= (b + c)\{a^2 + a(b + c) + bc\}$

$= (b + c)(a^2 + ab + ca + bc)$

$= (b + c)\{(ca + bc) + (a^2 + ab)\}$ [arranging in descending powers of c]

$= (b + c)\{c(a + b) + a(a + b)\}$

$= (a + b)(b + c)(c + a)$

(xiii) $\boldsymbol{a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc}$

Solution: $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$

$=a(b^2+2bc+c^2)+b(c^2+2ca+a^2)+c(a^2+2ab+b^2)-3abc$

$=ab^2+2abc+c^2a+bc^2+2abc+a^2b+ca^2+2abc+b^2c-3abc$

$=ab^2+c^2a+bc^2+a^2b+ca^2+b^2c+6abc-3abc$

$=ab^2+c^2a+bc^2+a^2b+ca^2+b^2c+3abc$

$=ab^2+c^2a+bc^2+a^2b+ca^2+b^2c+abc+abc+abc$

$=(a^2b+ab^2+abc)+(abc+b^2c+bc^2)+(ca^2+abc+c^2a)$

$=ab(a+b+c)+bc(a+b+c)+ca(a+b+c)$

$=(a+b+c)(ab+bc+ca)$

(xiv) $\boldsymbol{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}$

Solution: Let $x=b+c, y=c+a$ and $z=a+b$

Then $x+y+z=b+c+c+a+a+b$

$=2(a+b+c)$

Now, $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$

$=\{2(a+b+c)\}^3-(b+c)^3-(c+a)^3-(a+b)^3$

$=(x+y+z)^3-x^3-y^3-z^3$

$=3(x+y)(y+z)(z+x)$

$=3(b+c+c+a)(c+a+a+b)(a+b+b+c)$ [Restoring the values of x,y,z]

$=3(a+b+2c)(2a+b+c)(a+2b+c)$

$=3(2a+b+c)(a+2b+c)(a+b+2c)$

3. **Prove that** $\boldsymbol{(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)=0}$

Solution: Let $a=x-y, b=y-z$ and $c=z-x$

Then $a+b+c=x-y+y-z+z-x=0$

Now, $(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)$

$=a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$=0\times(a^2+b^2+c^2-ab-bc-ca)$ $[\because a+b+c=0]$

$=0$

Hence, $(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)=0$

4. ***If $a^3 + b^3 + c^3 = 3abc$, prove that either $a + b + c = 0$ or $a = b = c$.***

Solution: We have $a^3 + b^3 + c^3 = 3abc$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$$

$$\Rightarrow \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$$

Then, either $a + b + c = 0$ or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

But $(a - b)^2$, $(b - c)^2$ and $(c - a)^2$ are non-negative numbers as they are square numbers.

$\therefore$ $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$ is possible only when

$$(a - b)^2 = 0, (b - c)^2 = 0, (c - a)^2 = 0$$

$$\Rightarrow a - b = 0, b - c = 0, c - a = 0$$

$$\Rightarrow a = b, b = c, c = a$$

$$\therefore a = b = c.$$

Thus if $a^3 + b^3 + c^3 = 3abc$, then either $a + b + c = 0$ or $a = b = c$.

5. ***If $x + y + z = 9, xy + yz + zx = 26$ and $xyz = 24$, find the value of $x^2(y + z) + y^2(z + x) + z^2(x + y)$***

Solution: We have, $x + y + z = 9, xy + yz + zx = 26$ *and* $xyz = 24$

Now, $x^2(y + z) + y^2(z + x) + z^2(x + y)$

$$= x^2(y + z) + y^2(z + x) + z^2(x + y) + 3xyz - 3xyz$$

$$= (x + y + z)(xy + yz + zx) - 3xyz$$

$$= 9 \times 26 - 3 \times 24 = 234 - 72$$

$$= 162$$

6. ***If $x + y - z = 2, y + z - x = 4$ and $z + x - y = 6$, find the value of $2y^2z^2 + 2z^2x^2 + 2x^2y^2 - x^4 - y^4 - z^4$.***

Solution: We have, $x + y - z = 2, y + z - x = 4$ *and* $z + x - y = 6$

Then $(x + y - z) + (y + z - x) + (z + x - y) = 2 + 4 + 6$

$$\Rightarrow x + y - z + y + z - x + z + x - y = 12$$

$$\Rightarrow x + y + z = 12$$

Now, $2y^2z^2 + 2z^2x^2 + 2x^2y^2 - x^4 - y^4 - z^4$

$$= (x + y + z)(x + y - z)(y + z - x)(z + x - y)$$

$$= 12 \times 2 \times 4 \times 6$$

$$= 576$$

7. ***If $x+y+z=12$ and $x^2+y^2+z^2=44$, find the value of $(x+y+z)^3-x^3-y^3-z^3+3xyz$.***

Solution: We have, $x+y+z=12$ and $x^2+y^2+z^2=44$

Then $(x+y+z)^2=12^2$

$\Rightarrow x^2+y^2+z^2+2xy+2yz+2zx=144$

$\Rightarrow (x^2+y^2+z^2)+2(xy+yz+zx)=144$

$\Rightarrow 44+2(xy+yz+zx)=144$

$\Rightarrow 2(xy+yz+zx)=100$

$\Rightarrow xy+yz+zx=50$

Now, $(x+y+z)^3-x^3-y^3-z^3+3xyz$

$=(x+y+z)^3-[x^3+y^3+z^3-3xyz]$

$=(x+y+z)^3-(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$=(x+y+z)^3-(x+y+z)\{(x^2+y^2+z^2)-(xy+yz+zx)\}$

$=12^3-12(44-50)$

$=12^3-12\times(-6)$

$=1728+72$

$=1800$

8. ***Find the value of $xy(x+y)+yz(y+z)+zx(z+x)+3xyz$, when $x=a(b-c), y=b(c-a), z=c(a-b)$.***

Solution: We have, $x=a(b-c), y=b(c-a), z=c(a-b)$

Then, $x+y+z=a(b-c)+b(c-a)+c(a-b)$

$=ab-ca+bc-ab+ca-bc$

$=0$

Now $xy(x+y)+yz(y+z)+zx(z+x)+3xyz$

$=xy(x+y)+xyz+yz(y+z)+xyz+zx(z+x)+xyz$

$=xy(x+y+z)+yz(y+z+x)+zx(z+x+y)$

$=(x+y+z)(xy+yz+zx)$

$=0\times(xy+yz+zx)$

$=0$

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