CHAPTER – 2
POLYNOMIALS

- **Working rule to divide a polynomial by another polynomial**
  1. Write the dividend and divisor after arranging the term in the descending order of their degrees.
  2. Divide the highest degree term (first term) of the dividend by the highest degree term (first term) of the divisor to get the first term of the quotient.
  3. Multiply the divisor by the first term of the quotient and subtract this product from the dividend to get the remainder.
  4. Taking the remainder as the new dividend, keeping the divisor same, find the quotient and remainder to get the next quotient term.
  5. Continue the process till the degree of the remainder is less than the degree of the divisor.
- **Division Algorithm for Polynomials**

  If $p(x)$ and $d(x)$ are any two polynomials with $d(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that $p(x) = d(x) \times q(x) + r(x)$, where either $r(x) = 0$ or degree of $r(x) <$ degree of $d(x)$.

SOLUTIONS

EXERCISE 2.1

1. **Divide the polynomial $p(x)$ by the polynomial $d(x)$ and find the quotient and the remainder, and verify the division algorithm in each of the following:**

   (i) $p(x) = 2x^3 + x^2 - x, \quad d(x) = x$

Solution:-

$x)2x^3 + x^2 - x(2x^2 + x - 1$
$2x^3$
$x^2 - x$
x^2
$-x$
$-x$
0

Then quotient $= 2x^2 + x - 1$ and remainder $= 0$

Verification:-

Divisor× Quotient + Remainder $= x(2x^2 + x - 1) + 0$

$= 2x^3 + x^2 - x$

$=$ Dividend

Hence verified.

(ii) $\boldsymbol{p(x) = 3x^2 - x - 1,} \quad \boldsymbol{d(x) = -x}$

Solution:-

$$
\begin{array}{l}
-x)\,3x^2 - x - 1\,(-3x + 1 \\
\quad\ \underline{3x^2 \qquad\qquad} \\
\qquad\quad -x\ -1 \\
\qquad\quad \underline{-x \qquad} \\
\qquad\qquad\ \ -1
\end{array}
$$

Then quotient $= -3x + 1$ and remainder $= -1$

Verification:-

Divisor× Quotient + Remainder $= -x(-3x + 1) + (-1)$

$= 3x^2 - x - 1$

$=$ Dividend

Hence verified.

(iii) $\boldsymbol{p(x) = 3x^2 + 2x + 1,} \quad \boldsymbol{d(x) = x + 1}$

Solution:-

$$
\begin{array}{l}
x + 1)\,3x^2 + 2x + 1\,(3x - 1 \\
\qquad\ \underline{3x^2 + 3x \qquad} \\
\qquad\qquad\ -x + 1 \\
\qquad\qquad\ \underline{-x - 1} \\
\qquad\qquad\qquad\ 2
\end{array}
$$

Then quotient $= 3x - 1$ and remainder $= 2$

Verification:-

Divisor× Quotient + Remainder $= (x + 1)(3x - 1) + 2$

$= 3x^2 - x + 3x - 1 + 2$

$= 3x^2 + 2x + 1$

$=$ Dividend

Hence verified.

(iv) $\boldsymbol{p(x) = x^3 - 1}$, $\boldsymbol{d(x) = x - 1}$

Solution:-

$$\begin{array}{l} x-1)\,x^3 \qquad\qquad -1\,(\,x^2+x+1 \\ \quad\ \underline{x^3 - x^2\qquad\qquad} \\ \qquad\quad x^2 \qquad -1 \\ \qquad\quad \underline{x^2 - x\qquad} \\ \qquad\qquad\quad x - 1 \\ \qquad\qquad\quad \underline{x - 1} \\ \qquad\qquad\qquad 0 \end{array}$$

Then quotient $= x^2 + x + 1$ and remainder $= 0$

Verification:-

Divisor× Quotient + Remainder $= (x-1)(x^2 + x + 1\,) + 0$

$= x^3 + x^2 + x - x^2 - x - 1$

$= x^3 - 1$

= Dividend

Hence verified.

(v) $\boldsymbol{p(x) = 2x^2 + 3x + 1}$, $\boldsymbol{d(x) = 2 + x}$

Solution:-

$$\begin{array}{l} x+2)\,2x^2+3x+1\,(\,2x-1 \\ \quad\ \underline{2x^2+4x\qquad} \\ \qquad\quad -x+1 \\ \qquad\quad \underline{-x-2} \\ \qquad\qquad\quad 3 \end{array}$$

Then quotient $= 2x - 1$ and remainder $= 3$

Verification:-

Divisor× Quotient + Remainder $= (x+2)(2x-1\,) + 3$

$= 2x^2 - x + 4x - 2 + 3$

$= 2x^2 + 3x + 1$

= Dividend

Hence verified

(vi) $p(x) = x^3 - 3x^2 + 5x - 3,\quad d(x) = x^2 - 2$

Solution:-

$x^2 - 2)x^3 - 3x^2 + 5x - 3(\ x - 3$

$\underline{x^3 \qquad - 2x}$

$-3x^2 + 7x - 3$

$\underline{-3x^2 \qquad + 6}$

$7x - 9$

Then quotient $= x - 3$ and remainder $= 7x - 9$

Verification:-

Divisor× Quotient + Remainder $= (x^2 - 2)(x - 3\ ) + (7x - 9)$

$= x^3 - 3x^2 - 2x + 6 + 7x - 9$

$= x^3 - 3x^2 + 5x - 3$

= Dividend

Hence verified.

(vii) $p(x) = 4x^3 + 4x^2 - x + 1,\quad d(x) = x^2 + 2x$

Solution:-

$x^2 + 2x)4x^3 + 4x^2 - x + 1(\ 4x - 4$

$\underline{4x^3 + 8x^2}$

$-4x^2 - x + 1$

$\underline{-4x^2 - \ 8x}$

$7x + 1$

Then quotient $= 4x - 4$ and remainder $= 7x + 1$

Verification:-

Divisor× Quotient + Remainder $= (x^2 + 2x\ )(4x - 4\ ) + (7x + 1)$

$= 4x^3 - 4x^2 + 8x^2 - 8x + 7x + 1$

$= 4x^3 + 4x^2 - x + 1$

= Dividend

Hence verified.

(viii) $p(x) = 3x^2 - x^3 - 3x + 5, \quad d(x) = x - 1 - x^2$

Solution:-

$-x^2 + x - 1) -x^3 + 3x^2 - 3x + 5 (x - 2$

$-x^3 + x^2 - x$

$2x^2 - 2x + 5$

$2x^2 - 2x + 2$

3

Then quotient $= x - 2$ and remainder $= 3$

Verification:-

Divisor× Quotient + Remainder $= (-x^2 + x - 1)(x - 2) + 3$

$= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3$

$= -x^3 + 3x^2 - 3x + 5$

= Dividend

Hence verified.

(ix) $p(x) = x^4 - 3x^2 + 4x + 5, \quad d(x) = x^2 + 2 - 3x$

Solution:-

$x^2 - 3x + 2) x^4 \quad - 3x^2 + 4x + 5 (x^2 + 3x + 4$

$x^4 - 3x^3 + 2x^2$

$3x^3 - 5x^2 + 4x + 5$

$3x^3 - 9x^2 + 6x$

$4x^2 - 2x + 5$

$4x^2 - 12x + 8$

$10x - 3$

Then quotient $= x^2 + 3x + 4$ and remainder $= 10x - 3$

Verification:-

Divisor× Quotient + Remainder

$= (x^2 - 3x + 2)(x^2 + 3x + 4) + (10x - 3)$

$= x^4 + 3x^3 + 4x^2 - 3x^3 - 9x^2 - 12x + 2x^2 + 6x + 8) + 10x - 3$

$= x^4 - 3x^2 + 4x + 5$

= Dividend

Hence verified.

(x) $\boldsymbol{p(x) = x^4 - 5x^2 + 6,\quad d(x) = 2 - x^2}$

Solution:-

$$-x^2 + 2)x^4 - 5x^2 + 6(-x^2 + 3$$
$$\underline{x^4 - 2x^2}$$
$$-3x^2 + 6$$
$$\underline{-3x^2 + 6}$$
$$0$$

Then quotient $= -x^2 + 3$ and remainder $= 0$

Verification:-

Divisor× Quotient + Remainder $= (-x^2 + 2)(-x^2 + 3) + 0$

$= x^4 - 3x^2 - 2x^2 + 6$

$= x^4 - 5x^2 + 6$

= Dividend

Hence verified.

2. Check whether the first polynomial is a factor of the second polynomial by actual division:

(i) $\boldsymbol{x^2 - x + 1,\quad x^3 + 1}$

Solution:-

$$x^2 - x + 1)x^3 \qquad + 1(x + 1$$
$$\underline{x^3 - x^2 + x}$$
$$x^2 - x + 1$$
$$\underline{x^2 - x + 1}$$
$$0$$

Here, remainder $= 0$

$\therefore x^2 - x + 1$ is a factor of $x^3 + 1$.

(ii) $\boldsymbol{x^2 - 3,\quad 2x^4 + 3x^3 - 2x^2 - 9x - 5}$

Solution:-

$$x^2 - 3)2x^4 + 3x^3 - 2x^2 - 9x - 5(2x^2 + 3x + 4$$
$$\underline{2x^4 \qquad - 6x^2}$$
$$3x^3 + 4x^2 - 9x - 5$$
$$\underline{3x^3 \qquad - 9x}$$
$$4x^2 \qquad - 5$$
$$\underline{4x^2 \qquad - 12}$$
$$7$$

Here, remainder $\neq 0$

$\therefore x^2 - 3$ is not a factor of $2x^4 + 3x^3 - 2x^2 - 9x - 5$.

(iii) $\boldsymbol{t^2 + 3t + 1, \quad 3t^4 + 5t^3 + 2 + 2t - 7t^2}$

Solution:-

$$
\begin{array}{l}
t^2 + 3t + 1)\,3t^4 + 5t^3 - 7t^2 + 2t + 2\,(\,3t^2 - 4t + 2 \\
\qquad\qquad\quad \underline{3t^4 + 9t^3 + 3t^2} \\
\qquad\qquad\qquad -4t^3 - 10t^2 + 2t + 2 \\
\qquad\qquad\qquad \underline{-4t^3 - 12t^2 - 4t} \\
\qquad\qquad\qquad\qquad 2t^2 + 6t + 2 \\
\qquad\qquad\qquad\qquad \underline{2t^2 + 6t + 2} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Here, remainder $= 0$

$\therefore t^2 + 3t + 1$ is a factor of $3t^4 + 5t^3 - 7t^2 + 2t + 2$.

(iv) $\boldsymbol{1 + y^3 + 3y, \quad -1 - 4y^3 + y^5 + y^2 + 3y}$

Solution:-

$$
\begin{array}{l}
y^3 + 3y + 1)\,y^5 - 4y^3 + y^2 + 3y - 1\,(\,y^2 - 7 \\
\qquad\qquad\quad \underline{y^5 + 3y^3 + y^2} \\
\qquad\qquad\qquad -7y^3 \qquad + 3y - 1 \\
\qquad\qquad\qquad \underline{-7y^3 \qquad - 21y - 7} \\
\qquad\qquad\qquad\qquad\quad 24y + 6
\end{array}
$$

Here, remainder $\neq 0$

$\therefore y^3 + 3y + 1$ is not a factor of $y^5 - 4y^3 + y^2 + 3y - 1$.

(v) $\boldsymbol{7 + 3x, \quad 3x^3 + 7x}$

Solution:-

$$
\begin{array}{l}
3x + 7)\,3x^3 \qquad + 7x\,(\,x^2 - \frac{7}{3}x + \frac{70}{9} \\
\qquad \underline{3x^3 + 7x^2} \\
\qquad\quad -7x^2 + 7x \\
\qquad\quad \underline{-7x^2 - \frac{49}{3}x} \\
\qquad\qquad\quad \frac{70}{3}x \\
\qquad\qquad\quad \underline{\frac{70}{3}x + \frac{490}{9}} \\
\qquad\qquad\qquad -\frac{490}{9}
\end{array}
$$

Here, remainder $\neq 0$

$\therefore 3x + 7$ is not a factor of $3x^3 + 7x$.

3. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $f(x)$, the quotient and remainder are $x - 2$ and $4 - 2x$ respectively. Find $f(x)$.

Solution:-

We have, Dividend $= x^3 - 3x^2 + x + 2$

Divisor $= f(x)$

Quotient $= x - 2$

Remainder $= 4 - 2x$

By division algorithm, we know

Divisor × Quotient + Remainder = Dividend

$\Rightarrow f(x)(x-2) + (4-2x) = x^3 - 3x^2 + x + 2$

$\Rightarrow f(x)(x-2) = x^3 - 3x^2 + x + 2 - 4 + 2x$

$\Rightarrow f(x)(x-2) = x^3 - 3x^2 + 3x - 2$

$\Rightarrow f(x) = \frac{x^3-3x^2+3x-2}{x-2}$

Now, $x-2)x^3 - 3x^2 + 3x - 2(x^2 - x + 1$

$$\begin{array}{l} x^3 - 2x^2 \\ \hline \quad -x^2 + 3x - 2 \\ \quad -x^2 + 2x \\ \hline \qquad\qquad x - 2 \\ \qquad\qquad x - 2 \\ \hline \qquad\qquad\quad 0 \end{array}$$

$\therefore f(x) = x^2 - x + 1$

4. When a polynomial $p(x)$ is divided by $3x - 1$, the quotient and remainder are $x^2 + 2x - 3$ and 5 respectively. Find $p(x)$.

Solution:-

We have, Dividend $= p(x)$

Divisor $= 3x - 1$

Quotient $= x^2 + 2x - 3$

Remainder $= 5$

By division algorithm, we know

Dividend = Divisor × Quotient + Remainder

i.e. $p(x) = (3x-1)(x^2+2x-3) + 5$

$= 3x^3 + 6x^2 - 9x - x^2 - 2x + 3 + 5$

$= 3x^3 + 5x^2 - 11x + 8$

****

- **Remainder Theorem**

  Let $p(x)$ be any polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

  In case degree of the dividend $p(x)$ is less than that of the divisor $d(x)$, then we take $q(x) = 0$ and $r(x) = p(x)$.

- **Factor Theorem:-** If $p(x)$ is a polynomial of degree ≥ 1 and a is any real number, then $x - a$ is a factor of $p(x)$ if and only if $p(a) = 0$.

SOLUTIONS

EXERCISE 2.2

1. Find without actual division, the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

(i) $x - 1$

Solution:-

Let $p(x) = x^3 + 3x^2 + 3x + 1$

The zero of $x - 1$ is 1.

Hence, by Remainder Theorem, the remainder when $p(x)$ is divided by $x - 1$

$= p(1)$

$= 1^3 + 3 \times 1^2 + 3 \times 1 + 1$

$= 1 + 3 + 3 + 1$

$= 8$

(ii) $x + 1$

Solution:-

Let $p(x) = x^3 + 3x^2 + 3x + 1$

The zero of $x + 1$ is -1.

Hence, by Remainder Theorem, the remainder when $p(x)$ is divided by $x + 1$

$= p(-1)$

$= (-1)^3 + 3 \times (-1)^2 + 3 \times (-1) + 1$

$= -1 + 3 - 3 + 1$

$= 0$

(iii) $\boldsymbol{x-\frac{1}{2}}$

Solution:-

Let $p(x) = x^3 + 3x^2 + 3x + 1$

The zero of $x - \frac{1}{2}$ is $\frac{1}{2}$.

Hence, by Remainder Theorem, the remainder when $p(x)$ is divided by $x - \frac{1}{2}$

$$= p\left(\frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right) + 1$$

$$= \frac{1}{8} + 3 \times \frac{1}{4} + 3 \times \frac{1}{2} + 1$$

$$= \frac{1}{8} + \frac{3}{4} + \frac{3}{2} + 1$$

$$= \frac{1+6+12+8}{8}$$

$$= \frac{27}{8}$$

(iv) $\boldsymbol{2x+1}$

Solution:-

Let $p(x) = x^3 + 3x^2 + 3x + 1$

The zero of $2x + 1$ is $-\frac{1}{2}$.

Hence, by Remainder Theorem, the remainder when $p(x)$ is divided by $2x + 1$

$$= p\left(-\frac{1}{2}\right)$$

$$= \left(-\frac{1}{2}\right)^3 + 3 \times \left(-\frac{1}{2}\right)^2 + 3 \times \left(-\frac{1}{2}\right) + 1$$

$$= -\frac{1}{8} + 3 \times \frac{1}{4} - 3 \times \frac{1}{2} + 1$$

$$= -\frac{1}{8} + \frac{3}{4} - \frac{3}{2} + 1$$

$$= \frac{-1+6-12+8}{8}$$

$$= \frac{1}{8}$$

2. Determine whether $\boldsymbol{x+1}$ is a factor of :

(i) $\boldsymbol{x^3+x^2+x+1}$

Solution:-

Let $p(x) = x^3 + x^2 + x + 1$

The zero of $x + 1$ is -1.

Now, $p(-1) = (-1)^3 + (-1)^2 + (-1) + 1$

$$= -1 + 1 - 1 + 1$$

$$= 0$$

Hence, by Factor Theorem, $x + 1$ is a factor of $x^3 + x^2 + x + 1$.

(ii) $\boldsymbol{x^4 + x^3 + x^2 + x + 1}$

Solution:-

Let $p(x) = x^4 + x^3 + x^2 + x + 1$

The zero of $x + 1$ is -1.

Now, $p(-1) = (-1)^4 + (-1)^3 + (-1)^2 + (-1) + 1$

$= 1 - 1 + 1 - 1 + 1$

$= 1 \neq 0$

Hence, by Factor Theorem, $x + 1$ is not a factor of $x^4 + x^3 + x^2 + x + 1$.

(iii) $\boldsymbol{x^4 + 3x^3 + 3x^2 + x + 1}$

Solution:-

Let $p(x) = x^4 + 3x^3 + 3x^2 + x + 1$

The zero of $x + 1$ is -1.

Now, $p(-1) = (-1)^4 + 3 \times (-1)^3 + 3 \times (-1)^2 + (-1) + 1$

$= 1 + 3 \times (-1) + 3 \times 1 - 1 + 1$

$= 1 - 3 + 3 - 1 + 1$

$= 1 \neq 0$

Hence, by Factor Theorem, $x + 1$ is not a factor of $x^4 + 3x^3 + 3x^2 + x + 1$.

(iv) $\boldsymbol{x^3 + 3x^2 + (2 + \sqrt{2})x + \sqrt{2}}$

Solution:-

Let $p(x) = x^3 + 3x^2 + (2 + \sqrt{2})x + \sqrt{2}$

The zero of $x + 1$ is -1.

Now, $p(-1) = (-1)^3 + 3 \times (-1)^2 + (2 + \sqrt{2}) \times (-1) + \sqrt{2}$

$= -1 + 3 \times 1 - (2 + \sqrt{2}) + \sqrt{2}$

$= -1 + 3 - 2 - \sqrt{2} + \sqrt{2}$

$= 3 - 3$

$= 0$

Hence, by Factor Theorem, $x + 1$ is a factor of $x^3 + 3x^2 + (2 + \sqrt{2})x + \sqrt{2}$.

3. Use Factor Theorem to determine whether $q(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $\boldsymbol{p(x) = x^4 - 81,\ \ q(x) = x + 3}$

Solution:-

The zero of $x + 3$ is -3.

Putting $x = -3$ in $p(x)$, we get

$p(-3) = (-3)^4 - 81 = 81 - 81 = 0$

So, $q(x) = x + 3$ is a factor of $p(x) = x^4 - 81$.

(ii) $\boldsymbol{p(x) = 2x^3 + x^2 - 2x - 1,} \qquad \boldsymbol{q(x) = x + 1}$

Solution:-

The zero of $q(x) = x + 1$is -1.
Putting $x = -1$ in $p(x)$ in $p(x)$, we get

$$p(-1) = 2(-1)^3 + (-1)^2 - 2 \times (-1) - 1$$
$$= 2 \times (-1) + 1 + 2 - 1$$
$$= -2 + 1 + 2 - 1$$
$$= 0$$

So, $q(x) = x + 1$ is a factor of $p(x) = 2x^3 + x^2 - 2x - 1$.

(iii) $\boldsymbol{p(x) = x^3 - 3x^2 - 3x + 1,} \qquad \boldsymbol{q(x) = x - 1}$

Solution:-

The zero of $q(x) = x - 1$is 1.
Putting $x = 1$ in $p(x)$, we get

$$p(1) = 1^3 - 3 \times 1^2 - 3 \times 1 + 1$$
$$= 1 - 3 - 3 + 1$$
$$= 2 - 6$$
$$= -4 \neq 0$$

So, $q(x) = x - 1$ is not a factor of $p(x) = x^3 - 3x^2 - 3x + 1$.

(iv) $\boldsymbol{p(x) = x^3 - 4x^2 + x + 6,} \qquad \boldsymbol{q(x) = x - 4}$

Solution:-

The zero of $q(x) = x - 4$is 4.
Putting $x = 4$ in $p(x)$, we get

$$p(4) = 4^3 - 4 \times 4^2 + 4 + 6$$
$$= 64 - 64 + 4 + 6$$
$$= 10 \neq 0$$

So, $q(x) = x - 4$ is not a factor of $p(x) = x^3 - 4x^2 + x + 6$.

4. Using Factor Theorem, show that

(i) $\boldsymbol{x - 1}$ **is a factor of** $\boldsymbol{3x^5 - 2x^2 - 6x + 5}$

Solution:-

Let $p(x) = 3x^5 - 2x^2 - 6x + 5$
The zero of $x - 1$ is 1.

Now, $\quad p(1) = 3 \times 1^5 - 2 \times 1^2 - 6 \times 1 + 5$
$$= 3 - 2 - 6 + 5$$
$$= 0$$

Hence, $x - 1$ is a factor of $3x^5 - 2x^2 - 6x + 5$.

(ii) $\boldsymbol{x+1}$ **is a factor of** $\boldsymbol{2x^4-3x^2+6x+7}$

Solution:-

Let $p(x)=2x^4-3x^2+6x+7$

The zero of $x+1$ is -1.

Now, $p(-1)=2\times(-1)^4-3\times(-1)^2+6\times(-1)+7$

$=2\times1-3\times1-6+7$

$=2-3-6+7$

$=9-9$

$=0$

Hence, $x+1$ is a factor of $2x^4-3x^2+6x+7$.

(iii) $\boldsymbol{x-2}$ **is a factor of** $\boldsymbol{x^3-9x^2+26x-24}$

Solution:-

Let $p(x)=x^3-9x^2+26x-24$

The zero of $x-2$ is 2.

Now, $p(2)=2^3-9\times2^2+26\times2-24$

$=8-9\times4+52-24$

$=8-36+52-24$

$=60-60$

$=0$

Hence, $x-2$ is a factor of $x^3-9x^2+26x-24$.

(iv) $\boldsymbol{x+y,\ y+z,\ z+x}$ **are factors of** $\boldsymbol{(x+y+z)^3-(x^3+y^3+z^3)}$

Solution:-

The zero of $x+y$ is given by $x+y=0$ i.e. $x=-y$.

Putting $x=-y$, we get

$(x+y+z)^3-(x^3+y^3+z^3)=(-y+y+z)^3-\{(-y)^3+y^3+z^3\}$

$=(-y+y+z)^3-(-y^3+y^3+z^3)$

$=z^3-z^3$

$=0$

$\therefore x+y$ is a factor of $(x+y+z)^3-(x^3+y^3+z^3)$.

Again the zero of $y+z$ is given by $y+z=0$ i.e. $y=-z$.

Putting $y=-z$, we get

$$(x+y+z)^3-(x^3+y^3+z^3) = (x-z+z)^3-\{x^3+(-z)^3+z^3\}$$
$$= (x-z+z)^3-(x^3+-z^3+z^3)$$
$$= x^3-x^3$$
$$= 0$$

$\therefore y+z$ is also a factor of $(x+y+z)^3-(x^3+y^3+z^3)$.

And the zero of $z+x$ is given by $z+x=0$ i.e. $z=-x$.
Putting $z=-x$, we get

$$(x+y+z)^3-(x^3+y^3+z^3) = (x+y-x)^3-\{x^3+y^3+(-x)^3\}$$
$$= (x+y-x)^3-(x^3+y^3-x^3)$$
$$= y^3-y^3$$
$$= 0$$

$\therefore z+x$ is also a factor of $(x+y+z)^3-(x^3+y^3+z^3)$.

(v) $\boldsymbol{x-y, y-z, z-x}$ **are factors of** $\boldsymbol{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}$

Solution:-

The zero of $x-y$ is given by $x-y=0$ i.e. $x=y$.
Putting $x=y$, we get

$$x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$$
$$= y^3(y^2-z^2)+y^3(z^2-y^2)+z^3(y^2-y^2)$$
$$= y^5-y^3z^2+y^3z^2-y^5+0$$
$$= 0$$

$\therefore x-y$ is a factor of $x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$.

Again the zero of $y-z$ is given by $y-z=0$ i.e. $y=z$.

Putting $y=z$, we get

$$x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$$
$$= x^3(z^2-z^2)+z^3(z^2-x^2)+z^3(x^2-z^2)$$
$$= 0+z^5-z^3x^2+z^3x^2-z^5$$
$$= 0$$

$\therefore y-z$ is also a factor of $x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$.

And the zero of $z-x$ is given by $z-x=0$ i.e. $z=x$.

Putting $z=x$, we get

$$x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$$
$$= x^3(y^2-x^2)+y^3(x^2-x^2)+x^3(x^2-y^2)$$
$$= x^3y^2-x^5+0+x^5-x^3y^2$$
$$= 0$$

$\therefore z-x$ is also a factor of $x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)$.

5. Find the value of k, if $x-1$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x)=x^2-2x+k$

Solution:-

The zero of $x-1$ is 1.

Since $x-1$ is a factor of $p(x)=x^2-2x+k$,

$$p(1)=0$$
$$\Rightarrow 1^2-2\times 1+k=0$$
$$\Rightarrow 1-2+k=0$$
$$\Rightarrow -1+k=0$$
$$\therefore k=1$$

(ii) $p(x)=\sqrt{2}x^2+kx-1$

Solution:-

The zero of $x-1$ is 1.

Since $x-1$ is a factor of $p(x)=\sqrt{2}x^2+kx-1$,

$$p(1)=0$$
$$\Rightarrow \sqrt{2}\times 1^2+k\times 1-1=0$$
$$\Rightarrow \sqrt{2}+k-1=0$$
$$\therefore k=1-\sqrt{2}$$

(iii) $p(x)=kx^2-\sqrt{2}x+1$

Solution:-

The zero of $x-1$ is 1.

Since $x-1$ is a factor of $p(x)=kx^2-\sqrt{2}x+1$,

$$p(1)=0$$
$$\Rightarrow k\times 1^2-\sqrt{2}\times 1+1=0$$
$$\Rightarrow k-\sqrt{2}+1=0$$
$$\therefore k=\sqrt{2}-1$$

(iv) $p(x)=kx^2-3x+2k$

Solution:-

The zero of $x-1$ is 1.

Since $x-1$ is a factor of $p(x)=kx^2-3x+2k$,

$$p(1)=0$$
$$\Rightarrow k\times 1^2-3\times 1+2k=0$$
$$\Rightarrow k-3+2k=0$$
$$\Rightarrow 3k=3$$
$$\therefore k=1$$

6. If $x^2 + px + q$ and $x^2 + lx + m$ are both divisible by $x + a$, show that $a = \frac{m-q}{l-p}$.

Solution:-

The zero of $x + a$ is $-a$.

Since $x^2 + px + q$ is divisible by $x + a$,

$(-a)^2 + p \times (-a) + q = 0$ [Putting $x = -a$]

$\Rightarrow a^2 - pa + q = 0$ ---------------- (1)

Again, since $x^2 + lx + m$ is divisible by $x + a$,

$(-a)^2 + l \times (-a) + m = 0$ [Putting $x = -a$]

$\Rightarrow a^2 - la + m = 0$ ---------------- (2)

From equations (1) and (2), we get

$a^2 - pa + q = a^2 - la + m$

$\Rightarrow -pa + q = -la + m$

$\Rightarrow la - pa = m - q$

$\Rightarrow a(l - p) = m - q$

$\therefore a = \frac{m-q}{l-p}$

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SOLUTIONS

EXERCISE 2.3

1. Factorise by using Factor Theorem:

(i) $x^2 - 4x + 3$

Solution:-

Let $p(x) = x^2 - 4x + 3$

The factors of 3 are $\pm 1, \pm 3$.

Now, $p(1) = 1^2 - 4 \times 1 + 3$

$= 1 - 4 + 3$

$= 4 - 4$

$= 0$

So, by Factor Theorem, $x-1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x-1$, we have

$$x-1)x^2-4x+3(x-3$$
$$\underline{x^2-x}$$
$$-3x+3$$
$$\underline{-3x+3}$$

Hence, $x^2-4x+3=(x-1)(x-3)$

(ii) $\boldsymbol{x^2-6x+8}$

Solution:-

Let $p(x)=x^2-6x+8$

The factors of 8 are $\pm1, \pm2, \pm4, \pm8$.

Now, $p(2)=2^2-6\times2+8$

$=4-12+8$

$=12-12$

$=0$

So, $x-2$ is a factor of $p(x)$.

Dividing $p(x)$ by $x-2$, we have

$$x-2)x^2-6x+8(x-4$$
$$\underline{x^2-2x}$$
$$-4x+8$$
$$\underline{-4x+8}$$

Hence, $x^2-6x+8=(x-2)(x-4)$

(iii) $\boldsymbol{x^2+8x+15}$

Solution:-

Let $p(x)=x^2+8x+15$

The factors of 15 are $\pm1, \pm3, \pm5, \pm15$.

Now, $p(-3)=(-3)^2+8\times(-3)+15$

$=9-24+15$

$=24-24$

$=0$

So, $x+3$ is a factor of $p(x)$.

Dividing $p(x)$ by $x+3$, we have

$$x+3)x^2+8x+15(x+5$$
$$\underline{x^2+3x}$$
$$5x+15$$
$$\underline{5x+15}$$

Hence, $x^2+8x+15=(x+3)(x+5)$.

(iv) $\boldsymbol{x^2-6x-7}$

Solution:-

Let $p(x)=x^2-6x-7$

The factors of -7 are $\pm1, \pm7$.

Now, $p(-1)=(-1)^2-6\times(-1)-7$

$=1+6-7$

$=7-7$

$=0$

So, $x+1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x+1$, we have

$$x+1)x^2-6x-7(x-7$$
$$\underline{x^2+x}$$
$$-7x-7$$
$$\underline{-7x-7}$$

Hence, $x^2-6x-7=(x+1)(x-7)$.

(v) $\boldsymbol{x^2+3x-10}$

Solution:-

Let $p(x)=x^2+3x-10$

The factors of -10 are $\pm1, \pm2, \pm5, \pm10$.

Now, $p(2)=2^2+3\times2-10$

$=4+6-10$

$=10-10$

$=0$

So, $x-2$ is a factor of $p(x)$.

Dividing $p(x)$ by $x-2$, we have

$$x-2)x^2+3x-10(x+5$$
$$\underline{x^2-2x}$$
$$5x-10$$
$$\underline{5x-10}$$

Hence, $x^2+3x-10=(x-2)(x+5)$.

(vi) $\boldsymbol{x^3-3x^2-9x-5}$

Solution:-

Let $p(x)=x^3-3x^2-9x-5$

The factors of -5 are $\pm1, \pm5$.

Now, $p(-1)=(-1)^3-3\times(-1)^2-9\times(-1)-5$
$=-1-3\times1+9-5$
$=-1-3+9-5$
$=9-9$
$=0$

So, $x+1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x+1$, we have

$$x+1)x^3-3x^2-9x-5(x^2-4x-5$$
$$\underline{x^3+x^2}$$
$$-4x^2-9x-5$$
$$\underline{-4x^2-4x}$$
$$-5x-5$$
$$\underline{-5x-5}$$

Hence, $x^3-3x^2-9x-5=(x+1)(x^2-4x-5)$
$=(x+1)\{x^2-(5-1)x-5\}$
$=(x+1)(x^2-5x+x-5)$
$=(x+1)\{x(x-5)+1(x-5)\}$
$=(x+1)(x-5)(x+1)$
$=(x+1)^2(x-5)$

(vii) $\boldsymbol{x^3 + 13x^2 + 32x + 20}$

Solution:-

Let $p(x) = x^3 + 13x^2 + 32x + 20$
The factors of 20 are $\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$.

Now, $p(-1) = (-1)^3 + 13 \times (-1)^2 + 32 \times (-1) + 20$

$= -1 + 13 \times 1 - 32 + 20$

$= -1 + 13 - 32 + 20$

$= 33 - 33$

$= 0$

So, $x + 1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x + 1$, we have

$$
\begin{array}{r}
x+1)\,x^3 + 13x^2 + 32x + 20\,(x^2 + 12x + 20 \\
\underline{x^3 + \quad x^2 \qquad\qquad\quad} \\
12x^2 + 32x + 20 \\
\underline{12x^2 + 12x \qquad} \\
20x + 20 \\
\underline{20x + 20} \\
\end{array}
$$

Hence, $x^3 + 13x^2 + 32x + 20 = (x + 1)(x^2 + 12x + 20)$

$= (x + 1)\{x^2 + (2 + 10)x + 20\}$

$= (x + 1)(x^2 + 2x + 10x + 20)$

$= (x + 1)\{x(x + 2) + 10(x + 2)\}$

$= (x + 1)(x + 2)(x + 10)$

(viii) $\boldsymbol{x^3 - 7x + 6}$

Solution:-

Let $p(x) = x^3 - 7x + 6$

The factors of 6 are $\pm1, \pm2, \pm3, \pm6$.

Now, $p(1) = 1^3 - 7 \times 1 + 6$

$= 1 - 7 + 6$

$= 7 - 7$

$= 0$

So, $x - 1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x-1$, we have

$$x-1)x^3 \quad -7x+6(x^2+x-6$$
$$x^3-x^2$$
$$x^2-7x+6$$
$$x^2-x$$
$$-6x+6$$
$$-6x+6$$

Hence, $x^3-7x+6=(x-1)(x^2+x-6)$

$=(x-1)\{x^2+(3-2)x-6\}$

$=(x-1)(x^2+3x-2x-6)$

$=(x-1)\{x(x+3)-2(x+3)\}$

$=(x-1)(x-2)\ (x+3)$

(ix) $\boldsymbol{x^3+7x^2+14x+8}$

Solution:-

Let $p(x)=x^3+7x^2+14x+8$

The factors of 8 are $\pm1, \pm2, \pm4, \pm8$.

Now, $p(-1)=(-1)^3+7\times(-1)^2+14\times(-1)+8$

$=-1+7\times1-14+8$

$=-1+7-14+8$

$=15-15$

$=0$

So, $x+1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x+1$, we have

$$x+1)x^3+7x^2+14x+8(x^2+6x+8$$
$$x^3+x^2$$
$$6x^2+14x+8$$
$$6x^2+\ 6x$$
$$8x+8$$
$$8x+8$$

Hence, $x^3+7x^2+14x+8=(x+1)(x^2+6x+8)$

$=(x+1)\{x^2+(2+4)x+8\}$

$=(x+1)(x^2+2x+4x+8)$

$=(x+1)\{x(x+2)+4(x+2)\}$

$=(x+1)(x+2)(x+4)$

(x) $\boldsymbol{x^3 - 13x - 12}$

Solution:-

Let $p(x) = x^3 - 13x - 12$

The factors of -12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Now, $p(-1) = (-1)^3 - 13 \times (-1) - 12$

$= -1 + 13 - 12$

$= 13 - 13$

$= 0$

So, $x + 1$ is a factor of $p(x)$.

Dividing $p(x)$ by $x + 1$, we have

$$
\begin{array}{l}
x+1)\, x^3 \qquad - 13x - 12\,(x^2 - x - 12 \\
\qquad \underline{x^3 + x^2} \\
\qquad\quad -x^2 - 13x - 12 \\
\qquad\quad \underline{-x^2 - \quad x} \\
\qquad\qquad -12x - 12 \\
\qquad\qquad \underline{-12x - 12}
\end{array}
$$

Hence, $x^3 - 13x - 12 = (x + 1)(x^2 - x - 12)$

$= (x + 1)\{x^2 - (4 - 3)x - 12\}$

$= (x + 1)(x^2 - 4x + 3x - 12)$

$= (x + 1)\{x(x - 4) + 3(x - 4)\}$

$= (x + 1)(x + 3)(x - 4)$

(xi) $\boldsymbol{x^4 + 6x^3 + 13x^2 + 12x + 4}$

Solution:-

Let $p(x) = x^4 + 6x^3 + 13x^2 + 12x + 4$

The factors of 4 are $\pm 1, \pm 2, \pm 4$.

Now, $p(-1) = (-1)^4 + 6 \times (-1)^3 + 13 \times (-1)^2 + 12 \times (-1) + 4$

$= 1 + 6 \times (-1) + 13 \times 1 - 12 + 4$

$= 1 - 6 + 13 - 12 + 4$

$= 18 - 18$

$= 0$

So, $x + 1$ is a factor of $p(x)$.

Again, $p(-2) = (-2)^4 + 6 \times (-2)^3 + 13 \times (-2)^2 + 12 \times (-2) + 4$

$= 16 + 6 \times (-8) + 13 \times 4 - 24 + 4$

$= 16 - 48 + 52 - 24 + 4$

$= 72 - 72$

$= 0$

So, $x + 2$ is also a factor of $p(x)$.

Now, $(x + 1)(x + 2) = x^2 + 3x + 2$

Dividing $p(x)$ by $x^2 + 3x + 2$, we have

$$x^2 + 3x + 2\ )x^4 + 6x^3 + 13x^2 + 12x + 4\ (x^2 + 3x + 2$$

$$\underline{x^4 + 3x^3 + 2x^2}$$

$$3x^3 + 11x^2 + 12x + 4$$

$$\underline{3x^3 + 9x^2 + 6x}$$

$$2x^2 + 6x + 4$$

$$\underline{2x^2 + 6x + 4}$$

Hence, $x^4 + 6x^3 + 13x^2 + 12x + 4 = (x^2 + 3x + 2)(x^2 + 3x + 2)$

$= (x + 1)(x + 2)(x + 1)(x + 2)$

$= (x + 1)^2(x + 2)^2$

(xii) $\boldsymbol{x^4 - 1}$

Solution:-

Let $p(x) = x^4 - 1$

The factors of -1 are ± 1.

Now, $p(1) = 1^4 - 1$

$= 1 - 1$

$= 0$

So, $x - 1$ is a factor of $p(x)$.

Again, $p(-1) = (-1)^4 - 1$

$= 1 - 1$

$= 0$

So, $x + 1$ is also a factor of $p(x)$.

Now, $(x-1)(x+1) = x^2 - 1$

Dividing $p(x)$ by $x^2 - 1$, we have

$$
\begin{array}{r|l}
x^2 - 1 & x^4 \qquad\qquad - 1 \;(x^2 + 1 \\
 & x^4 \;\; - x^2 \\
\hline
 & \quad x^2 \;\; - 1 \\
 & \quad x^2 \;\; - 1 \\
\hline
\end{array}
$$

Hence $x^4 - 1 = (x^2 - 1)(x^2 + 1)$

$= (x+1)(x-1)(x^2+1)$

2. Factorise $x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$.

Solution:-

Putting $x = -y$, we have

$x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$

$= (-y)^2(y+z) + y^2(z-y) + z^2(-y+y) + 2(-y)yz$

$= y^2(y+z) + y^2(z-y) + 0 - 2y^2z$

$= y^3 + y^2z + y^2z - y^3 - 2y^2z$

$= 2y^2z - 2y^2z$

$= 0$

$\therefore x + y$ is a factor of $x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$.

Again, putting $y = -z$, we have

$x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$

$= x^2(-z+z) + (-z)^2(z+x) + z^2(x-z) + 2x(-z)z$

$= 0 + z^2(z+x) + z^2(x-z) - 2z^2x$

$= z^3 + z^2x + z^2x - z^3 - 2z^2x$

$= 2z^2x - 2z^2x$

$= 0$

$\therefore y + z$ is also a factor of $x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$.

And putting $z = -x$, we have

$$x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$$

$= x^2(y-x) + y^2(-x+x) + (-x)^2(x+y) + 2xy(-x)$

$= x^2(y-x)) + 0 + x^2(x+y) - 2x^2y$

$= x^2y - x^3 + x^3 + x^2y - 2x^2y$

$= 2x^2y - 2x^2y$

$= 0$

$\therefore z + x$ is also a factor of $x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz$.

Since the given polynomial is of degree 3, it has only three linear factors.

Then $x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz = (x+y)(y+z)(z+x)$

3. Prove that $x^n - a^n$ is divisible by $x + a$ only when n is even.

Solution:-

The zero of $x + a$ is $-a$.

Putting $x = -a$, we have

$$\begin{aligned} x^n - a^n &= (-a)^n - a^n \\ &= (-1)^n a^n - a^n \\ &= \{(-1)^n - 1\}a^n \\ &= 0 \text{ only when } n \text{ is even.} \end{aligned}$$

Hence, $x + a$ is a factor of $x^n - a^n$ (i.e. $x + a$ divides $x^n - a^n$) only when n is even.

Or

The zero of $x + a$ is $-a$.

Putting $x = -a$, we have

$$\begin{aligned} x^n - a^n &= (-a)^n - a^n \\ &= (-1)^n a^n - a^n \\ &= \{(-1)^n - 1\}a^n \end{aligned}$$

When n is odd, $\{(-1)^n - 1\}a^n = (-1-1)a^n$

$= -2a^n$

$\neq 0$

$\therefore\ x^n - a^n$ is not divisible by $x + a$.

When n is even, $\{(-1)^n - 1\}a^n = (1-1)a^n$

$$= 0 \times a^n$$

$$= 0$$

$\therefore\ x^n - a^n$ is divisible by $x + a$.

Hence $x^n - a^n$ is divisible by $x + a$ only when n is even.

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