DEPARTMENT OF EDUCATION (S)
Government of Manipur

CHAPTER – 1
BINARY OPERATIONS

SOLUTIONS

Textual Questions and answers

Exercise 1.1

1. If E is the set of all even natural numbers and F, the set of all odd natural numbers, answer the following:
   (a) Is addition a binary operation on F ?
   (b) Is Multiplication a binary operation on F ? If yes, find whether identity element exists or not.
   (c) Is addition a binary operation on E ? If yes, find whether identity element exists or not.
   (d) Is multiplication a binary operation on E ? If yes, find whether identity element exists or not.

   Soln: Here, E = the set of all even natural numbers and

   F = the set of all odd natural numbers.

   (a) No, addition is not a binary operation on F because sum of two odd natural number is not an odd natural number.
   (b) Yes, multiplication is a binary operation on F because the product of any two odd natural number is again an odd natural number. 1 is the identity element on F as $x \times 1 = 1 \times x = x$, for all $x \in F$.
   (c) Yes, addition is a binary operation on E. Identity element does not exist as the additive identity , $0 \notin E$.
   (d) Yes, multiplication is a binary operation on E. Identity element does not exist as the multiplicative identity, $1 \notin E$.

2. State whether each of the following definitions of * gives a binary operation on N or not. Give justification of your answer.

(i) $a * b = a - b$

Soln: * is not a binary operation for $1 * 3 = 1 - 3 = -2 \notin N; 1, 3 \in N$.

(ii) $a * b = |a - b|$

Soln: * is not a binary operation for $1 * 1 = |1 - 1| = 0 \notin N;\ 1 \in N$.

(iii) $a * b = a^2 b$

Soln: * is a binary operation for $a * b = a^2 b \in N$, a, b $\in N$.

(iv) $a*b=b$

Soln: * is a binary operation for $a*b=b\in N,\ a,\ \text{b}\ \in N.$

(v) $a*b=a+ab$

Soln: * is a binary operation for $a*b=a+ab\in N,\ \forall\ a,b\in N.$

(vi) $a*b=a^b$

Soln: * is a binary operation for $a*b=a^b\in N, \forall\ a,b\in N.$

(vii) $a*b=ab-1$

Soln: * is not a binary operation for $1*1=1.1-1=1-1=0\notin N;\ 1\ \ \in N$

(viii) $a*b=ab+1$

Soln: * is a binary operation for $a*b=ab+1\in N, \forall\ a,b\in N.$

3. Prove that the following binary operations on N are commutative but not associative:

(i) $a*b=2a+2b,$ where $a,b\in N.$

Soln: Let $a,b\in N.$

Then, $a*b=2a+2b$
$= 2b + 2a$
$=b*a$

So, * is commutative.

Let $a,b,c\in N.$

Then, $a*(b*c)=a*(2b+2c),$
$=2a+2(2b+2c),$
$=2a + 4b+4c.$

And, $(a*b)*c=(2a+2b)*c,$
$=2(2a+2b)+2c,$
$=4a + 4b+2c\neq a*(b*c).$

So, * is not associative.

(ii) $a*b=2^{ab}$

Soln: Let $a,b\in N$.

Then, $a*b=2^{ab}$

$=2^{ba}$

$=b*a$

So, * is commutative.

Let $a,b,c\in N$.

Then, $a*(b*c)=a*2^{bc}$,

$=2^{a2^{bc}}$.

And, $(a*b)*c=2^{ab}*c$,

$=2^{2^{ab}c}$,

$\neq a*(b*c)$.

So, * is not associative.

(iii) $a*b=(a-b)^2$

Soln: *Let* $a,b\in N$.

Then, $a*b=(a-b)^2$

$=(\mathrm{b}-a)^2=b*a$.

So, * is commutative.

Let $a,b,c\in N$.

Then, $a*(b*c)=a*(b-c)^2$,

$=[a-\ (b-c)^2]^2$.

And, $(a*b)*c=(a-b)^2*c$,

$=[(a-\ b)^2-c]^2$,

$\neq a*(b*c)$.

$\therefore$ * is not associative.

(iv) $a*b = ab+1$

Soln: $Let\ a,b \in N.$

$Then,\ a*b = ab+1$
$= ba+1\ = b*a.$
$So,$ * is commutative.

$Let\ a,b,c \in N.$
$Then,\ a*(b*c) = a*(bc+1),$
$= a(bc+1)+1,$
$= abc+a+1$
$And,\ (a*b)*c = (ab+1)*c,$
$= (ab+1)c+1,$
$= abc+c+1$
$\neq a*(b*c).$
$\therefore$ * is not associative.

4. Show that the binary operation on N defined by $a*b = b$ is associative but not commutative.

Soln:

$Let\ a,b,c \in N.$
$Then,\ a*(b*c) = a*c,$
$= c.$
$And,\ (a*b)*c = b*c,$
$= c$
$= a*(b*c).$
$\therefore$ * is associative.

And, $a*b = b$ and

$b*a = a \neq a*b$.

So, * is not commutative.

5. Show that each of the following binary operation * on Q is neither associative nor commutative.

(i) $x*y = x-y+1$

Soln: $Let\ x,\ y,z \in$ Q.

Then, $x * (y * z) = x * (y - z + 1)$

$= x - (y - z + 1) + 1$

$= x - y + z - 1 + 1$

$= x - y + z$

$And,\ (x * y) * z = (x - y + 1) * z,$

$= x - y + 1 - z + 1,$

$= x - y - z + 2.$

Thus, $x * (y * z) \neq (x * y) * z$

$\therefore$ * is not associative.

$And,\ x * y = x - y + 1$

$y * x = y - x + 1 \neq x * y$

$\therefore$ * is not commutative.

Hence, * is neither associative nor commutative.

(ii) $x * y = 2x + 3y$

Soln: $Let\ x,\ y, z \in$ Q.

$Then,\ x * (y * z) = x * (2y + 3z),$

$= 2x + 3(2y + 3z),$

$= 2x + 6y + 9z.$

$And,\ (x * y) * z = (2x + 3y) * z,$

$= 2(2x + 3y) + 3z,$

$= 4x + 6y + 3z \neq x * (y * z).$

$\therefore$ * is not associative.

$And,\ x * y = x - y + 1$

$y * x = y - x + 1 \neq x * y$

$\therefore$ * is not commutative.

Hence, * is neither associative nor commutative.

(iii) $x*y = x + xy$

Soln: $Let\ x,\ y, z \in$ Q.

$$Then,\ x*(y*z) = x*(y+yz),$$
$$= x + x(y+yz),$$
$$= x + xy + xyz.$$
$$And,\ (x*y)*z = (x+xy)*z,$$
$$= (x+xy) + (x+xy)z,$$
$$= x + xy + xz + xyz \neq x*(y*z).$$

$\therefore$ * is not associative.

$$And,\ x*y = x + xy$$
$$y*x = y + yx \neq x*y$$

$\therefore$ * is not commutative.

(iv) $x*y = xy^2$

Soln: $Let\ x,\ y, z \in$ Q.

$$Then,\ x*(y*z) = x*(yz^2),$$
$$= x(yz^2)^2,$$
$$= xy^2z^4.$$
$$And,\ (x*y)*z = (xy^2)*z,$$
$$= xy^2z^2 \neq x*(y*z).$$

$\therefore$ * is not associative.

$$And,\ x*y = xy^2$$
$$y*x = yx^2 \neq x*y$$

$\therefore$ * is not commutative.

*Hence, * is* neither associative nor commutative.

6. Prove that the binary operation $\circ$ on $\mathbb{Z}$ defined by $a \circ b = a + b - 5$, is associative as well as commutative.

Soln: *Let* $a, b, c \in \mathbb{Z}$.

$$Then,\ a \circ (b \circ c) = a \circ (b + c - 5),$$
$$= a + b + c - 5 - 5,$$
$$= a + b + c - 10.$$
$$And,\ (a \circ b) \circ c = (a + b - 5) \circ c,$$
$$= a + b - 5 + c - 5$$
$$= a + b + c - 10 = a \circ (b \circ c).$$

$\therefore\ \circ$ is associative.

And, $a \circ b = a + b - 5$

$b \circ a = b + a - 5 = a \circ b.$

$\therefore\ \circ$ is commutative.

Hence proved.

7. Prove that the binary operation * defined on $\mathbb{Z}$ by $a * b = 3a + 5b$ is neither associative nor commutative. Also, prove that the usual multiplication on $\mathbb{Z}$ distributes over *.

Soln: *Let* $a, b, c \in \mathbb{Z}$.

Then, $a * (b * c) = a * (3b + 5c)$

$$= 3a + 5(3b + 5c)$$
$$= 3a + 15b + 25c$$

Also, $(a * b) * c = (3a + 5b) * c$

$$= 3(3a + 5b) + 5c$$
$$= 9a + 15b + 5c$$

This shows that $a * (b * c) \neq (a * b) * c$.

$\therefore *$ is not associative.

Again, $a * b = 3a + 5b$

And $b * a = 3b + 5a$

This shows that $a * b \neq b * a$.

$\therefore *$ is not commutative.

Hence, $*$ is neither associative nor commutative.

2nd part:

Let $a\ ,b\ ,c\ \in Z$.

We are to show that $a \times (b * c) = (a \times b) * (a \times c)$.

Now, $a \times (b * c) = a \times (3b + 5c) = 3ab + 5ac$.

And,$(a \times b) * (a \times c) = ab * ac = 3ab + 5ac$.

Thus, $a \times (b * c) = (a \times b) * (a \times c)$.

Hence, usual multiplication distributes over $*$.

8. Let binary operations ∘ and * on R be defined by $x \circ y = 2x + 2y$ and $x * y = x$. Show that ∘ is commutative but not associative and * is associative but not commutative. Also show that ∘ distributes over *.

Soln: *Let* $x, y \in$ R.

$Then,\ x \circ y = 2x + 2y$
$y \circ x = 2y + 2x = x \circ y.$
$\therefore\ \circ$ is commutative.

$$Then,\ x \circ (y \circ z) = x \circ (2y + 2z),$$
$$= 2x + 2(2y + 2z),$$
$$= 2x + 4y + 4z.$$
$$And,\ (x \circ y) \circ z = (2x + 2y) \circ z,$$
$$= 2(2x + 2y) + 2z$$
$$= 4x + 4y + 2z \neq x \circ (y \circ z).$$
$\therefore\ \circ$ is not associative.

$Again,\ x * y = x$
$y * x = y \neq x * y.$
$\therefore$ * is not commutative.

$$x * (y * z) = x * y,$$
$$= x.$$
$$And,\ (x * y) * z = x * z,$$
$$= x$$
$$= x * (y * z).$$
$\therefore$ * is associative.

2nd part

We are to show that $a \circ (b * c) = (a \circ b) * (a \circ c)$.

Now, $a \circ (b * c) = a \circ b$

$= 2a + 2b$

$(a \circ b) * (a \circ c) = (2a + 2b) * (2a + 2c)$

$= 2a + 2b$

$= a \circ (b * c)$.

Hence proved.

9. Prove that the binary operation $\circ$ on N defined by $a \circ b =$ maximum of a and b is associative and commutative. Find the identity element and invertible elements of $(N, \circ)$.

Soln: *Let* $a, b, c \in N$.

Now, $ao(boc) = ao(maximum\ of\ b\ and\ c)$

$= maximum\ of\ a\ , b\ and\ c$

and $(aob)oc = (maximum\ of\ a\ and\ b)oc$

$= maximum\ of\ a\ , b\ and\ c$

Thus, $ao(boc) = (aob)oc$

$\therefore$ o is associative

Again, $aob = maximum\ of\ a\ and\ b$

And $boa = maximum\ of\ b\ and\ a$

$= maximum\ of\ a\ and\ b$

$= aob$

$\therefore$ o is commutative.

<u>Existence of Identity:</u>

Let e an identity element for the binary operation o.

Then , $aoe = a \quad \forall\ a \in N$

$\Rightarrow maximum\ of\ a\ and\ e = a$

$\Rightarrow e \leq a$ i.e. a natural number which is less than or equal to any natural number.

$\Rightarrow$ this relation holds only when $e = 1 \in N$

$\therefore$1 is the identity element.

Again, let b be the inverse of an element $a \in N$.

Then, $aob = 1$

$\Rightarrow$ the maximum of a and $b = 1$ i.e. the maximum between two natural number is 1.

This is possible only when $a = b = 1$.

$\therefore$ The invertible element of (N, o) is 1

10. Investigate the set of Integers, the set of rational numbers and the set of irrational numbers for closure under the following binary operations:
(i) Addition (ii) subtraction (iii) multiplication (iv) division.

Soln: Set of Integers (Z)

$Let\ a, b \in Z$ be any two elements.

Then, $a+b \in Z,\ a-b \in Z,\ ab \in Z$ and $a \div b$ may or may not be an integer.

Hence, the set of integers is closed under addition, subtraction, multiplication but not under division.

Set of rational numbers(Q):

$Let\ a, b \in Q.$
$Then,\ a+b \in Q,\ a-b \in Q,\ ab \in Q$ and $a \div b \notin Q$ since $0, 1 \in Q \Rightarrow 1 \div 0 \notin Q$

Hence, the set of rational is closed under addition, subtraction, multiplication but not under division.

Set of irrational (Q^c)

We know that $\sqrt{2}, 1-\sqrt{2} \in Q^0$ but

$\sqrt{2} + (1-\sqrt{2}) = 1 \notin Q^C$, $\sqrt{2} - \sqrt{2} = 0 \notin Q^C$

$\sqrt{2}.\sqrt{2} = 2 \notin Q^C$ and $\dfrac{\sqrt{2}}{\sqrt{2}} = 1 \notin Q^C$

Hence, the set of irrational is not closed under addition, subtraction, multiplication and division.

11. Prove that there is no non-empty finite subset of N closed under addition

Soln: Let $H = \{1, 2, 3, ..., n\}$ be a finite subset of N.

Then, $1+n \notin H$ (because $1+n > n$).

So, H is not closed under addition.

Hence proved.

12. Prove that the only non-empty finite subset of N closed under multiplication is $\{1\}$.

Soln: Let $A = \{1\} \subset N$.

Since, $1 \times 1 = 1 \in A$; the subset A is closed under multiplication.

Suppose $B\ (\neq A)$ be any non-empty finite subset of N closed under multiplication.

Since, B is non-empty and $B\ (\neq A)$, there exists at least a natural number $a \in B$ and $a \neq 1$.

Also, B is closed under multiplication, so $a \times a = a^2 \in B,\ a^2 \times a = a^3 \in B$ *and* so on.

$a, a^2, a^3, \ldots$ are distinct natural numbers. Since $a \neq 1$ and $a \in N$, $a < a^2 < a^3 < \ldots$

So, B is infinite, which is a contradiction.

Hence, the only non-empty finite subset of N closed under multiplication is $\{1\}$.

13. Find whether the identity element exists or not for each of the following algebraic structures:

(i) (N, +)

Soln: *Let* e be an identity element.

Then, $a + e = a = e + a \ \forall \ a \in N$.

$\Rightarrow e = a - a = 0 \notin \text{N}$

Identity element does not exist.

(ii) (N, .)

Soln: *Let* e be an identity element.

Then, $a.e = a = e.a \ \forall \ a \in N.$
$\Rightarrow \ (ae - a) = 0$
$\Rightarrow a(e-1) = 0$
$\Rightarrow e - 1 = 0$ or $a = 0$
$\Rightarrow e = 1 \in N$ (neglecting $a = 0$)

So, 1 is the identity element.

(iii) $(Z, +)$

Soln: *Let* e be an identity element.

Then, $a + e = a = e.+ \ \forall \ a \in Z.$
$\Rightarrow \ e = 0 \in$ Z

So, 0 is the identity element.

(iv) $(Z, .)$
Soln; Let e be an identity element.
Then, $ae = a = ea \ \forall \ a \in Z.$
$\Rightarrow ae = a$
$\Rightarrow a(e - 1) = 0$
$\Rightarrow e = a \epsilon Z$ (neglecting $a = 0$)
So, 1 is the identity element.

(v) $(Q, +)$

Soln: *Let* e be an identity element.

Then, $a + e = a = e + a \ \forall \ a \in Q.$
$\Rightarrow$ a+e = a
$\Rightarrow \ e = 0 \in$ Q

So, 0 is the identity element.

(vi) $(Q, .)$
Soln: Let e be an identity element.
Then, $ae = a = ea \ \forall \ a \in Q.$
$\Rightarrow ae = a$
$\Rightarrow a(e - 1) = 0$

$\Rightarrow e = a \epsilon Q$ (neglecting $a = 0$)

So, 1 is the identity element.

(vii) $(P(S), \cap)$

Soln: *Let* E be an identity element.

Then, $A \cap E = A = E \cap A, \ \forall \ A \in P(S)$.

$\Rightarrow \ A \subseteq E$

$\Rightarrow \ E = S \in P(S)$.

So, S is the identity element.

(viii) $(P(S), \cup)$

Soln: *Let* E be an identity element.

Then, $A \cup E = A = E \cup A, \ \forall \ A \in P(S)$.

$\Rightarrow \ E \subseteq A$

$\Rightarrow \ E = \phi \in P(S)$.

So, ϕ is the identity element.

14. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the identity element of the algebraic structure $(P(S), \cap)$. Also , find the inverse of $A = \{2, 4, 6\}$, if it exists.

Soln: *Let* E be an identity element.

Then, $C \cap E = C = E \cap C, \ \forall \ A \in P(S)$.

$\Rightarrow \ C \subseteq E, \ \forall \ C \in P(S)$.

$\Rightarrow \ E = S \in P(S)$.

Thus, identity element is S.

Let B be the inverse of $A = \{2, 4, 6\}$

Then, $A \cap B = S = B \cap A$.

But, for any element $B \in P(S)$, $A \cap B = B \cap A \subset S \ [\because A \subset S]$

$\therefore \ A \cap B = B \cap A \neq S, \forall \ B \in P(S)$.

Hence, the inverse does not exist.

15. Consider the binary operation * on Q defined by $x*y = x + y - xy$. Find the identity element of $(Q,*)$. Also find x^{-1} for $x \in Q$. For what value of x does the inverse not exist?

Soln: Let e be an identity element of the binary operation *.

$$Then,\ x*e = x = e*x,\ \forall\ x \in Q$$
$$\Rightarrow x + e - ex = x$$
$$\Rightarrow e - ex = 0$$
$$\Rightarrow e(1-x) = 0$$
$$\Rightarrow e = 0 \in Q \ \text{[neglecting } x = 4\text{]}$$

So, 0 is the identity element.

Let y be the inverse of an element $x \in Q$.

$$Then,\ x*y = e = y*x$$
$$\Rightarrow x*y = 0$$
$$\Rightarrow x + y - xy = 0$$
$$\Rightarrow y(1-x) = -x$$
$$\Rightarrow y = \frac{x}{x-1} \in Q,\ \forall\ x \neq 1$$
$$\Rightarrow y = x^{-1} = \frac{x}{x-1} \in Q$$

When $x = 1$, inverse does not exist.

16. Form the composition table for the set S={1, 2, 3,4, 5,6} with respect to the binary operation of multiplication modulo 7. Deduce that S is closed under the operation. From the table, find the identity element and the inverse of each element of S. Also calculate 2^6 in S.

Soln: Let * be binary operation defined on S={1, 2, 3,4, 5,6} as $x*y = x \otimes_7 y = xy(\text{mod } 7)$.

The composition table for * is given by

$\times_7$	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

From the table, for all $a \in S$,

$$a \times_7 1 = a = 1 \times_7 a$$

$\therefore$ identity element is 1.

and $1 \times_7 1 = 1 = 1 \times_7 1$

$2 \times_7 4 = 1 = 4 \times_7 2$

$3 \times_7 5 = 1 = 3 \times_7 5$

$6 \times_7 6 = 1$

$\therefore$ the inverse of 1, 2, 3, 4, 5, 6 are 1, 4, 5, 2, 3, 6.

Now, $2^6 = 64 = 7 \times 9 + 1$

$\therefore$ value of 2^6 in S is 1.

17. Form the composition table for the set S={0,1, 2, 3,4, 5} with respect to the binary operation of addition modulo 6. From the table, find the identity element and the inverse of each element of S.

Soln: The composition table for $+_6$ on $S = \{0,1,2,3,4,5\}$ is given by

$+_6$	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

From the table, for all $a \in S$, $a +_6 0 = a = 0 +_6 a$

$\therefore$ identity element is 0.

And $0 +_6 0 = 0$

$1 +_6 5 = 0 = 5 +_6 0$

$2 +_6 6 = 0 = 4 +_6 2$

$3 +_6 3 = 0$

$\therefore$ the inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1.

18. The composition table is given below

Soln: Given $a * b =$ H.C.F. of a and b : $a, b \in N$.

The composition table is given below

*	**1**	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

Since, all the entries of the table are elements of $H = \{1,2,3,4,5,6\}$, therefore H is closed under *.

19. Let a binary operation $\circ$ on N be defined by $a \circ b =$ LCM of a and b. Form a composition table for the set H = {1, 2, 3, 4, 5} with respect to $\circ$. State whether H is closed under $\circ$ or not.

Soln: We have, $a \circ b =$ LCM of a and b; $a, b \in N$.

The composition table for $\circ$ is given by

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

From the table, some elements 6, 10, 12, 15, 20 in the table are not member of H.

So, H is not closed under $\circ$.

20. Prove that the set $S = \{3n : n \in Z\}$ is closed under usual addition and multiplication. Examine the algebraic structures $(S,+)$ and $(S,.)$ for existence of identity and invertible elements.

Soln: Let $a, b \in S$. Then,

$a + b = 3n + 3m$, for $n, m \in Z$

$= 3(n+m) = 3q$, where $q = (n+m) \in Z$

$\Rightarrow a + b \in S$.

$ab = 3n \times 3m$, for $n, m \in Z$

$= 3p$, where $p = 3nm \in Z$

$\Rightarrow ab \in S$.

Thus, S is closed under addition and multiplication.

For (S, +):

Let e be an identity element.

Then, $a + e = a = e + a \; \forall a \in S$.

$\Rightarrow a + e = a$.

$\Rightarrow e = 0 \in S$.

Let 'b' be the inverse of an element $a = 3n \in S$.

Then, $a + b = 0 = b + a$

$\Rightarrow 3n + b = 0$

$\Rightarrow b = -3n = 3(-n)$

So, inverse of any element $3n$ is $3(-n)$.

For (S, .)

Let e be an identity element.

Then, $a.e = a = e.a \quad \forall\ a \in S$.

$\Rightarrow\ ae = a,$

$\Rightarrow\ e = 1 \notin S$.

The identity element does not exist and hence inverse of any element does not exist.

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