CHAPTER 6
FACTORISATION AND IDENTITIES

Factorisation: The process of expressing an algebraic expression as the product of its prime factors is known as factorisation.

SOME FACTORISATION RESULTS:

1. $a^2 - b^2 = (a+b)(a-b)$

2. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

3. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

4. $x^2 + (p+q)x + pq = (x+p)(x+q)$

5. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

6. $a^2(b-c) + b^2(c-a) + c^2(a-b) = bc(b-c) + ca(c-a) + ab(a-b)$

   $= -(b-c)(c-a)(a-b)$

7. $a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) + 2abc$

   $= a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc$

   $= bc(b+c) + ca(c+a) + ab(a+b) + 2abc$

   $= (b+c)(c+a)(a+b)$

8. $a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) + 3abc$

   $= a^2(b+c) + b^2((c+a) + c^2(a+b) + 3abc$

   $= bc(b+c) + ca(c+a) + ab(a+b) + 3abc$

   $= (a+b+c)(bc+ca+ab)$

9. $(a+b+c)^3 - a^3 - b^3 - c^3 = 3(b+c)(c+a)(a+b)$

10. $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$

    $= (a+b+c)(a+b-c)(b+c-a)(c+a-b)$

Factorisation by trial: We use factor theorem on polynomial which states that 'A polynomial $f(x)$is exactly divisible by $(x-a)$ if and only if $f(a)=0$'.

Notes:

1) If $f(a)=0$, then $f(x)=(x-a)\times g(x)$, where $g(x)$is a polynomial of degree oneless than that of $f(x)$.
2) If the sum of coefficients in any polynomial $f(x)$ is zero, then $f(1)=0$ and hence$(x-1)$ is a factor of $f(x)$.
3) If the sum of coefficients of odd powers of x in $f(x)$ is equal to the sum of the remaining coefficients, then $f(-1)=0$ and hence $x+1$ is a factor of $f(x)$.

Complete form of polynomial: A polynomial of degree 'n' in 'x' is said to be in its complete form if it involves all powers ofx^r of $x$for $0\le r\le n$.

Reciprocal (or recurring) expression: A complete polynomial is said to be a reciprocal expression if the coefficients of the terms equidistant from the beginning and the end are equal (the terms being in descending or ascending order of their degrees).

Notes:

1) A reciprocal expression of even degree can be factorised by grouping terms with equal coefficients.
2) A reciprocal expression of odd degree in x has in general $x+1$ as factor. Then, we express the expression $=(x+1)\times$ (a reciprocal expression of even degree), and which may be factorised by grouping terms with equal coefficients as stated in (1).

Factorisation of a polynomial expression in which the coefficients of the terms equidistant from the beginning and end are equal in magnitude but opposite in sign:

1) If such an expression in x is of odd degree, the sum of coefficients will be zero and hence it has $(x-1)$ as a factor.

   Thus, given expression$=(x-1)\times$ (reciprocal expression of even degree) and further factorisation can be done as already discuss.
2) If the degree of the expression in x is even ,say$2m$, then the coefficients ofx^m is zero and both$(x-1)$ and $(x+1)$ are factors of it.

   Thus, given expression $=(x^2-1)\times$ (reciprocal expression of even degree), which can be factorised as discuss above.

SOLUTIONS

EXERCISE 6.1

Resolve into factors:

1. $x^3 - 2x^2 - 5x + 6$

Solution: Let $f(x) = x^3 - 2x^2 - 5x + 6$

$f(1) = 1^3 - 2\times 1^2 - 5\times 1 + 6 = 0$

$\therefore x - 1$ is a factor.

Now, $f(x) = x^3 - 2x^2 - 5x + 6$

$= x^3 - x^2 - x^2 + x - 6x + 6$

$= x^2(x-1) - x(x-1) - 6(x-1)$

$= (x-1)(x^2 - x - 6)$

$= (x-1)(x^2 - 3x + 2x - 6)$

$= (x-1)[(x(x-3) + 2(x-3)]$

$= (x-1)(x-3)(x+2)$

2. $x^3 + 2x^2 - 5x - 6$

Solution: Let $f(x) = x^3 + 2x^2 - 5x - 6$.

$f(-1) = x^3 + 2x^2 - 5x - 6$

$= (-1)^3 + 2(-1)^2 - 5(-1) + 6$

$= -1 + 2 + 5 - 6 = 0$

$\therefore x + 1$ is a factor.

Now, $f(x) = x^3 + 2x^2 - 5x - 6$

$= x^3 + x^2 + x^2 + x - 6x - 6$

$= x^2(x+1) + x(x+1) - 6(x+1)$

$= (x+1)(x^2 + x - 6)$

$= (x+1)(x^2 + 3x - 2x - 6)$

$= (x+1)[x(x+3) - 2(x+3)]$

$= (x+1)(x+3)(x-2)$

3. $x^3 + 4x^2 - 2x - 20$

Solution: Let $f(x) = x^3 + 4x^2 - 2x - 20$

$f(2) = 2^3 + 4.2^2 - 2.2 - 20$

$= 8 + 16 - 4 - 20$

$= 0$

$\therefore x - 2$ is a factor

Now, $x^3+4x^2-2x-20 = x^3-2x^2+6x^2-12x+10x-20$

$$= x^2(x-2)+6x(x-2)+10(x-2)$$

$$= (x-2)(x^2+6x+10)$$

4. x^3+x^2-5x+3

Solution: Let $f(x)=x^3+x^2-5x+3$

$f(1)=1^3+1^2-5\times1+3$

$=1+1-5+3=0$

$\therefore x-1$ is a factor.

Now, $x^3+x^2-5x+3 = x^3-x^2+2x^2-2x-3x+3$

$$= x^2(x-1)+2x(x-1)-3(x-1)$$

$$= (x-1)(x^2+2x-3)$$

$$= (x-1)(x^2+3x-x-3)$$

$$= (x-1)[x(x+3)-1(x+3)]$$

$$= (x-1)(x+3)(x-1)$$

$$= (x-1)^2(x+3)$$

5. x^3+3x^2+4x+2

Solution: Let $f(x)=x^3+3x^2+4x+2$

$f(-1)=(-1)^3+3x(-1)^2+4(-1)+2$

$=-1+3-4+2$

$=0$

$\therefore (x+1)$ is a factor.

Now, $x^3+3x^2+4x+2 = x^3+x^2+2x^2+2x+2x+2$

$$= x^2(x+1)+2x(x+1)+2(x+1)$$

$$= (x+1)(x^2+2x+2)$$

6. $6x^3-11x^2+6x-1$

Solution: Let $f(x)=6x^3-11x^2+6x-1$

$f(1)=6.1^3-11.1^2+6.1-1$

$=6-11+6-1=0$

$\therefore x-1$ is a factor.

Now, $6x^3-11x^2+6x-1 = 6x^3-6x^2-5x^2+5x+x-1$

$$= 6x^2(x-1)-5x(x-1)+1(x-1)$$

$$= (x-1)(6x^2-5x+1)$$

$$= (x-1)(6x^2-3x-2x+1)$$

$$= (x-1)[3x(2x-1)-1(2x-1)]$$

$$= (x-1)(2x-1)(3x-1)$$

7. x^3-5x^2-2x+4

Solution: Let $f(x)=x^3-5x^2-2x+4$

$f(-1)=(-1)^3-5(-1)^2-2(-1)+4$

$=-1-5+2+4=0$

$\therefore x+1$ is a factor.

Now , $x^3-5x^2-2x+4=x^3+x^2-6x^2-6x+4x+4$

$=x^2(x+1)-6x(x+1)+4(x+1)$

$=(x+1)(x^2-6x+4)$

8. $x^3-6x^2+3x+10$

Solution: Let $f(x)=x^3-6x^2+3x+10$

$f(-1)=(-1)^3-6\times(-1)^2+3(-1)+10$

$=-1-6-3+10=0$

$\therefore x+1$ is a factor.

Now, $x^3-6x^2+3x+10=x^3+x^2-7x^2-7x+10x+10$

$=x^2(x+1)-7x(x+1)+10(x+1)$

$=(x+1)(x^2-7x+10)$

$=(x+1)(x^2-5x-2x+10)$

$=(x+1)[x(x-5)-2(x-5)]$

$=(x+1)(x-5)(x-2)$

9. x^3+2x^2-4x+1

Solution: Let $f(x)=x^3+2x^2-4x+1$

$f(1)=1^3+2\times1^2-4\times1+1$

$=1+2-4+1=0$

$\therefore x-1$ is a factor.

Now, $x^3+2x^2-4x+1=x^3-x^2+3x^2-3x-x+1$

$=x^2(x-1)+3x(x-1)-1(x-1)$

$=(x-1)(x^2+3x-1)$

10. x^3-2x^2+x-2

Solution: Let $f(x)=x^3-2x^2+x-2$

$f(2)=2^3-2\times2^2+2-2=8-8+2-2=0$

$\therefore x-2$ is a factor

Now, $x^3-2x^2+x-2=x^2(x-2)+1(x-2)$

$=(x-2)(x^2+1)$

11. $x^3 - 6x + 4$

Solution: Let $f(x) = x^3 - 6x + 4$

$f(2) = 2^3 - 6\times 2 + 4 = 8 - 12 + 4 = 0$

$\therefore x - 2$ is a factor

Now, $x^3 - 6x + 4 = x^3 - 2x^2 + 2x^2 - 4x - 2x + 4$

$= x^2(x-2) + 2x(x-2) - 2(x-2)$

$= (x-2)(x^2 + 2x - 2)$

12. $x^3 - 3x^2 + 4$

Solution: Let $f(x) = x^3 - 3x^2 + 4$

$f(2) = 2^3 - 3\times 2^2 + 4 = 8 - 12 + 4 = 0$

$\therefore x - 2$ is a factor

Now, $x^3 - 3x^2 + 4 = x^3 - 2x^2 - x^2 + 4$

$= x^2(x-2) - (x^2 - 2^2)$

$= x^2(x-2) - (x-2)(x+2)$

$= (x-2)(x^2 - x - 2)$

$= (x-2)(x^2 + x - 2x - 2)$

$= (x-2)[(x(x+1) - 2(x+1)]$

$= (x-2)(x+1)(x-2)$

$= (x+1)(x-2)^2$

13. $x^3 - 7x^2 + 36$

Solution: Let $f(x) = x^3 - 7x^2 + 36$

$f(3) = 3^3 - 7\times 3^2 + 36$

$= 27 - 7\times 9 + 36$

$= 27 - 63 + 36 = 0$

$\therefore x - 3$ is a factor

Now, $f(x) = x^3 - 3x^2 - 4x^2 + 36$

$= x^2(x-3) - 4(x^2 - 3^2)$

$= x^2(x-3) - 4(x-3)(x+3)$

$= (x-3)(x^2 - 4x - 12)$

$= (x-3)(x^2 - 6x + 2x - 12)$

$= (x-3)[x(x-6) + 2(x-6)]$

$= (x-3)(x-6)(x+2)$

14. $x^3 - 3x^2 - 6x + 8$

Solution: Let $f(x) = x^3 - 3x^2 - 6x + 8$

$f(1) = 1^3 - 3\times1^2 - 6\times1 + 8 = 1 - 3 - 6 + 8 = 0$

$\therefore x - 1$ is a factor

Now, $x^3 - 3x^2 - 6x + 8 = x^3 - x^2 - 2x^2 + 2x - 8x + 8$

$= x^2(x-1) - 2x(x-1) - 8(x-1)$

$= (x-1)(x^2 - 2x - 8)$

$= (x-1)(x^2 - 4x + 2x - 8)$

$= (x-1)[x(x-4) + 2(x-4)]$

$= (x-1)(x-4)(x+2)$

15. $8x^3 + 8x^2 - 1$

Solution: Let $f(x) = 8x^3 + 8x^2 - 1$

$= (2x)^3 + 2(2x)^2 - 1$

$= y^3 + 2y^2 - 1$, where $y = 2x$ [sum of odd coeff.= sum of even coeff

. $\therefore$ y+1 is a factor]

$= y^3 + y^2 + y^2 - 1$

$= y^2(y+1) + (y-1)(y+1)$

$= (y+1)(y^2 + y - 1)$

$= (2x+1)(4x^2 + 2x - 1)$ $[\because y = 2x]$

16. $8x^3 + 24x - 13$

Solution: Let $f(x) = 8x^3 + 24x - 13$

$= (2x)^3 + 12.2x - 13$

$= y^3 + 12.y - 13$ where $y = 2x$, [when $y = 1$, the expression vanishes.

$\therefore y - 1$ is a factor]

$= y^3 - y^2 + y^2 - y + 13y - 13$

$= y^2(y-1) + y(y-1) + 13(y-1)$

$= (y-1)(y^2 + y + 13)$

$= (2x-1)(4x^2 + 2x + 13)$ $[\because y = 2x]$

17. $27x^3 - 9x + 2$

Solution: $27x^3 - 9x + 2 = (3x)^3 - 3.3x + 2$

$= y^3 - 3y + 2$, where $y = 3x$ [when $y = 1$, the expression vanishes. $\therefore y - 1$ is a factor]

$= y^3 - y - 2y + 2$

$= y(y^2 - 1) - 2(y - 1)$

$= y(y-1)(y+1) - 2(y-1)$

$= (y-1)[y(y+1) - 2]$

$= (y-1)(y^2 + y - 2)$

$= (y-1)(y^2 - y + 2y - 2)$

$= (y-1)(y+2)(y-1)$

$= (y-1)^2(y+2)$

$= (3x-1)^2(3x+2)$ [$\because y = 3x$]

18. $27x^3 + 3x - 10$

Solution: $27x^3 + 3x - 10 = (3x)^3 + 3x - 10$

$= y^3 + y - 10$ where $y = 3x$, [when $y = 2$, the expression vanishes. $\therefore y - 2$ is a factor]

$= y^3 - 2y^2 + 2y^2 - 4y + 5y - 10$

$= y^2(y-2) + 2y(y-2) + 5(y-2)$

$= (y-2)(y^2 + 2y + 5)$

$= (3x-2)(9x^2 + 6x + 5)$ [$\because y = 3x$]

19. $x^4 - 2x^3 + 3x^2 - 2x + 1$

Solution: The given expression is a reciprocal expression of even degree.

$x^4 - 2x^3 + 3x^2 - 2x + 1$

$= (x^2)^2 + 1 - 2x(x^2 + 1) + 3x^2$

$= (x^2 + 1)^2 - 2x^2 - 2x(x^2 + 1) + 3x^2$

$= y^2 - 2xy + x^2$ where $y = x^2 + 1$

$= (y - x)^2$

$= (x^2 - x + 1)^2$ [$\because y = x^2 + 1$]

20. $x^4 - 5x^3 - 12x^2 - 5x + 1$

Solution: **Given expression is a reciprocal expression of even degree.**

$$x^4 - 5x^3 - 12x^2 - 5x + 1 = (x^2)^2 + 1 - 5x^3 - 5x - 12x^2$$

$$= (x^2+1)^2 - 2x^2 - 5x(x^2+1) - 12x^2$$

$$= y^2 - 5xy - 14x^2 \textbf{ where } y = x^2 + 1$$

$$= y^2 - 7xy + 2xy - 14x^2$$

$$= y(y-7x) + 2x(y-7x)$$

$$= (y-7x)(y+2x)$$

$$= (x^2+1-7x)(x^2+1+2x) \; (\because y = x^2+1)$$

$$= (x^2-7x+1)(x^2+2x+1)$$

$$= (x^2-7x+1)(x+1)^2$$

21. $x^4 - 6x^2 + 8x - 3$

Solution: Let $f(x) = x^4 - 6x^2 + 8x - 3$

$f(1) = 1 - 8 + 8 - 3 = 0$

$\therefore x - 1$ is a factor

Now, $x^4 - 6x^2 + 8x - 3 = x^4 - x^3 + x^3 - x^2 - 5x^2 + 5x + 3x - 3$

$$= x^3(x-1) + x^2(x-1) - 5x(x-1) + 3(x-1)$$

$$= (x-1)(x^3 + x^2 - 5x + 3)$$

$$= (x-1)(x^3 - x^2 + 2x^2 - 2x - 3x + 3)$$

$$= (x-1)[x^2(x-1) + 2x(x-1) - 3(x-1)]$$

$$= (x-1)(x-1)(x^2 + 2x - 3)$$

$$= (x-1)^2(x+3)(x-1)$$

$$= (x-1)^3(x+3)$$

22. $x^4 - 10x^3 + 26x^2 - 10x + 1$

Solution: Given expression is an even degree and it is a reciprocal expression.

$$x^4 - 10x^3 + 26x^2 - 10x + 1 = (x^2+1)^2 - 2x^2 - 10x(x^2+1) + 26x^2$$

$$= y^2 - 10xy + 24x^2, \text{ where } y = x^2 + 1$$

$$= y(y-6x) - 4x(y-6x)$$

$$= (y-6x)(y-4x)$$

$$= (x^2-6x+1)(x^2-4x+1) \; [\because y = x^2+1]$$

23. $x^4-3x^3+4x^2-3x+3$

Solution: $x^4-3x^3+4x^2-3x+3 = (x^2)^2+2.x^2.1+1^2-3x^3-3x+2x^2+2$

$= (x^2+1)^2-3x(x^2+1)+2(x^2+1)$

$= (x^2+1)(x^2+1-3x+2)$

$= (x^2+1)(x^2-3x+3)$

24. $x^4-7x^3+10x^2-35x+25$

Solution: $x^4-7x^3+10x^2-35x+25 = (x^2)^2+2.x^2.5+5^2-7x^3-35x$

$= (x^2+5)^2-7x(x^2+5)$

$= (x^2+5)(x^2+5-7x)$

$= (x^2+5)(x^2-7x+5)$

25. $x^4-6x^3+12x^2-2x-21$

Solution: $x^4-6x^3+12x^2-2x-21$ [sum of odd coeff.= sum of even coeff
. $\therefore x+1$ is a factor]

$= x^4+x^3-7x^3-7x^2+19x^2+19x-21x-21$

$= x^3(x+1)-7x^2(x+1)+19x(x+1)-21(x+1)$

$= (x+1)(x^3-7x^2+19x-21)$ [when $x=3$, the second factor vanishes. $\therefore x-3$is a factor]

$= (x+1)(x^3-3x^2-4x^2+12x+7x-21)$

$= (x+1)[x^2(x-3)-4x(x-3)+7(x-3)]$

$= (x+1)(x-3)(x^2-4x+7)$

26. $x^5+4x^4-13x^3-13x^2+4x+1$

Solution: $x^5+4x^4-13x^3-13x^2+4x+1$[a reciprocal expression of odd degree and so when
$x=-1$, the expression vanishes. $\therefore x+1$is a factor]

$= x^5+x^4+3x^4+3x^3-16x^3-16x^2+3x^2+3x+x+1$

$= x^4(x+1)+3x^3(x+1)-16x^2(x+1)+3x(x+1)+1(x+1)$

$= (x+1)[x^4+3x^3-16x^2+3x+1]$[second factor is a reciprocal expression of even degree].

$= (x+1)[(x^2+1)^2+3x(x^2+1)-18x^2]$

$= (x+1)[(x^2+1)(x^2+1+6x)-3x(x^2+1+6x)]$

$= (x+1)(x^2+6x+1)(x^2-3x+1)$

27. $2x^5-7x^4-x^3-x^2-7x+2$

Solution: $2x^5-7x^4-x^3-x^2-7x+2$ [a reciprocal expression of odd degree and so when $x=-1$, the expression vanishes. $\therefore x+1$ is a factor]

$=2x^5+2x^4-9x^4-9x^3+8x^3+8x^2-9x^2-9x+2x+2$

$=2x^4(x+1)-9x^3(x+1)+8x^2(x+1)-9x(x+1)+2(x+1)$

$=(x+1)(2x^4-9x^3+8x^2-9x+2)$ [second factor is a reciprocal expression of even degree].

$=(x+1)[2\{(x^2+1)^2-2x^2\}-9x(x^2+1)+8x^2]$

$=(x+1)[2y^2-9xy+4x^2]$, where $y=x^2+1$

$=(x+1)[2y^2-8xy-xy+4x^2]$

$=(x+1)[2y(y-4x)-x(y-4x)]$

$=(x+1)(2y-x)(y-4x)$

$=(x+1)(2x^2-x+2)(x^2-4x+1)$ [$\because y=x^2+1$]

28. $2x^5-15x^4+37x^3-37x^2+15x-2$

Solution: $2x^5-15x^4+37x^3-37x^2+15x-2$ [when $x=1$, the expression vanishes. $\therefore x-1$ is a factor]

$=2x^5-2x^4-13x^4+13x^3+24x^3-24x^2-13x^2+13x+2x-2$

$=2x^4(x-1)-13x^3(x-1)+24x^2(x-1)-13x(x-1)+2(x-1)$

$=(x-1)[2x^4-13x^3+24x^2-13x+2]$

$=(x-1)[2x^4-4x^3-9x^3+18x^2+6x^2-12x-x+2]$

$=(x-1)[2x^3(x-2)-9x^2(x-2)+6x(x-2)-1(x-2)]$

$=(x-1)(x-2)(2x^3-9x^2+6x-1)$

$=(x-1)(x-2)(2x^3-x^2-8x^2+4x+2x-1)$

$=(x-1)(x-2)[x^2(2x-1)-4x(2x-1)+1(2x-1)]$

$=(x-1)(x-2)(2x-1)(x^2-4x+1)$

29. $x^4-6x^3+15x^2-18x+5$

Solution: $x^4-6x^3+15x^2-18x+5=x^4-6x^3+9x^2+6x^2-18x+5$

$=(x^2)^2-2.x^2.3x+(3x)^2-6(x^2-3x)+5$

$=(x^2-3x)^2+6(x^2-3x)+5$

$=y^2+6y+5$ where $y=x^2-3x$

$=(y+1)(y+5)$

$=(x^2-3x+1)(x^2-3x+5)$

30. $6a^2+7ab+2b^2+11a+7b+3$

Solution: $6a^2+7ab+2b^2+11a+7b+3$

$=6a^2+(7b+11)a+2b^2+7b+3$

$=6a^2+(7b+11)a+(2b+1)(b+3)$

$=6a^2+2(2b+1)a+3(b+3)a+(2b+1)(b+3)$ [splitting$(7b+11)$into two parts whose product is $6(2b+1)(b+3)$].

$=2a(3a+2b+1)+(b+3)(3a+2b+1)$

$=(3a+2b+1)(2a+b+3)$

31. $a^2-4b^2-9c^2+12bc+4a-8b+12c$

Solution: $a^2-4b^2-9c^2+12bc+4a-8b+12c$

$=a^2-[(2b)^2+(3c)^2-2.2b.3c]+4(a-2b+3c)$

$=a^2-(2b-3c)^2+4(a-2b+3c)$

$=(a-2b+3c)(a+2b-3c)+4(a-2b+3c)$

$=(a-2b+3c)(a+2b-3c+4)$

32. $x^2-y^2-z^2-2yz+x-y-z$

Solution: $x^2-y^2-z^2-2yz+x-y-z=x^2-(y+z)^2+(x-y-z)$

$=(x-y-z)(x+y+z)+(x-y-z)$

$=(x+y+z+1)(x-y-z)$

33. $9x^2-4y^2-24zx+16z^2-15x+10y+20z$

Solution: $9x^2-4y^2-24zx+16z^2-15x+10y+20z$

$=(3x)^2-2.3x.4z+(4z)^2-4y^2-5(3x-2y-4z)$

$=(3x-4z)^2-(2y)^2-5(3x-2y-4z)$

$=(3x-4z-2y)(3x-4z+2y)-5(3x-2y-4z)$

$=(3x-2y-4z)(3x+2y-4z-5)$

34. $6x^2+7xy+2y^2+11xy+7yz+3z^2$

Solution: $6x^2+7xy+2y^2+11xy+7yz+3z^2$

$=6x^2+(7y+11z)x+\{2y^2+7yz+3z^2\}$ [on arranging in descending powers of x]

$=6x^2+(7y+11z)x+2y^2+6yz+yz+3z^2$ [splitting the coeff. $yz$of in the middle term]

$=6x^2+(7y+11z)x+(2y+z)(y+3z)$

$=6x^2+[2(2y+z)+3(y+3z)]x+(2y+z)(y+3z)$ [splitting the coeff. $x$of in the middle term into two parts whose product is $6(2y+z)(y+3z)$]

$=6x^2+2x(2y+z)+3x(y+3z)+(2y+z)(y+3z)$

$=2x(3x+2y+z)+(y+3z)(3x+2y+z)$

$=(3x+2y+z)(2x+y+3z)$

35. $a^2-3ab+2b^2-2bc-4c^2$

Solution: $a^2-3ab+2b^2-2bc-4c^2 = a^2-3ab+\{2b^2-2bc-4c^2\}$

$= a^2-3ab+2b^2-4bc+2bc-4c^2$

$= a^2-3b.a+(2b+2c)(b-2c)$

$= a^2-[2b+2c]a-(b-2c)a+(2b+2c)(b-2c)$

$= a(a-2b-2c)-(b-2c)(a-2b-2c)$

$= (a-2b-2c)(a-b+2c)$

36. $2x^2+5yz+zx-10xy-z^2$

Solution: $2x^2+5yz+zx-10xy-z^2$

$= 2x^2+(z-10y)x+z(5y-z)$

$= 2x^2-(10y-z)x+z(5y-z)$

$= 2x^2-\{z+2(5y-z)\}x+z(5y-z)$ [splitting the coeff. $x$of into two parts whose product is$2z(5y-z)$]

$= 2x^2-2(5y-z)x-zx+z(5y-z)$

$= 2x(x-5y+z)-z(x-5y+z)$

$= (x-5y+z)(2x-z)$

37. $x^2-2xy+y^2-5x+5y$

Solution: $x^2-2xy+y^2-5x+5y = x^2-2xy+y^2-5(x-y)$

$= (x-y)^2-5(x-y)$

$= (x-y)(x-y-5)$

38. $4x^2-4xy+y^2-6x+3y$

Solution: $4x^2-4xy+y^2-6x+3y = (2x)^2-2.2x.y+y^2-3(2x-y)$

$= (2x-y)^2-3(2x-y)$

$= (2x-y)(2x-y-3)$

39. $4x^2-12xy+9y^2+2x-3y-2$

Solution: $4x^2-12xy+9y^2+2x-3y-2 = (2x)^2-2.2x.3y+(3y)^2+2x-3y-2$

$= (2x-3y)^2+(2x-3y)-2$

$= m^2+m-2$ where $m = 2x-3y$

$= m^2+2m-m-2$

$= m(m+2)-1(m+2)$

$= (m+2)(m-1)$

$= (2x-3y+2)(2x-3y-1)$ [$\because m-2x-3y$]

40. $x^2-3x(2y-1)+4y(2y-3)$

Solution: $x^2-3x(2y-1)+4y(2y-3)$

$=x^2-(6y-3)x+4y(2y-3)$

$=x^2-4yx-(2y-3)x+4y(2y-3)$ [splitting the coeff. $x$of into two parts whose product is$4y(2y-3)$]

$=x(x-4y)-(2y-3)(x-4y)$

$=(x-4y)(x-2y+3)$

41. $x^4-2x^3y+2x^2y^2-2xy^3+y^4$

Solution: $x^4-2x^3y+2x^2y^2-2xy^3+y^4=(x^2)^2+(y^2)^2+2x^2y^2-2xy(x^2+y^2)$

$=(x^2+y^2)^2-2xy(x^2+y^2)$

$=(x^2+y^2)(x^2+y^2-2xy)$

$=(x^2+y^2)(x-y)^2$

42. $(a^2+b^2)x^2-a^2b(2a+b)+a(2bx^2-a^3)$

Solution: $(a^2+b^2)x^2-a^2b(2a+b)+a(2bx^2-a^3)=(a^2+b^2+2ab)x^2-a^4-a^2b^2-2a^3b$

$=(a+b)^2x^2-a^2(a^2+b^2+2ab)$

$=(a+b)^2x^2-a^2(a+b)^2$

$=(a+b)^2(x^2-a^2)$

$=(a+b)^2(x-a)(x+a)$

43. $(2x^2+3b^2)a-(2a^2b+3x^2)b$

Solution: $(2x^2+3b^2)a-(2a^2b+3x^2)b=2ax^2+3ab^2-2a^2b-3bx^2$

$=(2a-3b)x^2-ab(2a-3b)$

$=(2a-3b)(x^2-ab)$

44. $a^4-b^3c+a^2b^2-b^2c^2$

Solution: $a^4-b^3c+a^2b^2-b^2c^2=(a^2)^2-(bc)^2+b^2(a^2-bc)$

$=(a^2-bc)(a^2+bc)+b^2(a^2-bc)$

$=(a^2-bc)(a^2+bc+b^2)$

45. $a^3-7a^2b+14ab^2-8b^3$

Solution: $a^3-7a^2b+14ab^2-8b^3=a^3-(2b)^3-7ab(a-2b)$

$=(a-2b)(a^2+2ab+4b^2)-7ab(a-2b)$

$=(a-2b)(a^2+2ab+4b^2-7ab)$

$=(a-2b)(a^2-5ab+4b^2)$

$=(a-2b)(a^2-4ab-ab+4b^2)$

$=(a-2b)(a-4b)(a-b)$

46. $a^3+6a^2-24a-64$

Solution: $a^3+6a^2-24a-64 = a^3-4^3-6a(a-4)$

$$= (a-4)(a^2+4a+4^2)+6a(a-4)$$
$$= (a-4)(a^2+4a+16+6a)$$
$$= (a-4)(a^2+10a+16)$$
$$= (a-4)(a^2+8a+2a+16)$$
$$= (a-4)[a(a+8)+2(a+8)]$$
$$= (a-4)(a+8)(a+2)$$

47. $3x^3-(5a+3b)x^2+(3a+5ab)x-5a^2$

Solution: $3x^3-(5a+3b)x^2+(3a+5ab)x-5a^2 = 3x^3-3bx^2+3ax-5ax^2+5abx-5a^2$

$$= 3x[x^2-bx+a]-5a(x^2-bx+a)$$
$$= (3x-5a)(x^2-bx+a)$$

48. $x^4+4x^3y-10x^2y^2+4xy^3+y^4$

Solution: $x^4+4x^3y-10x^2y^2+4xy^3+y^4 = (x^2)^2+(y^2)^2+4x^3y+4xy^3-10x^2y^2$

$$= (x^2+y^2)-2x^2y^2+4xy(x^2+y^2)-10x^2y^2$$
$$= (x^2+y^2)+4xy(x^2+y^2)-12x^2y^2$$
$$= m^2+4mxy-12x^2y^2,\ \text{where } m = x^2+y^2$$
$$= m^2+6mxy-2mxy-12x^2y^2$$
$$= m(m+6xy)-2xy(m+6xy)$$
$$= (m+6xy)(m-2xy)$$
$$= (x^2+y^2+6xy)(x^2+y^2-2xy)\ [\because m = x^2+y^2]$$
$$= (x-y)^2(x^2+6xy+y^2)$$

49. $x^4-5x^3y+6x^2y^2-5xy^3+y^4$

Solution: $x^4-5x^3y+6x^2y^2-5xy^3+y^4 = (x^2+y^2)^2-2x^2y^2-5xy(x^2+y^2)+6x^2y^2$

$$= (x^2+y^2)^2-5xy(x^2+y^2)+4x^2y^2$$
$$= m^2-5mxy+4x^2y^2,\ \text{where } x^2+y^2 = m$$
$$= m^2-4mxy-mxy+4x^2y^2$$
$$= m(m-4xy)-xy(m-4xy)$$
$$= (m-4xy)(m-xy)$$
$$= (x^2+y^2-4xy)(x^2+y^2-xy)\ [\because m = x^2+y^2]$$

50. $a^4b^4+a^2b^2-c^2+2abc+1$

Solution: $a^4b^4+a^2b^2-c^2+2abc+1 = (a^2b^2)^2+1^2+a^2b^2-c^2+2abc$

$=(a^2b^2+1)^2-2a^2b^2+a^2b^2-c^2+2abc$

$=(a^2b^2+1)^2-a^2b^2-c^2+2abc$

$=(a^2b^2+1)^2-[(ab)^2+c^2-2abc]$

$=(a^2b^2+1)^2-(ab-c)^2$

$=(a^2b^2+1-ab+c)(a^2b^2+1+ab-c)$

$=(a^2b^2-ab+c+1)(a^2b^2+ab-c+1)$

51. $a^3(b-c)+b^3(c-a)+c^3(a-b)$

Solution: $a^3(b-c)+b^3(c-a)+c^3(a-b)$ [a cyclic expression]

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$

$=(b-c)[a^3-a(b^2+bc+c^2)+b^2c+bc^2]$

$=(b-c)[b^2(c-a)+bc(c-a)-a(c^2-a^2)]$

$=(b-c)(c-a)[b^2+bc-ac-a^2]$

$=(b-c)(c-a)[b^2-a^2+bc-ac]$

$=(b-c)(c-a)(b-a)(a+b+c)$

$=-(a-b)(b-c)(c-a)(a+b+c)$

52.(i) $(x+1)(x+2)(x-3)(x-4)+6$

Solution: $(x+1)(x+2)(x-3)(x-4)+6=\{(x+1)(x-3)\}\{(x+2)(x-4)\}+6$

$=(x^2-2x-3)(x^2-2x-8)+6$

$=(y-3)(y-8)+6$, where $y=x^2-2x$

$=y^2-11y+30$

$=y^2-6y-5y+30$

$=(y-6)(y-5)\backslash$

$=(x^2-2x-6)(x^2-2x-5)$ [$\because y=x^2-2x$]

(ii) $(x-1)(x-2)(x+4)(x+5)+8$

Solution: $(x-1)(x-2)(x+4)(x+5)+8 =\{(x-1)(x+4)\}\{(x-2)(x+5)\}+8$

$=(x^2+3x-4)(x^2+3x-10)+8$

$=(y-4)(y-10)+8$, where $y=x^2+3x$

$=y^2-14y+48$

$=y^2-8y-6y+48$

$=y(y-8)-6(y-8)$

$=(y-8)(y-6)$

$=(x^2+3x-8)(x^2+3x-6)$ [$\because y=x^2+3x$]

(iii) $(x-1)(x-3)(x+4)(x+6)+13$

Solution: $(x-1)(x-3)(x+4)(x+6)+13 = \{(x-1)(x+4)\}\{(x-3)(x+6)\}+13$

$= (x^2+3x-4)(x^2+3x-18)+13$

$= (y-4)(y-18)+13$ where $y = x^2+3x$

$= y^2-22y+85$

$= y^2-17y-5y+85$

$= y(y-17)-5(y-17)$

$= (y-17)(y-5)$

$= (x^2+3x-17)(x^2+3x-5)$ $[\because y = x^2+3x]$

(iv) $(x+1)(x+2)(x+3)(x+4)-3$

Solution: $(x+1)(x+2)(x+3)(x+4)-3 = \{(x+1)(x+4)\}\{(x+2)(x+3)\}-3$

$= (x^2+5x+4)(x^2+5x+6)-3$

$= (y+4)(y+6)-3$ where $y = x^2+5x$

$= y^2+10y+21$

$= y(y+7)+3(y+7)$

$= (y+7)(y+3)$

$= (x^2+5x+7)(x^2+5x+3)$ $[\because y = x^2+5x]$

(v) $x(x-2)(2x+1)(2x-3)-63$

Solution: $x(x-2)(2x+1)(2x-3)-63 = x(2x-3)(x-2)(2x+1)-63$

$= (2x^2-3x)(2x^2-3x-2)-63$

$= y(y-2)-63$ where $y = 2x^2-3x$

$= y^2-2y-63$

$= y^2-9y+7y-63$

$= y(y-9)+7(y-9)$

$= (y-9)(y+7)$

$= (2x^2-3x-9)(2x^2-3x+7)$ $[\because y = 2x^2-3x]$

$= (2x^2-6x+3x-9)(2x^2-3x+7)$

$= (x-3)(2x+3)(2x^2-3x+7)$

53. $2x^3-x^2y-y^3$

Solution: $2x^3-x^2y-y^3 = x^3-y^3+x^3-x^2y$

$= (x-y)(x^2+xy+y^2)+x^2(x-y)$

$= (x-y)(x^2+xy+y^2+x^2)$

$= (x-y)(2x^2+xy+y^2)$

54. $x^3 - 6xy^2 + 9y^3$

Solution: $x^3 - 6xy^2 + 9y^3 = x^3 + 27y^3 - 6xy^2 - 18y^3$

$= x^3 + (3y)^3 - 6y^2(x+3y)$

$= (x+3y)(x^2 - 3xy + 9y^2) - 6y^2(x+3y)$

$= (x+3y)(x^2 - 3xy + 9y^2 - 6y^2)$

$= (x+3y)(x^2 - 3xy + 3y^2)$

55. $x^2 + bx - (a^2 - 3ab + 2b^2)$

Solution: $x^2 + bx - (a^2 - 3ab + 2b^2) = x^2 + bx - (a^2 - 2ab - ab + 2b^2)$

$= x^2 + bx - [a(a-2b) - b(a-2b)]$

$= x^2 + bx - (a-2b)(a-b)$

$= x^2 + \{(2b-a) + (a-b)\}x + (a-b)(2b-a)$

$= x^2 + (2b-a)x + (a-b)x + (a-b)(2b-a)$

$= x(x+2b-a) + (a-b)(x+2b-a)$

$= (x-a+2b)(x+a-b)$

56. $x^2 + 2xy - 5zx - 4yz + 6z^2$

Solution: $x^2 + 2xy - 5zx - 4yz + 6z^2$ [homogeneous expression of second degree]

$= x^2 + (2y-5z)x - (2y-3z)2z$

$= x^2 + \{(2y-3z) - 2z\}x - 2z(2y-3z)$

$= x^2 + (2y-3z)x - 2zx - 2z(2y-3z)$

$= x[x+2y-3z] - 2z(x+2y-3z)$

$= (x+2y-3z)(x-2z)$

57. $a^2x^2 - b^2y^2 - bcyz + cazx$

Solution: $a^2x^2 - b^2y^2 - bcyz + cazx = (ax-by)(ax+by) + cz(ax-by)$

$= (ax-by)(ax+by+cz)$

58. $(a^2+b^2)(x^2-y^2) + 2ab(x^2+y^2)$

Solution: $(a^2+b^2)(x^2-y^2) + 2ab(x^2+y^2) = (a^2+b^2)x^2 + 2abx^2 - (a^2+b^2)y^2 + 2aby^2$

$= x^2(a^2+b^2+2ab) - y^2(a^2+b^2-2ab)$

$= x^2(a+b)^2 - y^2(a-b)^2$

$= [(a+b)x - (a-b)y][(a+b)x + (a-b)y]$

59. Find the value of $x^4 - x^3 + x^2 + 2$, when $x^2 + 2 = 2x$.

Solution: Given, $x^2 + 2 = 2x$

$\Rightarrow x^2 = 2x - 2$ ………… (1)

Now, $x^4 - x^3 + x^2 + 2$

$= (x^2)^2 - x.x^2 + x^2 + 2$

$= (2x-2)^2 - x(2x-2) + 2x - 2 + 2$ [from (1)]

$= 4x^2 - 8x + 4 - 2x^2 + 2x + 2x$

$= 2x^2 - 4x + 4$

$= 2(2x-2) - 4x + 4$ [from (1)]

$= 4x - 4 - 4x + 4$

$= 0$

60. Find the value of $xy(x+y) + yz(y+z) + zx(z+x) + 3xyz$ when $x = a(b-c)$, $y = b(c-a)$, $z = c(a-b)$

Solution: Given, $x = a(b-c) = ab - ac$

$y = b(c-a) = bc - ab$

$z = c(a-b) = ca - cb$

$\therefore x + y + z = ab - ac + bc - ab + ca - cb = 0$

Then, $xy(x+y) + yz(y+z) + zx(z+x) + 3xyz = (x+y+z)(xy+yz+zx)$

$= 0 \times (xy + yz + zx)$

$= 0$

****

Identity: An algebraic identity is a statement that two algebraic expressions are equal for all values of the letters or variables involved.

Note: The following procedure is to be noted for proving an identity.

i) We reduce one of the sides (preferably, the more complex side) to the form of the other by simplification using known formulae.

ii) If both sides are complex, we reduce each side to its simplest form and establish their equality.

iii) Sometimes an identity follows easily by transposition of terms or addition of terms to both sides.

iv) Sometimes an identity becomes trivial when new letter(s) are substituted for a group of letters occurring in the identity. Necessary substitutions to be made whenever required.

Conditional Identities

The relations which hold under some condition(s) imposed on the symbols (or variables) involved are called conditional identities.

For example: If $a+b+c=0$,then

i) $a^2+b^2+c^2=-2(ab+bc+ca)$

ii) $a^3+b^3+c^3=3abc$

iii) $(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2=\frac{1}{4}(a^2+b^2+c^2)^2$

iv) $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)=\frac{1}{2}(a^2+b^2+c^2)^2$

SOLUTIONS

EXERCISE 6.2

Prove that (1 – 15)

1. $(y-z)^3+(y-x)^3+(x-y)^3=3(x-y)(y-z)(z-x)$

Solution: Putting $y-z=a$, $z-x=b$, $x-y=c$

$\therefore a+b+c=y-z+z-x+x-y=0$

$\therefore a^3+b^3+c^3=3abc$

$\Rightarrow (y-z)^3+(z-x)^3+(x-y)^3=3(x-y)(y-z)(z-x)$

2. $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3+24abc$

$=(2a+b-c)^3+(b+c)^3-(a+b-c)^3-6a(a+b)(a-2c)$

Solution: L.H.S. $=(b+c-a)^3+(c+a-b)^3+(a+b-c)^3+24abc$

$=(b+c-a+c+a-b+a+b-c)^3-3(b+c-a+c+a-b)$

$(c+a-b+a+b-c)(a+b-c+b+c-a)+24abc$

$[\because a^3+b^3+c^3=(a+b+c)^3-3(a+b)(b+c)(c+a)]$

$=(a+b+c)^3-3.2c2a.2b+24abc$

$=((a+b+c)^3-24abc+24abc$

$=(a+b+c)^3$

R.H.S. $=(2a+b-c)^3+(b+c)^3-(a+b-c)^3-6a(a+b)(a-2c)$

$=(2a+b-c+b+c-a-b+c)^3+3(2a+b-c+b+c)$

$(b+c-a-b+c)(a+b-c-2a-b+c)-6a(a+b)(a-2c)$

$[\because a^3+b^3-c^3=(a+b-c)^3+3(a+b)(b-c)(c-a)]$

$=(a+b+c)^3+3.2a(a+b)(a-2c)-6a(a+b)(a-2c)$

$=(a+b+c)^3+6a(a+b)(a-2c)-6a(a+b)(a-2c)$

$=(a+b+c)^3=$ L.H.S.

3. $(ax+by+cz)=(a+b+c)(x+y+z)$ if $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$

Solution: L.H.S .$=ax+by+cz$

$=a(a^2-bc)+(b^2-ca)+c(c^2-ab)$

$=a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2-bc+b^2-ca+c^2-ab)$

$=(a+b+c)(x+y+z)=R.H.S.$

4. $x^3+y^3+z^3-3xyz=(a^3+b^3+c^3-3abc)^2$ if $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$

Solution: We have, $x-y=a^2-b^2+ca-bc$

$=(a-b)(a+b)+c(a-b)$

$=(a-b)(a+b+c)$

Similarly, $y-z=(b-c)(a+b+c)$

$z-x=(c-a)(a+b+c)$

L.H.S.$= x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)[(x-y)^2 + (y-z)^2 + (z-x)^2]$

$= \frac{1}{2}(a^2+b^2+c^2-ab-bc-ca)(a+b+c)^2[(a-b)^2+(b-c)^2(c-a)^2]$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \times \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

$= (a^3+b^3+c^3-3abc)(a^3+b^3+c^3-3abc)$

$= (a^3+b^3+c^3-3abc)^2$ = R.H.S.

5. $a^2x+b^2y+c^2z = (x+y+z)(a^2+b^2+c^2)$ if $a^2 = x^2 - yz$, $b^2 = y^2 - zx$ and $c^2 = z^2 - xy$

Solution: L.H.S. $= a^2x+b^2y+c^2z$

$= x(x^2-yz)+y(y^2-zx)+z(z^2-xy)$

$= x^3+y^3+z^3-3xyz$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$= (x+y+z)(x^2-yz+y^2-zx+z^2-xy)$

$= (x+y+z)(a^2+b^2+c^2)$

= R.H.S.

6. $s(s-a)(s-b)+s(s-c)(s-a)+s(s+a)(s-c)+c(s+a)(s+b)$

$= (s+a)(s+b)(s+c)$ if $s = a+b+c$

Solution: L.H.S. $= s(s-a)(s-b)+s(s-c)(s-a)+s(s+a)(s-c)+c(s+a)(s+b)$

$= s(s-a)(s-b+s-c)+(s+a)(s^2-cs+cs+bc)$

$= s(s-a)(s+a)+(s+a)(s^2+bc)$ $[\because s-b-c=a]$

$= (s+a)[s^2-as+s^2+bc]$

$= (s+a)[s^2-s^2+bs+cs+s^2+bc]$ $[\because a=s-b-c]$

$= (s+a)[s(s+b)+c(s+b)]$

$= (s+a)(s+b)(s+c)$

= R.H.S.

7. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$

$=\frac{1}{2}(a^3+b^3+c^3-3abc)$ if $2s=a+b+c$

Solution: Given, $2s=a+b+c$

L.H.S. $=(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$

$=\frac{1}{2}(s-a+s-b+s-c)[(s-a-s+b)^2+(s-b-s+c)^2+(s-c-s+a)^2]$

$=\frac{1}{2}[s+a+b+c-a-b-c][(a-b)^2+(b-c)^2+(c-a)^2]$

$=\frac{1}{2}\times s.(a-b)^2+(b-c)^2+(c-a)^2$

$=\frac{1}{2}\times\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

$=\frac{1}{2}(a^3+b^3+c^3-3abc)$ = R.H.S.

8. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(a-b)(b-c)(c-a)$

Solution: L.H.S. $=a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

$=(ab-ac)^3+(bc-ab)^3+(ca-bc)^3$

$=3(ab-ac)(bc-ab)(ca-bc)$ $[\because ab-ac+bc-ab+ca-bc=0]$

$=3abc(b-c)(c-a)(a-b)$ = R.H.S.

9. $(x-y)(x+y-2z)^3+(y-z)(y+z-2x)^3+(z-x)(z+x-2y)^3=0$

Solution: L.H.S. $=(x-y)(x+y-2z)^3+(y-z)(y+z-2x)^3+(z-x)(z+x-2y)^3$

$=(x-y)[(y-z)-(z-x)]^3+(y-z)[(z-x)-(x-y)]^3+(z-x)[(x-y)-(y-z)]^3$

$=a(b-c)^3+b(c-a)^3+c(a-b)^3$,

where $a=x-y$, $b=y-z$, $c=z-x$ and $a+b+c=0$

$=ab^3-ac^3+bc^3-ba^3+ca^3-cb^3-3abc(b-c+c-a+a-b)$

$=a(b^3-c^3)-a^3(b-c)-bc(b^2-c^2)-3abc.0$

$=(b-c)[ab^2+abc+ac^2-a^3-b^2c-bc^2]$

$=(b-c)[-b^2(c-a)-bc(c-a)+a(c^2-a^2)]$

$=-(b-c)(c-a)[b^2+bc-ac-a^2]$

$=-(b-c)(c-a)[(b-a)(b+a)+c(b-a)]$

$=-(b-c)(c-a)[(b-a)(a+b+c)]$

$=-(b-c)(c-a)(b-a)\times 0$ $[\because a+b+c=0]$

$=0$ = R.H.S.

10. $(s-a)^3+(s-b)^3+(s-c)^3=s^3-3abc \text{ if } 2s=a+b+c$

Solution: L.H.S. $=(s-a)^3+(s-b)^3+(s-c)^3$

$$=(s-a+s-b+s-c)^3-3(s-a+s-b)(s-b+s-c)(s-c+s-a)$$

$$[\because a^3+b^3+c^3=(a+b+c)^3-3(a+b)(b+c)(c+a)]$$

$$=[3s-(a+b+c)]^3-3(2s-(a+b))[2s-(b+c)][2s-(c+a)]$$

$$=(3s-2s)^3-3.c.a.b\ [\because 2s=a+b+c]$$

$$=s^3-3abc = \text{R.H.S.}$$

11. $2a(b+c-a)+(c+a-b)(a+b-c)$

$=2b(c+a-b)+(a+b-c)(b+c-a)$

$=2c(a+b-c)+(b+c-a)(c+a-b)$

$=(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b)$

Solution: We have, $(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b)$

$$=(c+a-b)(a+b-c)+(b+c-a)(a+b-c+c+a-b)$$

$$=(c+a-b)(a+b-c)+(b+c-a).2a$$

$$=2a(b+c-a)+(c+a-b)(a+b-c) \text{ ....(i)}$$

Again, $(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b)$

$$=(a+b-c)(b+c-a)+(c+a-b)(a+b-c+b+c-a)$$

$$=(a+b-c)(b+c-a)+(c+a-b).2b$$

$$=2b(c+a-b)+(a+b-c)(b+c-a) \text{ ....(ii)}$$

Also, $(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b)$

$$=(b+c-a)(c+a-b)+(a+b-c)(c+a-b+b+c-a)$$

$$=(b+c-a)(c+a-b)+(a+b-c).2c$$

$$=2c(a+b-c)+(b+c-a)(c+a-b) \text{ ....(iii)}$$

From (i) , (ii) and (iii) , we can conclude that

$$2a(b+c-a)+(c+a-b)(a+b-c)$$

$$=2b(c+a-b)+(a+b-c)(b+c-a)$$

$$=2c(a+b-c)+(b+c-a)(c+a-b)$$

$$=(c+a-b)(a+b-c)+(a+b-c)(b+c-a)+(b+c-a)(c+a-b)$$

12. $a(b-c)^3+b(c-a)^3+c(a-b)^3=(b-c)(c-a)(a-b)(a+b+c)$

Solution: L.H.S. $=a(b-c)^3+b(c-a)^3+c(a-b)^3$

$=ab^3-ac^3+bc^3-a^3b+ca^3-cb^3-3abc(b-c+c-a+a-b)$

$=a(b^3-c^3)-a^3(b-c)-bc(b^2-c^2)$

$=(b-c)[a(b^2+bc+c^2)-a^3-bc(b+c)]$

$=(b-c)[ab^2+abc+ac^2-a^3-b^2c-bc^2]$

$=(b-c)[-b^2(c-a)-bc(c-a)+a(c^2-a^2)]$

$=(b-c)(c-a)[-b^2-bc+ac+a^2]$

$=-(b-c)(c-a)[b^2-a^2+bc-ac]$

$=-(b-c)(c-a)[(b-a)(b+a)+c(b-a)]$

$=-(b-c)(c-a)(b-a)(a+b+c)$

$=(a-b)(b-c)(c-a)(a+b+c)$

= R.H.S.

13. $(a^2+b^2+c^2)(p^2+q^2+r^2)-(ap+bq+cr)^2$
$=(aq-bp)^2+(br-cq)^2+(cp-ar)^2$

Solution: L.H.S. $=(a^2+b^2+c^2)(p^2+q^2+r^2)-(ap+bq+cr)^2$

$=a^2p^2+a^2q^2+a^2r^2+b^2p^2+b^2q^2+b^2r^2+c^2p^2+c^2q^2+c^2r^2$
$-a^2p^2-b^2q^2-c^2r^2-2abpq-2bcqr-2capr$

$=[(aq)^2+(bp)^2-2abpq)]+[(br)^2+(cq)^2-2brcq)]+[(cp)^2+(ar)^2-2cp.ar]$

$=(aq-bp)^2+(br-cq)^2+(cp-ar)^2$

14. $x(y-z)(1+xy)(1+zx)+y(z-x)(1+yz)(1+yx)+z(x-y)(1+zx)(1+yz)$
$=xyz(y-z)(z-x)(x-y)$

Solution: Putting $1+xy=a$, $1+yz=b$, $1+zx=c$, we get

$b-a=1+yz-(1+xy)=y(z-x)$

Similarly, $c-b=z(x-y)$, $a-c=x(y-z)$

$\therefore$ L.H.S. $=x(y-z)(1+xy)(1+zx)+y(z-x)(1+yz)(1+yx)+z(x-y)(1+zx)(1+yz)$

$=ac(a-c)+ba(b-a)+bc(c-b)$

$=a^2c-ac^2+b^2a-a^2b+bc^2-b^2c$

$=a^2(c-b)-a(c^2-b^2)+bc(c-b)$

$=(c-b)[a^2-ac-ab+bc]$

$=(c-b)[a(a-c)-b(a-c)]$

$=(c-b)(a-c)(a-b)$

$=-(b-a)(c-b)(a-c)$

$=-z(x-y)x(y-z)y(z-x)$

$=-xyz(x-y)(y-z)(z-x)$

15. $(b-c)(1+a^2b)(1+a^2c)+(c-a)(1+b^2c)(1+b^2a)+(a-b)(1+c^2a)(1+c^2b)$

$=-abc(a+b+c)(b-c)(c-a)(a-b)$

Solution: $(b-c)(1+a^2b)(1+a^2c)=(b-c)(1+a^2c+a^2b+a^4bc)$

$=b+a^2bc+a^2b^2+a^4b^2c-c-a^2c^2-a^2bc-a^4bc^2$ …………(1)

$(c-a)(1+b^2c)(1+b^2a)=(c-a)(1+ab^2+b^2c+ab^4c)$

$=c+ab^2c+b^2c^2+ab^4c^2-a-a^2b^2-ab^2c-a^2b^4c$ ……………(2)

$(a-b)(1+c^2a)(1+c^2b)=(a-b)(1+c^2a+c^2b+abc^4)$

$=a+c^2a^2+abc^2+a^2bc^4-b-abc^2-b^2c^2-ab^2c^4$ ………….. (3)

Adding (1), (2) & (3)

$$(b-c)(1+a^2b)(1+a^2c)+(c-a)(1+b^2c)(1+b^2a)+(a-b)(1+c^2a)(1+c^2b)$$

$$=a^4b^2c+ab^4c^2+a^2bc^4-a^4bc^2-a^2b^4c-ab^2c^4$$

$$=abc[a^3b+b^3c+ac^3-a^3c-ab^3-bc^3]$$

$$=abc[a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)]$$

$$=abc(b-c)[a^3-a(b^2+bc+c^2)+bc(b+c)]$$

$$=abc(b-c)[b^2(c-a)+bc(c-a)-a(c^2-a^2)]$$

$$=abc(b-c)(c-a)[b^2+bc-ac-a^2]$$

$$=abc(b-c)(c-a)[b^2-a^2+bc-ac]$$

$$=abc(b-c)(c-a)[(b-a)(b+a)+c(b-a)]$$

$$=abc(b-c)(c-a)(b-a)(a+b+c)$$

$$=-abc(b-c)(c-a)(a-b)(a+b+c)$$

16. If $x+y+z=a$, $yz+zx+xy=b$ and $xyz=c$, prove that $a^3-3ab+3c=x^3+y^3+z^3$

Solution: We have

$$a^3-3ab+3c=(x+y+z)^3-3(x+y+z)(xy+yz+zx)+3xyz$$

$$=(x+y+z)^3-3[\{(x+y)+z\}\{xy+z(x+y)\}-xyz]$$

$$=(x+y+z)^3-3[xy(x+y)+z(x+y)^2+xyz+z^2(x+y)-xyz]$$

$$=(x+y+z)^3-3(x+y)[xy+zx+yz+z^2]$$

$$=(x+y+z)^3-3(x+y)[x(y+z)+z(y+z)]$$

$$=(x+y+z)^3-3(x+y)(y+z)(x+z)$$

$$=x^3+y^3+z^3 = \text{R.H.S.}$$

17. If $2s = a+b+c$ and $2t^2 = a^2+b^2+c^2$, show that
$(t^2-a^2)(t^2-b^2)+(t^2-c^2)(t^2-b^2)+(t^2-c^2)(t^2-a^2)$
$= 4s(s-a)(s-b)(s-c)$

Solution: L.H.S. $= (t^2-a^2)(t^2-b^2)+(t^2-c^2)(t^2-b^2)+(t^2-c^2)(t^2-a^2)$

$$= t^4 - a^2t^2 - b^2t^2 + a^2b^2 + t^4 - b^2t^2 - c^2t^2 + b^2c^2 + t^4 - a^2t^2 - c^2t^2 + c^2a^2$$
$$= 3t^4 - 2t^2(a^2+b^2+c^2) + a^2b^2 + b^2c^2 + c^2a^2$$
$$= 3t^4 - 2t^2.2t^2 + (ab)^2 + (bc)^2 + (ca)^2 \ [\because 2t^2 = a^2+b^2+c^2]$$
$$= -t^4 + (ab)^2 + (bc)^2 + (ca)^2$$
$$= (ab)^2 + (bc)^2 + (ca)^2 - \left(\frac{a^2+b^2+c^2}{2}\right)^2$$
$$= \frac{1}{4}[4a^2b^2 + 4b^2c^2 + 4c^2a^2 - a^4 - b^4 - c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2]$$
$$= \frac{1}{4}[2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4]$$
$$= \frac{1}{4}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$= \frac{1}{4} \times 2s(2s-c-c)(2s-a-a)(2s-b-b) \ [\because 2s = a+b+c]$$
$$= \frac{1}{2} \times s(2s-2c)(2s-2a)(2s-2b)$$
$$= \frac{s}{2} \times 8(s-c)(s-a)(s-b) = 4s(s-c)(s-a)(s-b)$$

18. If $s = a+b+c$, prove that
$(s-3a)^2 + (s-3b)^2 + (s-3c)^2 = 3\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

Solution: L.H.S. $= (s-3a)^2 + (s-3b)^2 + (s-3c)^2$

$$= s^2 - 6as + 9a^2 + s^2 - 6sb + 9b^2 + s^2 - 6sc + 9c^2$$
$$= 3s^2 - 6s(a+b+c) + 9(a^2+b^2+c^2)$$
$$= 3s^2 - 6s.s + 9(a^2+b^2+c^2) \ [\because a+b+c = s]$$
$$= 9(a^2+b^2+c^2) - 3s^2$$
$$= 9(a^2+b^2+c^2) - 3(a+b+c)^2$$
$$= 3[3a^2 + 3b^2 + 3c^2 - a^2 - b^2 - c^2 - 2ab - 2bc - 2ca]$$
$$= 3[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$
$$= 3[(a^2+b^2-2ab) + (b^2+c^2-2bc) + (c^2+a^2-2ca)]$$
$$= 3[(a-b)^2 + (b-c)^2 + (c-a)^2]$$
= R.H.S.

19. If $a+b+c=1$, prove that

$$(a+bc)(b+c)=(b+ca)(c+a)=(c+ab)(a+b)=(1-a)(1-b)(1-c)$$

Solution: Given $a+b+c=1$

Then, we have

$$(a+bc)(b+c)=(1-b-c+bc)(1-a)$$
$$=[1(1-b)-c(1-b)](1-a)$$
$$=(1-b)(1-c)(1-a) \text{ ....(i)}$$

Also, $(b+ca)(c+a)=(1-c-a+ca)(1-b)$
$$=[1(1-c)-a(1-c)](1-b)$$
$$=(1-c)(1-a)(1-b) \text{ ....(ii)}$$

Again, $(c+ab)(a+b)=(1-a-b+ab)(1-c)$
$$=[1(1-a)-b(1-a)](1-c)$$
$$=(1-a)(1-b)(1-c) \text{ ....(iii)}$$

From (i), (ii) and (iii) we can conclude that

$$(a+bc)(b+c)=(b+ca)(c+a)=(c+ab)(a+b)=(1-a)(1-b)(1-c)$$

20. If $s=a+b+c$ show that

$$(s-a)(s-b)(s-c)=(a+b+c)(bc+ca+ab)-abc$$

Solution: Given $s=a+b+c$

We have

R.H.S.$=(a+b+c)(bc+ca+ab)-abc$
$$=abc+ca^2+a^2b+b^2c+abc+ab^2+bc^2+c^2a+abc-abc$$
$$=a^2(b+c)+a(b^2+c^2+2bc)+bc(b+c)$$
$$=a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$=(b+c)[a^2+a(b+c)+bc]$$
$$=(b+c)[a(a+b)+c(a+b)]$$
$$=(b+c)(a+b)(a+c)$$
$$=(s-a)(s-c)(s-b) \quad [\because s=a+b+c]$$
$$=\text{L.H.S.}$$

21. If $a+b+c=0$ prove that

i) $a(a+b)(a+c)=b(b+c)(b+a)=c(c+a)(c+b)=abc$

Solution: Given, $a+b+c=0$

We have

$$a(a+b)(a+c)=a(-b)(-c)=abc$$
$$b(b+c)(b+a)=b(-a)(-c)=abc$$
$$c(c+a)(c+b)=c(-b)(-a)=abc$$

Hence, $a(a+b)(a+c)=b(b+c)(b+a)=c(c+a)(c+b)=abc$

ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)=3(b+c)(c+a)(a+b)$

Solution: Given, $a+b+c=0$

L.H.S. $=a^2(b+c)+b^2(c+a)+c^2(a+b)$

$=a^2(-a)+b^2(-b)+c^2(-c)$

$=-(a^3+b^3+c^3)$

$=-3abc\ [\because a^3+b^3+c^3=3abc]$

$=-3(-b-c)(-c-a)(-a-b)$ [using the given condition $a+b+c=0$]

$=3(a+b)(b+c)(c+a)$

= R.H.S.

iii) $a(b-c)^3+b(c-a)^3+c(a-b)^3=0$

Solution: Given, $a+b+c=0$

L.H.S. $=a(b-c)^3+b(c-a)^3+c(a-b)^3$

$=ab^3-ac^3+bc^3-a^3b+ca^3-cb^3-3abc(b-c+c-a+a-b)$

$=a(b^3-c^3)-a^3(b-c)-bc(b^2-c^2)-3abc.0$

$=(b-c)[a(b^2+bc+c^2)-a^3-bc(b+c)]$

$=(b-c)[ab^2+abc+ac^2-a^3-b^2c-bc^2]$

$=(b-c)[b^2(a-c)+bc(a-c)-a(a^2-c^2)]$

$=(b-c)(a-c)[b^2+bc-a^2-ac]$

$=(b-c)(a-c)[(b^2-a^2)+c(b-a)]$

$=(b-c)(a-c)[(b-a)(b+a)+c(b-a)]$

$=(b-c)(a-c)(b-a)(a+b+c)$

$=(b-c)(a-c)(b-a)\times 0\ [\because a+b+c=0]$

$=0$

= R.H.S.

iv) $a^5+b^5+c^5=-5abc(bc+ca+ab)$

$=\dfrac{5}{6}(a^3+b^3+c^3)(a^2+b^2+c^2)$

Solution: Given, $a+b+c=0$

$\Rightarrow (a+b)^5=(-c)^5$

$\Rightarrow a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5$

$\Rightarrow a^5+b^5+c^5=-5ab[a^3+b^3+2a^2b+2ab^2]$

$=-5ab[(a+b)(a^2-ab+b^2)+2ab(a+b)]$

$=-5ab(a+b)(a^2-ab+b^2+2ab)$

$=-5ab(-c)(a^2+ab+b^2)$

$$= -5ab(-c)(a^2 + ab + b.b)$$
$$= -5ab(-c)[a(a+b) + b(-c-a)]$$
$$= +5abc[a(-c) - bc - ab]$$
$$= -5abc[ac + bc + ab] \text{ ....(i)}$$

Also, $\frac{5}{6}(a^3 + b^3 + c^3)(a^2 + b^2 + c^2)$

$$= \frac{5}{6} \times 3abc(-2)(ab + bc + ca) \; [\because a^3 + b^3 + c^3 = 3abc, a^2 + b^2 + c^2 = -2(ab + bc + ca)]$$
$$= -5abc(ab + bc + ca) \text{ ....(ii)}$$

From (i) and (ii), we can conclude that

$$a^5 + b^5 + c^5 = -5abc(bc + ca + ab) = \frac{5}{6}(a^3 + b^3 + c^3)(a^2 + b^2 + c^2)$$

v) $\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right) = 9$

Solution: Given, $a + b + c = 0$

L.H.S. $= \left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)$

$$= 1 + \frac{b(b-c)}{a(c-a)} + \frac{c(b-c)}{a(a-b)} + \frac{a(c-a)}{b(b-c)} + 1 + \frac{c(c-a)}{b(a-b)} + \frac{a(a-b)}{c(b-c)} + \frac{b(a-b)}{c(c-a)} + 1$$
$$= 3 + \frac{b}{c-a}\left[\frac{b-c}{a} + \frac{a-b}{c}\right] + \frac{c}{a-b}\left[\frac{b-c}{a} + \frac{c-a}{b}\right] + \frac{a}{b-c}\left[\frac{c-a}{b} + \frac{a-b}{c}\right]$$
$$= 3 + \frac{b}{c-a}\left[\frac{bc - c^2 + a^2 - ab}{ca}\right] + \frac{c}{a-b}\left[\frac{b^2 - bc + ac - a^2}{ab}\right] + \frac{a}{b-c}\left[\frac{c^2 - ac + ab - b^2}{bc}\right]$$
$$= 3 + \frac{b}{c-a}\left[\frac{(c-a)(b-c-a)}{ca}\right] + \frac{c}{a-b}\left[\frac{(a-b)(c-a-b)}{ab}\right] + \frac{a}{b-c}\left[\frac{(b-c)(a-b-c)}{bc}\right]$$
$$= 3 + \frac{b(b+b)}{ca} + \frac{c(c+c)}{ab} + \frac{a(a+a)}{bc}$$
$$= 3 + 2.\left(\frac{b^3 + c^3 + a^3}{abc}\right)$$
$$= 3 + 2 \times \frac{3abc}{abc} \; [\because a^3 + b^3 + c^3 = 3abc]$$
$$= 3 + 6$$
$$= 9$$

= R.H.S.

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