CHAPTER 4
BINOMIAL THEOREM

BINOMIAL EXPRESSIONS:

Algebraic expressions having two terms are called binomial expressions.

E.g.: $(x+3), (3x-2y)$ etc.

BINOMIAL THEOREM:

If a and x be any two real numbers and n be any positive integer, then

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots. + {}^nC_n x^n$$

DEDUCTIONS:

In the binomial expansion,

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots. + {}^nC_n x^n$$

(i) If x is replaced by $-x$

$$(a-x)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 - \ldots\ldots + (-1)^r\,{}^nC_r a^{n-r}x^r + \ldots. + (-1)^n\,{}^nC_n x^n$$

(ii) If $a=1$, we get

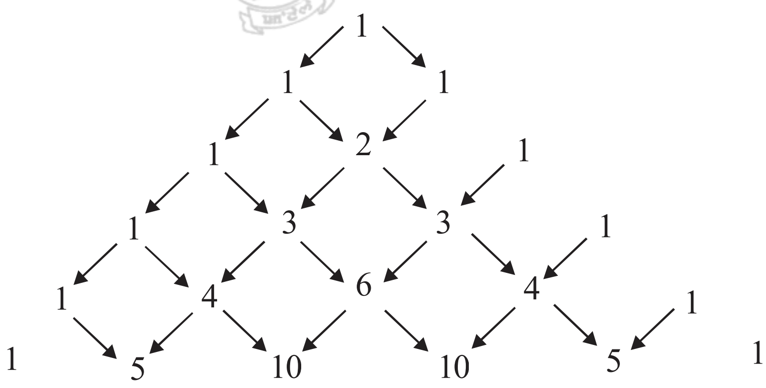
$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_r x^r + \ldots. + x^n$$

(iii) If $a=1$ and x is replaced by $-x$, we get

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - \ldots\ldots + (-1)^r\,{}^nC_r x^r + \ldots. + (-1)^n\, x^n$$

PASCAL'S TRIANGLE :

The binomial co-efficients can be arranged in the form of triangle as follows:



General term: In the expansion $(a+x)^n$, the general term is denoted by T_{r+1} and is given by

$T_{r+1} = {}^nC_r a^{n-r} x^r$

Middle term of a binomial expansion:

1. When n is even

When n is even, the number of terms in the expansion of $(a+x)^n$ is $(n+1)$ which is odd.

$\therefore$ The middle term is $\left(\frac{n}{2}+1\right)^{th}$ term. Hence $T_{\frac{n}{2}+1}$ is the middle term.

2. When n is odd.

The number of terms $(n+1)$ being even, there are two middle terms which are the $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Thus, $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$ are two middle terms.

Properties of binomial co-efficients:

i) The sum of all the binomial co-efficientsis 2^n.

i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

ii) The sum of the binomial co-efficients of odd terms is equal to that of even terms, each being equal to 2^{n-1}.

i.e. ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

SOLUTIONS

EXERCISE – 4.1

Q1. How many terms are there in the expansion of

(i) $(2a-x)^7$

Ans: The no. of terms in the expansion of $(2a-x)^7$ is 7+1=8.

(ii) $(x+4)^{10}$

Ans: The no. of terms in the expansion of $(x+4)^{10}$ is 10+1=11.

(iii) $(1-3x)^{15}$

Ans: The no. of terms in the expansion of $(1-3x)^{15}$ is 15+1=16.

(iv) $(2+5y)^{20}$

Ans: The no. of terms in the expansion of $(2+5y)^{20}$ is 20+1=21.

Q2. Using Pascal's triangle, expand

(i) $(x+y)^5$

Solution: The no. of terms is 5+1=6

$\therefore$ By Pascal's triangle, the 6th row is

1 5 10 10 5 1

$$\therefore (x+y)^5 = 1.x^5 + 5..x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1.y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

(ii) $(a+2x)^4$

Solution: The no. of terms in the expansion of $(a+2x)^4$ is 4+1=5

$\therefore$ By Pascal's triangle, the 5th row is

1 4 6 4 1

Now, $(a+2x)^4 = 1.a^4 + 4.a^3.(2x)^1 + 6.a^2(2x)^2 + 4.a(2x)^3 + 1.(2x)^4$

$$= a^4 + 8a^3x + 24a^2x^2 + 32ax^3 + 16x^4$$

(iii) $(3-2x)^5$

Solution: The no. of terms in the expansion of $(3-2x)^5$ is 5+1=6

$\therefore$ By Pascal's triangle, the 6th row is

1 5 10 10 5 1

Now,

$$(3-2x)^5 = 1\times 3^5 - 5\times 3^4.(2x) + 10\times 3^3.(2x)^2 - 10\times 3^2(2x)^3 + 5\times 3^1\times(2x)^4 - (2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

Q3. Expand the following using Binomial theorem.

(i) $(1+x)^5$

Solution: $(1+x)^5 = {}^5C_0.1^5 + {}^5C_1.1^4x^1 + {}^5C_2.1^3.x^2 + {}^5C_3.1^2.x^3 + {}^5C_4.1.x^4 + {}^5C_5.x^5$

$= 1.1 + 5.1.x + 10.1.x^2 + 10.1.x^3 + 5.x^4 + x^5$

$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

(ii) $(a-2x)^6$

Solution: $(a-2x)^6 = {}^6c_0.a^6 - {}^6C_1.a^5(2x) + {}^6C_2a^4(2x)^2 - {}^6C_3a^3(2x)^3 +$
${}^6C_4a^2(2x)^4 - {}^6C_5a.(2x)^5 + {}^6C_6.(2x)^6$

$= 1.a^6 - 6.a^5.2x + 15.a^4.4x^2 - 20.a^3.8x^3 + 15.a^2.16x^4 - 6.a.32x^5 + 64x^6$

$= a^6 - 12a^5x + 60a^4x^2 - 160a^3x^3 + 240a^2x^4 - 192ax^5 + 64x^6$

(iii) $(x+2y)^4$

Solution: $(x+2y)^4 = {}^4C_0x^4 + {}^4C_1x^3(2y) + {}^4C_2x^2(2y)^2 + {}^4C_3x(2y)^3 + {}^4C_4.(2y)^4$

$= 1.x^4 + 4.x^3.2y + 6.x^2.4y^2 + 4.x.8y^3 + 1.16y^4$

$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

(iv) $\left(x+\frac{1}{2}\right)^7$

Solution: $\left(x+\frac{1}{2}\right)^7 =$

$${}^7C_0x^7 + {}^7C_1x^6.\left(\frac{1}{2}\right) + {}^7C_2x^5\left(\frac{1}{2}\right)^2 + {}^7C_3x^4\left(\frac{1}{2}\right)^3 + {}^7C_4x^3\left(\frac{1}{2}\right)^4 + {}^7C_5x^2\left(\frac{1}{x}\right)^5 + {}^7C_6x.\left(\frac{1}{x}\right)^6 + {}^7C_7\left(\frac{1}{x}\right)^7$$

$$= 1.x^7 + 7.x^5 + 21.x^3 + 35.x^4.\frac{1}{x^3} + 35.x^3.\frac{1}{x^4} + 21.x^2.\frac{1}{x^5} + 7.x.\frac{1}{x^6} + \frac{1}{x^7}$$

$$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

(v) $(2x-y)^5$

Solution: $(2x-y)^5 = {}^5C_0(2x)^5 - {}^5C_1(2x)^4.y + {}^5C_2(2x)^3.y^2 - {}^5C_3(2x)^2.y^3 + {}^5C_4(2x).y^4 - {}^5C_5.y^5$

$= 1.32x^5 - 5.16x^4y + 10.8x^3.y^2 - 10.4x^2.y^3 + 5.2y^4 + y^5$

$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

(vi) $\left(\frac{x}{a}+\frac{a}{x}\right)^6$

Solution: $\left(\frac{x}{a}+\frac{a}{x}\right)^6$

$= {}^6C_0.\left(\frac{x}{a}\right)^6 + {}^6C_1\left(\frac{x}{a}\right)^5\left(\frac{a}{x}\right) + {}^6C_2\left(\frac{x}{a}\right)^4\left(\frac{a}{x}\right)^2 + {}^6C_3\left(\frac{x}{a}\right)^3\left(\frac{a}{x}\right)^3 + {}^6C_4\left(\frac{x}{a}\right)^2\left(\frac{a}{x}\right)^4 + {}^6C_5\left(\frac{x}{a}\right)\left(\frac{a}{x}\right)^5 + {}^6C_6.\left(\frac{a}{x}\right)^6$

$= \left(\frac{x}{a}\right)^6 + 6.\left(\frac{x}{a}\right)^4 + 15.\left(\frac{x}{a}\right)^2 + 20 + 15.\left(\frac{a}{x}\right)^2 + 6.\left(\frac{a}{x}\right)^4 + \left(\frac{a}{x}\right)^6$

$= \frac{x^6}{a^6} + \frac{6x^4}{a^4} + \frac{15x^2}{a^2} + 20 + \frac{15a^2}{x^2} + \frac{6a^4}{x^4} + \frac{a^6}{x^6}$

Q4. Find the first four terms in the expansion of $(x-2y)^{10}$.

Solution: The first four term in the expansion of $(x-2y)^{10}$ are

${}^{10}C_0x^{10}, -{}^{10}C_1x^9(2y), {}^{10}C_2x^8(2y)^2, -{}^{10}C_3x^7(2y)^3$

i.e. $x^{10}, -10x^9(2y)\ 45.x^8.4y^2, -120.x^7.8y^3$

i.e. $x^{10}, -20x^9y, 180x^8y^2, -960x^7y^3$

Q5. Find the 6th term in the expansion of $(1+x)^{10}$.

Solution: We have, 6th term $= T_6 = T_{5+1}$

$= {}^{10}C_5.1^{10-5}.x^5$

$= \frac{10\times9\times8\times7\times6}{1\times2\times3\times4\times5}.1^5.x^5$

$= 252x^5$

Q6. Find the 7th term in the expansion of $\left(x-\frac{1}{x^2}\right)^8$

Solution: We have, 7th term $= T_7 = T_{6+1}$

$$= {}^8C_6 . x^{8-6}\left(-\frac{1}{x^2}\right)^6$$

$$= 28.x^2.\frac{1}{x^{12}}$$

$$= \frac{28}{x^{10}}$$

Q7. Find the 11th term in the expansion of $(x+2y)^{15}$

Solution: We have, 11th term $= T_{11} = T_{10+1}$

$$= {}^{15}C_{10} . x^{15-10}(2y)^{10}$$

$$= {}^{15}C_{10} . x^5 . 2^{10}\ y^{10}$$

$$= {}^{15}C_{10} . 2^{10} . x^5 . y^{10}$$

Q8. Find the term containing x^6 **in the expansion of** $(1+x^2)^6$

Solution: Let T_{r+1} be the term containing x^6

Then, $T_{r+1} = {}^6C_r . 1^{6-r} . (x^2)^r$

$$= {}^6C_r . 1 . x^{2r}$$

Since, T_{r+1} contains x^6,

$\therefore\ x^{2r} = x^6$

$\Rightarrow 2r = 6$

$\Rightarrow r = 3$

Hence, the term $= {}^6C_3 . 1 . x^6$

$$= \frac{6\times5\times4}{1\times2\times3}.x^6 = 20x^6$$

Q9. Find the term containing x^9 in the expansion of $\left(x^2+\frac{1}{x}\right)^9$

Solution: Let T_{r+1} be the term containing x^9

Then, $T_{r+1} = {}^9C_r.(x^2)^{9-r}\left(\frac{1}{x}\right)^r$

$= {}^9C_r x^{18-2r}.x^{-r} = {}^9C_r x^{18-3r}$

Since, T_{r+1} contains x^9,

$\therefore\ x^9 = x^{18-3r}$

$\Rightarrow 9 = 18-3r$

$\Rightarrow 3r = 9$

$\Rightarrow r = 3$

Hence, the term $= {}^9C_3.x^9$

$= \frac{9\times8\times7}{1\times2\times3}.x^9 = 84x^9$

Q10. Find the co-efficient of x^4 in the expansion of $\left(x-\frac{1}{x}\right)^{10}$.

Solution: Let T_{r+1} be the term containing x^4

Then, $T_{r+1} = (-1)^r\ {}^{10}C_r x^{10-r}\left(\frac{1}{x}\right)^r$

$= (-1)^r.{}^{10}C_r.x^{10-2r}$

Since, T_{r+1} contains x^4,

$\therefore\ x^4 = x^{10-2r}$

$\Rightarrow 4 = 10-2r$

$\Rightarrow 2r = 6$

$\Rightarrow r = 3$

Hence, the term $= (-1)^3 . {}^{10}C_3 x^4$

$$= (-1).\frac{10\times 9\times 8}{1\times 2\times 3}.x^4$$

$$= -120x^4$$

$\therefore$ The co-efficient $= -120$

Q11. Find the term independent of x in the expansion of

(i) $\left(x^2 + \frac{1}{x}\right)^4$

Solution: Let T_{r+1} the term independent of x, i.e. containing x^0.

Then, $T_{r+1} = {}^9C_r (x^2)^{9-r} \left(\frac{1}{x}\right)^r$

$$= {}^9C_r x^{18-3r}$$

Since, T_{r+1} contains x^0,

$\therefore x^0 = x^{18-3r}$

$\Rightarrow 18 - 3r = 0$

$\Rightarrow r = \frac{18}{3} = 6$

Hence, the term $= {}^9C_6 .x^0$

$$= \frac{9\times 8\times 7}{1\times 2\times 3} = 84$$

(ii) $\left(x - \frac{1}{x^2}\right)^{12}$

Solution: Let T_{r+1} the term independent of x.

Then, $T_{r+1} = {}^{12}C_r .x^{12-r} . \left(-\frac{1}{x^2}\right)^r$

$$= (-1)^r . {}^{12}C_r x^{12-3r}$$

Since, T_{r+1} contains x^0,

$\therefore\ x^0 = x^{12-3r}$

$\Rightarrow 0 = 12 - 3r$

$\Rightarrow r = \frac{12}{3} = 4$

Hence, the 5^{th} term = $(-1)^4.{}^{12}C_{4.}.x^0$

$= 1.\frac{12\times 11\times 10\times 9}{1\times 2\times 3\times 4}$

$= 495$

(iii) $\left(x+\frac{2}{x^2}\right)^{15}$

Solution: Let T_{r+1} the term independent of x.

Then, $T_{r+1} = {}^{15}C_r.x^{15-r}.\left(\frac{2}{x^2}\right)^r = {}^{15}C_r x^{15-r}.2^r.x^{-2r}$

$= 2^r.{}^{15}C_r x^{15-3r}$

Since, T_{r+1} contains x^0,

$\Rightarrow 15 - 3r = 0$

$\Rightarrow r = \frac{15}{3} = 5$

Hence, the 6^{th} term is independent of x.

$\therefore$ The term $= 2^5.{}^{15}C_5.x^0$

$= {}^{15}C_5.2^5$

Q12. Find the middle terms in the expansion of

(i) $(1+x)^6$

Solution: Here, the index 6 is even.

$\therefore$ There is only one middle term and it is the $\left(\frac{6}{2}+1\right)^{th}$ term i.e. T_4

Now, $T_4 = T_{3+1}$

$= {}^6C_3x^3$

$= \frac{6\times5\times4}{1\times2\times3}x^3$

$= 20x^3$

(ii) $(x-y)^8$

Solution: Here, the index 8 is even.

$\therefore$ There is only one middle term and it is the $\left(\frac{8}{2}+1\right)^{th}$ term i.e. 5^{th} term.

$\therefore$ 5^{th} term $= T_5 = T_{4+1}$

$= (-1)^4.{}^8C_4.x^{8-4}.y^4$

$= 1.{}^8C_4x^4y^4$

$= \frac{8\times7\times6\times5}{1\times2\times3\times4}x^4y^4$

$= 70x^4y^4$

(iii) $(2x+3y)^{10}$

Solution: Here, the index 10 is even.

$\therefore$ There is only one middle term and it is the $\left(\frac{10}{2}+1\right)^{th}$ term i.e. 6^{th} term.

$\therefore$ 6^{th} term $= T_6 = T_{5+1}$

$= {}^{10}C_5(2x)^{10-5}.(3y)^5$

$= {}^{10}C_5(2x)^5(3y)^5$

$= {}^{10}C_5.2^5.3^5x^5y^5$

Q13. Find the middle term in the expansion of

(i) $(x+y)^7$

Solution: Here, the index 7 is odd.

$\therefore$ There are two middle terms. They are $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+3}{2}\right)^{th}$ terms. i.e. 4^{th} and 5^{th} terms.

$\therefore$ 4^{th} term $= T_4 \quad = T_{3+1}$

$= {}^7C_3 x^{7-3} y^3$

$= \frac{7\times6\times5}{1\times2\times3} x^4 y^3$

$= 35x^4y^3$

& 5^{th} term $= T_5 \quad = T_{4+1}$

$= {}^7C_4 x^{7-4} y^4$

$= \frac{7\times6\times5}{1\times2\times3} x^3 y^4 = 35x^3y^4$

(ii) $\left(x+\frac{1}{x}\right)^9$

Solution: Here, the index 9 is odd.

$\therefore$ There are two middle terms. They are $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ terms. i.e. 5^{th} and 6^{th} terms.

$\therefore$ 5^{th} term $= T_5 \quad = T_{4+1}$

$= {}^9C_4 x^{9-4} \left(\frac{1}{x}\right)^4$

$= \frac{9\times8\times7\times6}{1\times2\times3\times4} .x^5 . \frac{1}{x^4}$

$= 126x$

& 6^{th} term $= T_6 \quad = T_{5+1}$

$= {}^9C_5 x^{9-5} \left(\frac{1}{x}\right)^5$

$= \frac{9\times8\times7\times6}{1\times2\times3\times4} x^4 . \frac{1}{x^5}$

$= \frac{126}{x}$

(iii) $(x-2y)^{11}$

Solution: Here, the index 11 is odd.

$\therefore$ There are two middle terms. They are $\left(\frac{11+1}{2}\right)^{th}$ and $\left(\frac{11+3}{2}\right)^{th}$ terms. i.e. 6^{th} and 7^{th} terms.

$$\therefore\ 6^{th} \text{ term} = T_6 = T_{5+1}$$
$$= (-1)^5\ {}^{11}C_5 x^{11-5}(2y)^5$$
$$= (-1)^5\ {}^{11}C_5 x^6.2^5.y^5$$
$$= -\ {}^{11}C_5 2^5 x^6 y^5$$

$$\&\ 7^{th} \text{ term} = T_7 = T_{6+1}$$
$$= (-1)^6\ {}^{11}C_6 x^{11-6}(2y)^6$$
$$= 1.\ {}^{11}C_6 .x^5.2^6.y^6$$
$$= {}^{11}C_6 .2^6.x^5 y^6$$

Q14. If C_r denotes the binomial co-efficient. ${}^nC_r$, prove that

(i) $C_0 + 2C_1 + 4C_2 + ... + 2^n C_n = 3^n$

Solution: We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$$

Putting $x = 2$, we get

$$(1+2)^n = {}^nC_0 + {}^nC_1.2 + {}^nC_2.2^2 + ... + {}^nC_n.2^n$$
$$\Rightarrow 3^n = {}^nC_0 + 2.{}^nC_1 + 4\,{}^nC_2 + ... + 2^n.{}^nC_n$$
$$\text{i.e. } C_0 + 2.C_1 + 4C_2 + ... + 2^n.C_n = 3^n$$

(ii) $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = (n+1)2^n$

Solution: We have, $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

$$2^n = C_0 + C_1 + C_2 + ... + C_n$$

Now, $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n$

$$= \left(C_0 + C_1 + C_2 + ... + C_n\right) + \left(2C_1 + 4C_2 + ... + 2_n C_n\right)$$

$$= 2^n + 2\left(C_1 + 2C_2 + 3C_3 + ... +^n C_n\right)$$

$$= 2^n + 2\left(n + 2.\frac{n(n-1)}{1.2} + 3.\frac{n(n-1)(n-2)}{1.2.3} + ... + n.1\right)$$

$$= 2^n + 2n\left[1 + \left(n-1\right) + \frac{(n-1)(n-2)}{1.2} + ... +^{n-1} C_{n-1}\right]$$

$$= 2^n + 2n.2^{n-1}$$

$$= 2^n + n.2^n$$

$$= 2^n(1+n)$$

$$= (n+1)2^n$$

(iii) $C_0 + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

Solution: We have, $C_0 + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}}$

$$= n + \frac{2.\frac{n(n-1)}{1.2}}{n} + \frac{3.\frac{n(n-1)(n-2)}{1.2.3}}{\frac{n(n-1)}{1.2}} + ... + \frac{n.1}{n}$$

$$= n + (n-1) + (n-2) + ... + 1$$

$$= 1 + 2 + 3 + ... + n$$

$$= \frac{n(n+1)}{2}$$

(iv) $C_1 - 2C_2 + 3C_3 - \ldots\ldots\ldots\ldots + (-1)^{n-1} n.C_n = 0$

Solution: We have, $C_1 - 2C_2 + 3C_3 - \ldots\ldots\ldots\ldots + (-1)^{n-1} n.C_n$

$$= n - 2\left\{\frac{n(n-1)}{1.2}\right\} + 3\left\{\frac{n(n-1)(n-2)}{1.2.3}\right\} + \ldots\ldots\ldots + (-1)^{n-1}.n.1$$

$$= n\left\{1 - \frac{n-1)}{1} + \frac{(n-1)(n-2)}{1.2} - \ldots\ldots\ldots\ldots + (-1)^{n-2}\right\}$$

$$= n\left\{{}^{n-1}C_0 - {}^{n-1}C_1 + {}^{n-1}C_2 - \ldots\ldots\ldots\ldots\ldots\ldots + (-1)^{n-1}\,{}^{n-1}C_{n-1}\right\}$$

$$= n(1-1)^{n-1}$$

$$= n.o^{n-1}$$

$$= n.0$$

$$= 0$$

Q15. Evaluate the following by using Binomial theorem.

(i) 99^4

Solution: $99^4 = (100-1)^4$

$$= {}^4C_0.100^4 - {}^4C_1.100^3.1 + {}^4C_2.100^2.1^2 - {}^4C_3.100.1^3 + {}^4C_4.1^4$$

$$= 100^4 - 4.100^3 + 6.100^2 - 4.100 + 1$$

$$= 100000000 - 4000000 + 60000 - 400 + 1$$

$$= 100060001 - 4000400$$

$$= 96059601$$

(ii) 102^5

Solution: $102^5 = (100+2)^5$

$$= {}^5C_0.100^5 + {}^5C_1.100^4.2 + {}^5C_2 100^3.2^2 + {}^5C_3.100^2.2^3 + {}^5C_4 100^1.2^4 + {}^5C_5 2^5.$$

$$= 1.10000000000 + 5.100000000.2 + 10.1000000.4 + 10.10000.8 + 5.100.16 + 1.32$$

$$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$$

$$= 11040808032.$$

(iii) 1001^3

Solution: $1001^3 = (1000+1)^3$

$$= {}^3C_0.1000^3 + {}^3C_1.1000^2.1 + {}^3C_2.1000.1^2 + {}^3C_3.1^3$$

$$= 1000000000 + 3\times1000000 + 3\times1000 + 1$$

$$= 1000000000 + 3000000 + 3000 + 1$$

$$= 1003003001$$

Q16. Show that $(2+\sqrt{5})^4 + (2-\sqrt{5^4})$ **is rational.**

Solution: We have

$$(2+\sqrt{5})^4 = {}^4C_0.2^4 + {}^4C_1.2^3.\sqrt{5} + {}^4C_2.2^2.(\sqrt{5})^2 + {}^4C_3.2.(\sqrt{5})^3 + {}^4C_4(\sqrt{5})^4$$

$$= 1\times16 + 4\times8\sqrt{5} + 6\times4\times5 + 4\times2\times5\sqrt{5} + 25$$

$$= 16 + 32\sqrt{5} + 120 + 40\sqrt{5} + 25$$

$$= 161 + 72\sqrt{5} \qquad - \qquad \text{(i)}$$

and $(2-\sqrt{5})^4 = {}^4C_0.2^4 - {}^4C_1.2^3.\sqrt{5} + {}^4C_2.2^2(\sqrt{5})^2 - {}^4C_3.2(\sqrt{5})^3 + {}^4C_2(\sqrt{5})^4$

$$= 16 - 4\times8\sqrt{5} + 6\times4\times5 - 4\times2\times5\sqrt{5} + 25$$

$$= 16 - 32\sqrt{5} + 120 - 40\sqrt{5} + 25$$

$$= 161 - 72\sqrt{5} \qquad - \qquad \text{(ii)}$$

Adding (i) & (ii), we get

$$(2+\sqrt{5})^4 + (2-\sqrt{5})^4 = 161+161$$

$$= 322 \text{ which is rational.}$$

Q17. Using Binomial theorem, prove that $4^n - 3n - 1$ is divisible by 9 for $n \in N$.

Solution: We have

$$4^n = (1+3)^3$$

$$= {}^nC_0 + {}^nC_1.3^1 + {}^nC_2.3^2 + {}^nC_3.3^3 + \dots\dots\dots + {}^nC_n.3^n$$

$$= 1 + 3n + {}^nC_2.3^2 + {}^nC_3.3^3 + \dots\dots\dots + 3^n \qquad \left[\because {}^nC_1 = n\right]$$

$$4^n - 3n - 1 = {}^nC_2.3^2 + {}^nC_3.3^3 + \dots\dots\dots + {}^nC_n.3^n$$

$$= 3^2\left({}^nC_2 + {}^nC_3.3 + \dots\dots\dots + {}^nC_n.3^{n-2}\right)$$

$$= 9 \times \text{(an integer)}$$

$\therefore$ $4^n - 3n - 1$ is divisible by 9 for all $n \in N$.

Q18. Using Binomial theorem, prove that $2^{3n} - 7n$ $(n \in N)$ always leaves the remainder 1 when divided by 49.

Solution: We have,

$$2^{3n} = (2^3)^n = 8^n$$

$$= (1+7)^n$$

$$= {}^nC_0 + {}^nC_1.7 + {}^nC_2.7^2 + {}^nC_3 7^3 + \dots\dots\dots + {}^nC_n 7^n$$

$$= 1 + {}^nC_1.7 + {}^nC_2.7^2 + {}^nC_3.7^3 + \dots\dots\dots + 7^n$$

$$\Rightarrow 2^{3n} - 7n = 1 + 7n + {}^nC_2.7^2 + {}^nC_3.7^3 + \dots\dots\dots + 7^n - 7n$$

$$= 1 + 7^2({}^nC_2 + {}^nC_3.7 + \dots\dots\dots + 7^{n-2})$$

$$= 1 + \text{(a multiple of 49)}$$

Hence, the $2^{3n} - 7n$ always leaves the remainder 1 when divided by 49.

******